Siberian Mathematical Journal, Vol. 54, *No.* 2, *pp.* 205–211, 2013
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SEQUENCES OF INDEPENDENT WALSH FUNCTIONS IN BMO

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Abstract: Under examination are the sequences of independent Walsh functions in the space of functions of bounded mean oscillation. We study geometric properties of the subspaces spanned by the sequences; in particular, some necessary and sufficient conditions are found for such a subspace to be complemented.

DOI: 10.1134/S0037446613020031

Keywords: Rademacher functions, Walsh functions, space of functions of bounded mean oscillation, complemented subspace

1. Introduction and Statement of the Main Results

The space BMO $=$ BMO$[0,1]$ consists of the functions $f \in L_1[0,1]$ of bounded mean oscillation, that is, satisfying the condition

$$\|f\|_{\mathrm{BMO}} := \sup_I \frac{1}{|I|} \int\limits_I |f(u) - f_I|\, du < \infty,$$

where the supremum is taken over all intervals $I \subset [0,1]$ and $f_I := \frac{1}{|I|}\int_I f(u)\,du$. If we use only the dyadic intervals $I_m^i = ((i-1)2^{-m}, i2^{-m}]$ $(m = 0,1,2,\dots,\ i = 1,\dots,2^m)$ in the definition then we obtain the dyadic space BMO_d. Clearly, $\mathrm{BMO} \subset \mathrm{BMO}_d$ and $\|f\|_d := \|f\|_{\mathrm{BMO}_d} \le \|f\|_{\mathrm{BMO}}$ for all $f \in \mathrm{BMO}$. Moreover, $L_\infty[0,1] \subset \mathrm{BMO}$ and $\|f\|_{\mathrm{BMO}} \le \|f\|_{L_\infty}$ for all $f \in L_\infty = L_\infty[0,1]$. At the same time, $\mathrm{BMO} \ne L_\infty$ and $\mathrm{BMO}_d \ne \mathrm{BMO}$. For example, $\log|s-1/2|\chi_{[0,1]}(s) \in \mathrm{BMO}\setminus L_\infty$ and $\log|s-1/2|\chi_{[1/2,1]}(s) \in \mathrm{BMO}_d \setminus \mathrm{BMO}$.

Given an increasing sequence of positive integers $1 \le p_1 < p_2 < \cdots$, consider the sequence of Walsh functions

$$f_k = \prod_{i\in A_k} r_i, \quad A_k \subset \{p_k+1, p_k+2, \dots, p_{k+1}\}, \tag{1}$$

where $r_i(t)$ are the Rademacher functions on $[0,1]$; i.e., $r_i(t) = \mathrm{sgn}[\sin(2^i\pi t)]$ $(i \in \mathbb{N})$.

In view of [1, Theorem 1], the system $\{f_k\}_{k=1}^\infty$ defined by (1) is equivalent in BMO_d to the canonical basis of l_2; more exactly, for every finite sequence $a = (a_k)_{k=1}^\infty$, we have

$$\frac{1}{\sqrt{2}}\|a\|_2 \le \left\|\sum_{k=1}^\infty a_k f_k\right\|_d \le \sqrt{2}\|a\|_2, \tag{2}$$

where $\|a\|_2 := \left(\sum_{k=1}^\infty a_k^2\right)^{1/2}$. In this article it is proven that the subspace $[f_k]$ generated by a sequence $\{f_k\}_{k=1}^\infty$ (i.e. the closed linear span of $\{f_k\}$) is complemented in BMO_d. Recall that the closed subspace Y of a Banach space X is called *complemented* in X if there exists a bounded linear projection $P : X \to X$ whose range coincides with Y.

The main aim of the present article is the study of a system $\{f_k\}$ defined in (1) in the "usual" BMO space. It is important to note that the last space is invariant under translations in contrast to its

The first author was supported by the Russian Foundation for Basic Research (Grant 10–01–00077).

Samara and Luleå. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 54, No. 2, pp. 270–278, March–April, 2013. Original article submitted January 19, 2012.

dyadic version BMO_d and so it plays a more important role in analysis. We demonstrate that (2) may fail in BMO, while geometric properties of the subspace $[f_k]$ (in particular, its complementability) are essentially determined by evenness of the number of elements in the sets A_k. The present results are a natural continuation of those of the article [2] in which similar questions are studied for the Rademacher system. In particular, Theorems 1(a) and 3 can be considered as an extension of Theorems 2 and 4 in [2].

We present the main results of the article, where an expression of the form $f \asymp g$ means that $cf \le g \le Cf$ for some constants $c > 0$ and $C > 0$ independent of all or some part of arguments of the norms (seminorms) f and g. Moreover, we assume in what follows that A_k and f_k are defined by (1).

Theorem 1. (a) *If every set A_k contains an odd number of elements then, for some universal constants and every sequence $a = (a_k)_{k=1}^{\infty} \in l_2$, we have*

$$\Big\|\sum_{k=1}^{\infty} a_k f_k\Big\|_{\mathrm{BMO}} \asymp \|a\|_2 + \sup_{s\ge 1}\Big|\sum_{k=1}^{s} a_k\Big|. \tag{3}$$

(b) *If every set A_k contains an even number of elements then, for some universal constants and every sequence $a = (a_k)_{k=1}^{\infty} \in l_2$, we have*

$$\Big\|\sum_{k=1}^{\infty} a_k f_k\Big\|_{\mathrm{BMO}} \asymp \|a\|_2. \tag{4}$$

Theorem 2. *If $\{k_i\}$ $(k_1 < k_2 < \cdots)$ is the set of all $k \in \mathbb{N}$ for which the number of elements of all A_k is odd then, for some universal constants and every sequence $a = (a_k)_{k=1}^{\infty} \in l_2$, we have*

$$\Big\|\sum_{k=1}^{\infty} a_k f_k\Big\|_{\mathrm{BMO}} \asymp \|a\|_2 + \sup_{s\ge 1}\Big|\sum_{i=1}^{s} a_{k_i}\Big|. \tag{5}$$

Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ of elements in a Banach space X is called a *basic sequence* if it is a basis of the closed linear span $[x_n]$. Moreover, a basic sequence $\{x_n\}_{n=1}^{\infty}$ is referred to as *unconditional* whenever, for every permutation π of positive integers, the sequence $\{x_{\pi(k)}\}_{k=1}^{\infty}$ is a basic sequence in X as well. As is known (see, for instance, [3, Theorem 1.1]), a basic sequence $\{x_n\}_{n=1}^{\infty}$ is unconditional in X if and only if the convergence of a series $\sum_{n=1}^{\infty} a_n x_n$ $(a_n \in \mathbb{R})$ ensures the convergence of a series $\sum_{n=1}^{\infty} \theta_n a_n x_n$, where $\{\theta_n\}_{n=1}^{\infty}$ is an arbitrary sequence of signs, i.e., $\theta_n = \pm 1$. Thereby, Theorem 2 yields

Corollary 1. *The following are equivalent:*
(1) *$\{f_k\}_{k=1}^{\infty}$ is an unconditional basic sequence in* BMO*;*
(2) *$\{f_k\}_{k=1}^{\infty}$ is equivalent in* BMO *to the canonical basis of l_2;*
(3) *all sets A_k, except possibly finitely many, contain an even number of elements.*

The last result gives a necessary and sufficient condition for the subspace $[f_k]$ to be complemented in BMO.

Theorem 3. *The following are equivalent:*
(1) *the subspace $[f_k]$ generated by $\{f_k\}_{k=1}^{\infty}$ is complemented in* BMO*;*
(2) *all sets A_k, except possibly finitely many, contain an even number of elements.*

2. Proofs

To prove Theorem 1, we need two auxiliary propositions; the former of them is the contents of Problem 12(b) in [4, p. 266] (see its proof with the constants below in [2, Proposition 1]).

Define the functional $A(f) = \sup_{I_1,I_2} |f_{I_1} - f_{I_2}|$, where I_1 and I_2 are adjacent dyadic intervals of the same length.

Proposition 1. *For all $f \in L_1[0,1]$, we have*

$$\frac{1}{3}(\|f\|_d + A(f)) \le \|f\|_{\mathrm{BMO}} \le 32(\|f\|_d + A(f)). \tag{6}$$

Proposition 2. *Let functions f_k be defined in* (1) *and let a sequence of reals $a = (a_k)$ be finite. Put $f := \sum_{k\ge1} a_k f_k$.*

(i) *If every set A_k contains an odd number of elements, then*

$$\frac{2}{3}\max_{s\ge1}\left|\sum_{k=1}^{s} a_k\right| \le A(f) \le 8\max_{s\ge1}\left|\sum_{k=1}^{s} a_k\right|. \tag{7}$$

(ii) *If every set A_k contains an even number of elements, then*

$$A(f) = 2\max_{s\ge1}|a_s|. \tag{8}$$

Proof. Let I be an arbitrary dyadic interval of length 2^{-r}, i.e., $I = I_r^i = ((i-1)2^{-r}, i2^{-r}]$. Put $m_k := \min A_k$, $M_k := \max A_k$, and $s = s(r) = \max\{k : M_k \le r\}$. If $M_k > r$ for all k then we set $s = 0$. In this case

$$(f_k)_I = \begin{cases} \operatorname{sgn}(f_k|_I), & k \le s, \\ 0, & k > s. \end{cases}$$

Indeed, if $k \le s$ then f_k is constant on I. In the case of $k > s$, we have $f_k = \prod_{i\in B_k\cup C_k} r_i$, and r_i are constant on I for all $i \in B_k$, while $\int_I \prod_{i\in C_k} r_i = 0$. Hence, since r_i, $i \in C_k$, are independent with respect to I, we infer

$$(f_k)_I = \frac{1}{|I|}\int_I f_k(t)\,dt = \frac{c}{|I|}\int_I \prod_{i\in C_k} r_i(t)\,dt = \frac{c}{|I|}\prod_{i\in C_k}\int_I r_i(t)\,dt = 0,$$

where c is the value of $\prod_{i\in B_k} r_i$ on I.

Therefore, if I_1 and I_2 are adjacent dyadic intervals of length 2^{-r} then

$$f_{I_1} - f_{I_2} = \sum_{k=1}^{s} a_k \operatorname{sgn}(f_k|_{I_1}) - \sum_{k=1}^{s} a_k \operatorname{sgn}(f_k|_{I_2}) = \sum_{k=1}^{s} a_k[\operatorname{sgn}(f_k|_{I_1}) - \operatorname{sgn}(f_k|_{I_2})]. \tag{9}$$

Let I be the least dyadic interval containing the union of I_1 and I_2. Assume that its length is 2^{-j}. The definition of I implies that the union $I_1 \cup I_2$ lies in the middle of I (let I_1 lie on the left). As is easily seen, the index j runs over all integers from 0 to $r-1$ in dependence on I_1 and I_2.

Below, we need the following notation: If $a, b \in \mathbb{N}$ and $a \le b$ then $[a,b] := \{i \in \mathbb{N} : a \le i \le b\}$. In view of the definition of s, we have $\bigcup_{k=1}^{s} A_k \subset [1,r]$. Put

$$B := \bigcup_{l=1}^{s}[m_k, M_k] \quad \text{and} \quad B' := [1,r]\setminus B = \bigcup_{l=1}^{s+1}[M_{l-1}+1, m_l - 1],$$

where $M_0 = 0$ and $m_{s+1} = r+1$. Moreover, in what follows, $|A|$ stands for the number of elements in a set $A \subset \mathbb{N}$.

Let us compare the values of f_k, $1 \le k \le s$, on the intervals I_1 and I_2. If $m_k > j+1$, then all r_i with indices from A_k change their signs under the passage from I_1 to I_2; in this case $r_i|_{I_1} = -1$ and $r_i|_{I_2} = 1$. Hence, for these k, we have $f_k|_{I_1} = (-1)^{|A_k|}$ and $f_k|_{I_2} = 1$. If $M_k \le j$ then all r_i with indices from A_k are constant on I and so $f_k|_{I_1} = f_k|_{I_2}$. In dependence on which set B or B' contains the number $j+1$, we examine two cases.

(a) $j+1 \in B'$. In other words, $M_{l-1}+1 \le j+1 \le m_l - 1$ for some $l = 1, 2, \dots, s+1$. If $l = s+1$ then $j \ge M_s$ and so from (9) and the above remarks it follows that $f_{I_1} - f_{I_2} = 0$. In the case when the previous inequality is fulfilled for $l = 1, 2, \dots, s$, we see that $M_{l-1} \le j < m_l - 2$ and, by the same reasons,

$$f_{I_1} - f_{I_2} = \sum_{k=l}^{s} a_k \left(f_k|_{I_1} - f_k|_{I_2}\right) = \sum_{k=l}^{s} a_k \left((-1)^{|A_k|} - 1\right). \tag{10}$$

Since j takes all values from 0 to $r-1$, equation (10) is true for $l \in \mathscr{F}_1$, where the set $\mathscr{F}_1$ consists of those numbers k for which the segment $[M_{k-1}+1, m_k - 1]$ is nonempty (if $2 \le k \le s$, then the latter is equivalent to the fact that there is a "gap" between the sets A_{k-1} and A_k; $1 \in \mathscr{F}_1$ if $1 \notin A_1$).

(b) $j+1 \in B$. In this case $m_l \le j+1 \le M_l$ for some $l = 1, 2, \dots, s$ and the functions r_i change their signs under the passage from I_1 to I_2 if $i \in A_l^{(j)} := \{p \in A_l : p > j\}$ and they are constant on I if $i \in A_l \setminus A_l^{(j)}$. Hence, $f_l|_{I_2} = (-1)^{|A_l^{(j)}|} f_l|_{I_1}$. Since $j+1 < m_{l+1}$ and $j \ge M_{l-1}$, taking into account the above remarks, by (9), we establish

$$f_{I_1} - f_{I_2} = a_l f_l|_{I_1}\left(1 - (-1)^{|A_l^{(j)}|}\right) + \sum_{k=l+1}^{s} a_k\left((-1)^{|A_k|} - 1\right). \tag{11}$$

Assume that all numbers $|A_k|$ are odd. First of all, (10) implies that

$$|f_{I_1} - f_{I_2}| = 2\left|\sum_{k=l}^{s} a_k\right|, \quad l \in \mathscr{F}_1. \tag{12}$$

If there exists j such that $m_l \le j+1 \le M_l$ and either $f_l|_{I_1} = -1$ or $\left|A_l^{(j)}\right|$ is even, then (11) ensures (12) (in the second case l should be replaced with $l+1$). Denote by $\mathscr{F}$ the set of l such that (12) holds. Clearly, $\mathscr{F} \supset \mathscr{F}_1$.

It is easy to see that, for $l = 1, 2, \dots, s$, we have one more possibility. If $j = m_l - 1$ then $A_l^{(j)} = A_l$. Since $r_{m_l}|_{I_1} = 1$ and $r_i|_{I_1} = -1$, if $i \in A_l$, $i \ne m_l$ then $f_l|_{I_1} = 1$. Hence, (11) yields

$$|f_{I_1} - f_{I_2}| = 2\left|\sum_{k=l+1}^{s} a_k - a_l\right|, \quad l = 1, 2, \dots, s. \tag{13}$$

For a given pair of adjacent dyadic intervals I_1 and I_2, the quantity $|f_{I_1} - f_{I_2}|$ is determined by equalities (12) or (13). Therefore, according to the definition of the functional $A(f)$ we obtain

$$A(f) = 2\max\left(\max_{1\le l\le s<\infty}\left|\sum_{k=l+1}^{s} a_k - a_l\right|, \max_{\substack{1\le l\le s<\infty,\\ l\in\mathscr{F}}}\left|\sum_{k=l}^{s} a_k\right|\right).$$

Moreover, every finite sequence (a_k) satisfies the inequality

$$\frac{1}{3}\max_{1\le l\le s<\infty}\left|\sum_{k=l}^{s} a_k\right| \le \max_{1\le l\le s<\infty}\left|\sum_{k=l+1}^{s} a_k - a_l\right| \le 2\max_{1\le l\le s<\infty}\left|\sum_{k=l}^{s} a_k\right|. \tag{14}$$

Indeed, on the one hand, for arbitrary $1 \le l \le s$, we have

$$\left|\sum_{k=l}^{s} a_k\right| \le \left|\sum_{k=l+1}^{s} a_k - a_l\right| + 2|a_l| \le 3\max_{1\le l\le s<\infty}\left|\sum_{k=l+1}^{s} a_k - a_l\right|$$

(letting in the case $s = l$ the sum $\sum_{k=l+1}^{s} a_k$ to be equal to zero). Conversely, for $1 \le l \le s$, we infer

$$\left|\sum_{k=l+1}^{s} a_k - a_l\right| \le \left|\sum_{k=l+1}^{s} a_k\right| + |a_l| \le 2\max_{1\le l\le s<\infty}\left|\sum_{k=l}^{s} a_k\right|.$$

Moreover,

$$\max_{s\ge 1}\Bigl|\sum_{k=1}^{s} a_k\Bigr| \le \max_{1\le l\le s<\infty}\Bigl|\sum_{k=l}^{s} a_k\Bigr| \le 2\max_{s\ge 1}\Bigl|\sum_{k=1}^{s} a_k\Bigr|. \tag{15}$$

The left inequality in (15) is obvious and the right one follows from the estimate

$$\Bigl|\sum_{k=l}^{s} a_k\Bigr| \le \Bigl|\sum_{k=1}^{s} a_k - \sum_{k=1}^{l} a_k\Bigr| \le \Bigl|\sum_{k=1}^{s} a_k\Bigr| + \Bigl|\sum_{k=1}^{l} a_k\Bigr| \le 2\max_{s\ge 1}\Bigl|\sum_{k=1}^{s} a_k\Bigr|$$

which is valid for arbitrary $1 \le l \le s$. As a result, the left inequality in (7) follows from the above inequality for $A(f)$ and the left inequality in (14). To obtain the right inequality in (7), it suffices to use the right-hand sides in (14) and (15).

The case when every set A_k contains an even number of elements is much simpler. In fact, equalities (10) and (11) ensure that either $f_{I_1} - f_{I_2} = 0$ or $|f_{I_1} - f_{I_2}| = 2|a_l|$, $l = 1, 2, \dots, s$. Thereby, (8) is proved.

Proof of Theorem 1. It suffices to apply Propositions 1 and 2 together with (2).

Proof of Theorem 2. Let $f := \sum_{k=1}^{\infty} a_k f_k$. Relations (10) and (11) and the same arguments as in the proof of Proposition 2 imply that

$$A(f) \asymp \max\Bigl(\sup_{k=1,2,\dots} |a_k|,\ \sup_{s\ge 1}\Bigl|\sum_{i=1}^{s} a_{k_i}\Bigr|\Bigr),$$

where $\{k_i\}$ ($k_1 < k_2 < \cdots$) is the set of all $k \in \mathbb{N}$ such that $|A_k|$ is odd. Thereby, Proposition 1 and (2) justify the claim.

To prove Theorem 3, we need one more auxiliary statement about the block bases of a sequence $\{f_k\}_{k=1}^{\infty}$ in BMO.

Denote by $U = \{u_n\}_{n=1}^{\infty}$ an arbitrary block basis of the sequence $\{f_k\}_{k=1}^{\infty}$, i.e.,

$$u_n = \sum_{k=s_n+1}^{s_{n+1}} a_k f_k, \quad n = 1, 2, \dots,$$

where $1 \le s_1 < s_2 < \cdots$ and $a_k \in \mathbb{R}$. Moreover, let

$$\gamma_n(U) = \sum_{k=s_n+1}^{s_{n+1}} a_k, \quad n = 1, 2, \dots.$$

Proposition 3. *If a sequence $\{f_k\}_{k=1}^{\infty}$ contains infinitely many elements with an odd $|A_k|$, then the closed linear span $[f_k] \subset \mathrm{BMO}$ contains a subspace E isomorphic to c_0 and complemented in $[f_k]$.*

Proof. Let $\mathscr{E}$ be the set of all $k \in \mathbb{N}$ for which $|A_k|$ is odd. By condition, $|\mathscr{E}| = \infty$. Propositions 1 and 2 together with (2) ensure the existence of a block basis $U = \{u_n\}_{n=1}^{\infty}$ satisfying the conditions

(a) $\|u_n\|_{\mathrm{BMO}} = 1$, $n = 1, 2, \dots$;

(b) $\|u_n\|_d \le \sqrt{2}\bigl(\sum_{k=s_n+1}^{s_{n+1}} a_k^2\bigr)^{1/2} \le 2^{-n-5}$, $n = 1, 2, \dots$;

(c) $\gamma_n(U) = 0$, $n = 1, 2, \dots$;

(d) $\{k : a_k \ne 0\} \subset \mathscr{E}$.

Indeed, assume that we already constructed $u_1, \dots, u_{n-1}$. Given $n \in \mathbb{N}$, there exist a sufficiently large $m_n \in \mathbb{N}$ and numbers $b_k \ge 0$, $k = 1, 2, \dots, m_n$ such that

$$\Bigl(\sum_{k=1}^{m_n} b_k^2\Bigr)^{1/2} \le \frac{2}{9}\cdot 2^{-n-7} \quad \text{and} \quad \sum_{k=1}^{m_n} b_k = 1. \tag{16}$$

Put $a'_k = b_k$ if $k = 1, 2, \dots, m_n$ and $a'_k = -b_{k-m_n}$, if $k = m_n + 1, \dots, 2m_n$. Choose a sufficiently large number $s_{n+1} > s_n$ such that the segment $\{s_n + 1, \dots, s_{n+1}\}$ of positive integers contains at least $2m_n$ elements of $\mathscr{E}$, and introduce new indices for a'_k from the intersection $\{s_n+1, \dots, s_{n+1}\} \cap \mathscr{E}$ preserving the order. For the remaining numbers k from the set $\{s_n+1, \dots, s_{n+1}\}$, we put $a'_k = 0$. If $u'_n := \sum_{k=s_n+1}^{s_{n+1}} a'_k f_k$, then, in view of (2) and the first relation in (16), we find that

$$\|u'_n\|_d \le \sqrt{2} \cdot 2 \left(\sum_{k=1}^{m_n} b_k^2 \right)^{1/2} \le \frac{2}{9} \cdot 2^{-n-5}.$$

Moreover, since the definition of a'_k and the second equality in (16) imply that

$$\max_{s_n+1 \le s \le s_{n+1}} \left| \sum_{k=s_n+1}^{s} a'_k \right| = \sum_{k=1}^{m_n} b_k = 1,$$

Propositions 1 and 2 yield

$$\frac{2}{3} \le A(u'_n) \le 8 \quad \text{and} \quad \frac{2}{9} \le \|u'_n\|_{\mathrm{BMO}} \le 288.$$

Thereby, it is easy to see that the function $u_n := \frac{u'_n}{\|u'_n\|_{\mathrm{BMO}}}$ meets (a), (b), and (d). Condition (c) holds since $\sum_{k=s_n+1}^{s_{n+1}} a'_k = 0$ by construction.

Let us show that the subspace $E := [u_n]$, spanned by the functions of this block basis in BMO, is isomorphic to c_0.

Take $f \in [u_n]$. If $f = \sum_{n=1}^{\infty} \beta_n u_n$ ($\beta_n \in \mathbb{R}$) then

$$f = \sum_{n=1}^{\infty} \left(\sum_{k=s_n+1}^{s_{n+1}} \beta_n a_k f_k \right) = \sum_{k=1}^{\infty} \gamma_k f_k,$$

where $\gamma_k = \beta_n a_k$ for $k = s_n + 1, \dots, s_{n+1}$. Assuming that $p, q \in \mathbb{N}$ satisfy the inequalities $s_{n-1} \le p < s_n < s_{n+l} < q \le s_{n+l+1}$ with some positive integers n and l, we estimate the sum $\sum_{k=p}^{q} \gamma_k$. In view of (d), (c), and (a), together with Propositions 1 and 2(i), we infer

$$\begin{aligned}
\left| \sum_{k=p}^{q} \gamma_k \right| &= \left| \sum_{k=p}^{s_n} \gamma_k + \sum_{k=s_n+1}^{s_{n+l}} \gamma_k + \sum_{k=s_{n+l}+1}^{q} \gamma_k \right| \\
&= \left| \sum_{k=p}^{s_n} \beta_{n-1} a_k + \sum_{i=n}^{n+l-1} \sum_{k=s_i+1}^{s_{i+1}} \beta_i a_k + \sum_{k=s_{n+l}+1}^{q} \beta_{n+l} a_k \right| \\
&\le |\beta_{n-1}| \left| \sum_{k=p}^{s_n} a_k \right| + \sum_{i=n}^{n+l-1} |\beta_i| \left| \sum_{k=s_i+1}^{s_{i+1}} a_k \right| + |\beta_{n+l}| \left| \sum_{k=s_{n+l}+1}^{q} a_k \right| \\
&\le \sup_n |\beta_n| \left(\left| \sum_{k=p}^{s_n} a_k \right| + \left| \sum_{k=s_{n+l}+1}^{q} a_k \right| \right) \le \sup_n |\beta_n| \left(\frac{3}{2} A(u_{n-1}) + \frac{3}{2} A(u_{n+l}) \right) \\
&\le \frac{9}{2} (\|u_{n-1}\|_{\mathrm{BMO}} + \|u_{n+l}\|_{\mathrm{BMO}}) \|\{\beta_n\}\|_{c_0} = 9 \|\{\beta_n\}\|_{c_0}.
\end{aligned}$$

Applying Theorem 1(a), inequality (2), and properties (a) and (b), we find that

$$\begin{aligned}
\|f\|_{\mathrm{BMO}} &\le C \left(\|f\|_d + \sup_{q \ge 1} \left| \sum_{k=1}^{q} \gamma_k \right| \right) \le C \left(\left\| \sum_{n=1}^{\infty} \beta_n u_n \right\|_d + 9 \|\{\beta_n\}\|_{c_0} \right) \\
&\le C \left(\sum_{n=1}^{\infty} 2^{-n} + 9 \right) \|\{\beta_n\}\|_{c_0} = 10C \|\{\beta_n\}\|_{c_0}.
\end{aligned}$$

On the other hand, Proposition 1, (a), and (b) imply that

$$A(u_n) \geq \frac{1}{32}\|u_n\|_{\mathrm{BMO}} - \|u_n\|_d \geq \frac{1}{32} - 2^{-n-5} \geq \frac{1}{64}$$

for all $n \in \mathbb{N}$. Hence, applying Propositions 1 and 2 once again (also see inequalities (7) and (15)), we arrive at the inequality

$$\|f\|_{\mathrm{BMO}} \geq \frac{1}{3}A(f) \geq \frac{1}{9}\sup_{n\in\mathbb{N}}\sup_{s_n+1\leq p<q\leq s_{n+1}}\left|\sum_{k=p}^{q}\beta_n a_k\right| \geq \frac{1}{72}A(u_n)\|\{\beta_n\}\|_{c_0} \geq \frac{1}{72\cdot 64}\|\{\beta_n\}\|_{c_0}.$$

Thus, the system $\{u_n\}$ is equivalent in BMO to the canonical basis of c_0, and so E is isomorphic to c_0. Since $[f_k]$ is separable, by the Sobczyk theorem (see, for instance, [5, Corollary 2.5.9]), E is complemented in $[f_k]_{k=1}^{\infty}$ and the proposition is proved.

Proof of Theorem 3. First, assume that all sets A_k, except possibly finitely many, contain an even number of elements. By Corollary 1, $\{f_k\}_{k=1}^{\infty}$ is equivalent in BMO to the canonical basis of l_2. On the other hand, one of the consequences of the classical John–Nirenberg inequality (see, for instance, [6, Remark 3.19]) is the embedding $\mathrm{BMO} \subset L_p[0,1]$ for $1 \leq p < \infty$. Since $\{f_k\}_{k=1}^{\infty}$ is an orthonormal system in $L_2[0,1]$, the corresponding orthogonal projection P is bounded in $L_2[0,1]$, its range is the subspace $[f_k]$, and the sequence $\{f_k\}_{k=1}^{\infty}$ in $L_2[0,1]$ is also equivalent to the canonical basis of l_2. Hence, for every function $f \in \mathrm{BMO}$, we have

$$\|Pf\|_{\mathrm{BMO}} \asymp \|Pf\|_{L_2} \leq \|P\|\|f\|_{L_2} \leq C\|P\|\|f\|_{\mathrm{BMO}},$$

i.e., P is bounded in BMO and its range coincides with the closed linear span $[f_k]$ in this space. Thereby, the subspace $[f_k]$ is complemented in BMO.

We prove the converse. Assume that $[f_k]$ is complemented in BMO and the set $\mathscr{E}$ of the numbers $k \in \mathbb{N}$ with an odd $|A_k|$ is infinite. Let $P_1 : \mathrm{BMO} \to [f_k]$ be a bounded linear projection whose range coincides with $[f_k]$. By Proposition 3, there exists a subspace E complemented in $[f_k]$ and isomorphic to c_0. Let $P_2 : [f_k] \to E$ be a bounded linear projection whose range is E. In this case $P := P_2 \circ P_1$ is a linear projection bounded in BMO with the range E. Hence, BMO contains a complemented subspace isomorphic to c_0. Since BMO is a dual space (more exactly, $\mathrm{BMO} = (\mathrm{Re}\, H_1)^*$; see, for instance, [3, Theorem 5.5]), it contradicts to the known Bessaga–Pełczyński result that a dual space does not contain a complemented subspace isomorphic to c_0 (see [7, Corollary 4]). Thus, if the set $\mathscr{E}$ of numbers $k \in \mathbb{N}$ with an odd $|A_k|$ is infinite, the subspace $[f_k]$ is not complemented in BMO, and the theorem is proved. $\square$

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