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Spaces defined by the Paley function

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Abstract. The paper is concerned with Haar and Rademacher series in symmetric spaces, and also with the properties of spaces defined by the Paley function. In particular, the symmetric hull of the space of functions with uniformly bounded Paley function is found.

Bibliography: 27 titles.

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§ 1. Introduction and preliminaries

A Banach space E of measurable functions on $[0,1]$ is called a *symmetric space* or a *rearrangement invariant space* if:

1) E is an ideal lattice; that is, if $|x(t)| \leqslant |y(t)|$ for almost all $t \in [0,1]$ and $y \in E$, then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$;

2) if $y \in E$ and if functions x and y are equimeasurable; that is,

$$\bigl|\{t \in [0,1] : |y(t)| > u\}\bigr| = \bigl|\{t \in [0,1] : |x(t)| > u\}\bigr|, \qquad u > 0,$$

where $|a|$ is the Lebesgue measure of a set $a \subset \mathbb{R}$, then $x \in E$ and $\|x\|_E = \|y\|_E$.

In particular, any measurable function $x(t)$ on $[0,1]$ is equimeasurable with its nonincreasing left-continuous rearrangement

$$x^*(t) := \inf\bigl\{u \geqslant 0 : |\{s \in [0,1] : |x(s)| > u\}| < t\bigr\}, \qquad 0 < t \leqslant 1.$$

Given a symmetric space X on $[0,1]$, its *associate space* (the *Köthe dual*) X' consists of all y for which

$$\|y\|_{X'} = \sup\biggl\{\int_0^1 x(t)y(t)\,dt : \|x\|_X \leqslant 1\biggr\} < \infty.$$

The space X', which is also symmetric, is isometrically embedded in the (Banach) dual X^*; here, $X' = X^*$ if and only if X is separable. A symmetric space X is *maximal* if the conditions $x_n \in X$, $n = 1, 2, \dots$, $\sup_{n=1,2,\dots} \|x_n\|_X < \infty$ and $x_n \to x$ almost everywhere on $[0,1]$ imply that $x \in X$ and $\|x\|_X \leqslant \limsup_{n\to\infty} \|x_n\|_X$. Throughout the paper we will follow [1] in assuming that all the symmetric spaces

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in question are separable or maximal. We also assume that the normalization condition $\|\kappa_{[0,1]}\|_E = 1$ is satisfied, where $\kappa_e(t)$ is the characteristic function of a measurable subset $e \subset [0,1]$. We note that $L_\infty \subset E \subset L_1$ and $\|x\|_{L_1} \leqslant \|x\|_E \leqslant \|x\|_{L_\infty}$ for any $x \in L_\infty$. In any symmetric space E, the dilation operator $\sigma_\tau x(t) = x(t/\tau)\kappa_{[0,1]}(t/\tau)$, $0 \leqslant t \leqslant 1$, $\tau > 0$, is bounded. Also, $\|\sigma_\tau\|_{E\to E} \leqslant \max(1,\tau)$ (see Theorems 2.4.4 and 2.4.5 in [2]). The numbers

$$\alpha_E = \lim_{\tau\to 0} \frac{\ln\|\sigma_\tau\|_E}{\ln\tau}, \qquad \beta_E = \lim_{\tau\to\infty} \frac{\ln\|\sigma_\tau\|_E}{\ln\tau}$$

are known as the (upper and lower) *Boyd indices* of a symmetric space E. The inequality $0 \leqslant \alpha_E \leqslant \beta_E \leqslant 1$ always holds. For example, $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $p \in [1,\infty]$.

Orlicz spaces give an important class of symmetric spaces. Let $M(u)$ be a strictly increasing convex function on $[0,\infty)$, $M(0) = 0$. By definition, the Orlicz space L_M consists of all $x \in L_1$ for which

$$\int_0^1 M\biggl(\frac{|x(t)|}{\lambda}\biggr)\,dt \leqslant 1$$

for sufficiently large $\lambda > 0$. The functional $\|x\|_{L_M} = \inf\lambda$ gives a norm on L_M, the infimum being taken over all $\lambda > 0$ satisfying the previous inequality. We let $\exp L_2$ denote the Orlicz space constructed from the function $M(u) = e^{u^2} - 1$, and let G denote the closure of L_∞ in $\exp L_2$. The following characterization of these spaces is well known (see [3] or [4], Lemma 3.2): a measurable function x lies in $\exp L_2$ (in G) if and only if

$$\sup_{0<t\leqslant 1} \frac{x^*(t)}{\log_2^{1/2}(2/t)} < \infty \qquad \biggl(\text{respectively,}\quad \lim_{t\to 0} \frac{x^*(t)}{\log_2^{1/2}(2/t)} = 0\biggr). \tag{1.1}$$

Let $\mathbb{D}$ be the family of *dyadic* intervals in $[0,1]$; these are intervals of the form $\Delta_n^k = ((k-1)2^{-n}, k2^{-n})$, $1 \leqslant k \leqslant 2^n$, $n = 0,1,\dots$. The functions $\chi_0^0(t) = 1$,

$$\chi_n^k(t) = \chi_{\Delta_n^k}(t) = \begin{cases} 1, & t \in \Delta_{n+1}^{2k-1}, \\ -1, & t \in \Delta_{n+1}^{2k}, \\ 0 & \text{for the remaining } t \in [0,1] \end{cases}$$

form the *Haar system*. We let Ω denote the index set of the Haar functions; that is, $\Omega := \{(0,0), (n,k),\ 1 \leqslant k \leqslant 2^n,\ n = 0,1,\dots\}$. The formula $m = 2^n + k$ establishes a one-to-one correspondence between Ω and the set of natural numbers $\mathbb{N}$, letting us represent this system using only one index: $\{\chi_m\}_{m\in\mathbb{N}}$. The functions $2^{n/2}\chi_n^k$, $(n,k) \in \Omega$, form a complete orthogonal system which is a monotone basis in any separable symmetric space. A necessary and sufficient condition that the Haar system forms an unconditional basis in a separable symmetric space E is that $0 < \alpha_E \leqslant \beta_E < 1$ (see [1], Theorem 2.c.6, or [2], Theorem 2.9.6). An equivalent condition is that $\|x\|_E \asymp \|Px\|_E$, where

$$x(t) = \sum_{(n,k)\in\Omega} c_n^k \chi_n^k(t), \qquad Px(t) = \biggl(\sum_{(n,k)\in\Omega} (c_n^k \chi_n^k(t))^2\biggr)^{1/2}$$

(see [5], Theorem 1.11; if f and g are two nonnegative functions or quasi-norms defined on a set T, then, in what follows, the expression $f \asymp g$, $t \in T$ means that $C^{-1}f(t) \leqslant g(t) \leqslant Cf(t)$ for some constant $C > 0$ and all $t \in T$).

The *Fourier-Haar coefficients* $c_n^k = c_n^k(x)$ of a function $x \in L_1[0,1]$ are defined by

$$c_n^k = 2^n \int_0^1 x(s)\chi_n^k(s)\,ds, \qquad (n,k) \in \Omega.$$

The function $Px(t)$ is called a *Paley function*. Given a symmetric space E, we let $P(E)$ denote the space of all functions $x \in L_1[0,1]$ such that $Px \in E$; we equip this space with the norm

$$\|x\|_{P(E)} := \|Px\|_E.$$

This paper is devoted to the study of the spaces $P(E)$, with special emphasis on the space $P(L_\infty)$. Note that in the case $E = L_1$ we have the well-known dyadic space H^1 (see [6], § 1.2; in § 1.3 the book looks at some properties of the space $P(L_\infty)$, which is denoted by SL_∞ there). In what follows, particular attention will be paid to the relation between $P(E)$ and symmetric spaces. We will obtain necessary and sufficient conditions for $P(E)$ to be symmetric (both in isometric and isomorphic statements of the problem). Moreover, by strengthening the aforementioned results, we shall prove that the operator P is bounded in a symmetric space E if and only if its Boyd indices are nontrivial; that is, $0 < \alpha_E \leqslant \beta_E < 1$. In particular, the space $P(L_\infty)$ is not symmetric (and not even ideal). We shall show that its symmetric hull (considered in a certain sense) coincides with the exponential Orlicz space $\exp L_2$. The fundamental property the spaces in question have is that the Rademacher system 'spans' a 1-complemented subspace in $P(E)$ for any symmetric space E. In the final part of the paper we shall examine interpolation properties of the couple $(L_\infty, P(L_\infty))$ and prove that the Rademacher system generates a $\mathscr{K}$-closed subcouple of the couple $(L_\infty, P(L_\infty))$ in a natural way.

Recall that the Rademacher functions are defined by $r_k(t) = \operatorname{sign}[\sin(2^k\pi t)]$, $0 \leqslant t \leqslant 1$, $k \in \mathbb{N}$. If $a = (a_k)_{k=1}^\infty \in \ell_2$, then by Rademacher's theorem, the series $\sum_{n=1}^\infty a_n r_n(t)$ converges almost everywhere. Let Rad be the class of all such functions. We have

$$r_{n+1}(t) = \sum_{k=1}^{2^n} \chi_n^k(t), \qquad n = 0, 1, 2, \dots,$$

and hence $P(\sum_{n=1}^\infty a_n r_n)(t) = (\sum_{n=1}^\infty a_n^2)^{1/2}$. It follows that Rad $\subset P(E)$ and

$$\left\|\sum_{n=1}^\infty a_n r_n\right\|_{P(E)} = \|(a_n)\|_{\ell_2} \tag{1.2}$$

for any symmetric space E. Consequently, along with Rad, the space $P(E)$ contains unbounded functions. More on symmetric spaces can be found in the books [1] and [2], for Haar and Rademacher systems, see [4]–[8].

Some results in this paper were announced in [9].

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§ 2. Elementary properties of $P(E)$-spaces

Proposition 1. *For any symmetric space E the space $P(E)$ is complete.*

Proof. Let $x \in P(E)$, $x = \int_0^1 x(s)\,ds + \sum_{I\in\mathbb{D}} a_I\chi_I$. Following [6], pp. 39–40, we set $x_1 := \int_0^1 x(s)\,ds$ and $x_n := \sum_{I\in\mathbb{D},\,|I|=2^{-n+2}} a_I\chi_I$, $n = 2,3,\dots$. Now the mapping $Ux := \{x_n\}_{n=1}^\infty$ is an isometric embedding of $P(E)$ into the Banach space $E(\ell_2)$, which consists of all sequences $\{f_n\}_{n=1}^\infty$ of functions in E such that

$$\|\{f_n\}\|_{E(\ell_2)} := \biggl\|\biggl(\sum_{n=1}^\infty f_n^2\biggr)^{1/2}\biggr\|_E < \infty.$$

Standard arguments show that the image under U is closed in $E(\ell_2)$. This proves Proposition 1.

Proposition 2. *The space $P(E)$ is separable if and only if the symmetric space E is.*

Proof. Assume first that E is separable and $\sum_{m=1}^\infty c_m\chi_m \in P(E)$. In this case $(\sum_{m=1}^\infty (c_m\chi_m)^2)^{1/2} \in E$ and

$$\lim_{k\to\infty}\biggl\|\biggl(\sum_{m=k}^\infty (c_m\chi_m)^2\biggr)^{1/2}\biggr\|_E = 0.$$

Consequently, the set of functions of the form $\sum_{m=1}^k c_m\chi_m$, where $k\in\mathbb{N}$ and c_m are rational, forms a countable dense subset of $P(E)$, and hence $P(E)$ is separable.

Assume now that E is nonseparable. Then there exists a function $x \in E$ such that $|x| = x^*$ and $\|x\kappa_{(0,\varepsilon)}\|_E > 1$ for any $\varepsilon > 0$. It is easily seen, that we can assume without loss of generality that $x = \sum_{n=1}^\infty c_n\chi_n^2 \in P(E)$, $0 \leqslant c_1 \leqslant c_2 \leqslant \cdots$. We have $P(x) = |x|$, and so $x \in P(E)$. By the choice of x, there exists a sequence $m_1 < m_2 < \cdots$ such that

$$\biggl\|\sum_{n=m_i+1}^{m_{i+1}} c_n\chi_n^2\biggr\|_{P(E)} > 1.$$

To any sequence $\theta = (\theta_i)_{i=1}^\infty$, $\theta_i = \pm 1$, $i = 1,2,\dots$, we assign the function

$$y_\theta := \sum_{i=1}^\infty \theta_i \sum_{n=m_i+1}^{m_{i+1}} c_n\chi_n^2.$$

Clearly, $y_\theta \in P(E)$ and $\|y_\theta\|_{P(E)} = \|x\|_{P(E)}$. Since $\|y_{\theta^1} - y_{\theta^2}\|_{P(E)} > 1$ for arbitrary $\theta^1 \neq \theta^2$, and since the set of functions $\{y_\theta\}$ has the cardinality of the continuum, $P(E)$ is nonseparable.

Now we shall examine the question when the space $P(E)$ is symmetric. The following theorem shows that the answer depends largely on the way the question is posed.

Theorem 1. *The following assertions hold for an arbitrary symmetric space E.*

1. *$P(E)$ is symmetric if and only if $E = L_2$ and $\|x\|_E = \|x\|_{L_2}$ for all $x \in L_2$.*

2. *$P(E)$ coincides (up to equivalence of norms) with some symmetric space if and only if $0 < \alpha_E \leqslant \beta_E < 1$.*

Proof. 1. Sufficiency, of course, follows from Parseval's equality. We will prove necessity. Let $x(t)$ be a step function of the form

$$x(t) = \sum_{k=1}^{n} a_k \kappa_{\Delta(k)}(t), \tag{2.1}$$

where the dyadic intervals $\Delta(k)$ are pairwise disjoint. If a function $\varepsilon(t)$, $|\varepsilon(t)| = 1$, $0 \leqslant t \leqslant 1$, assumes the value $+1$ on the left part of the interval $\Delta(k)$, and -1, on the right part of $\Delta(k)$, then $\varepsilon \cdot \kappa_{\Delta(k)}$ is some Haar function $\chi_{i(k)}$ for each $k = 1, 2, \ldots, n$. Hence, $\varepsilon \cdot x = \sum_{k=1}^{n} a_k \chi_{i(k)}$ and

$$P(\varepsilon \cdot x) = \left(\sum_{k=1}^{n} (a_k \chi_{i(k)})^2 \right)^{1/2} = \sum_{k=1}^{n} |a_k| \kappa_{\Delta(k)} = |x|. \tag{2.2}$$

Therefore, by the hypothesis,

$$\|Px\|_E = \|P(\varepsilon \cdot x)\|_E = \big\| |x| \big\|_E = \|x\|_E. \tag{2.3}$$

Let S be the averaging operator on the set of constants; that is,

$$Sy(t) := \int_0^1 y(s)\, ds \cdot \kappa_{[0,1]}(t).$$

Since any polynomial in the Haar system $x = \sum_{k=1}^{n} c_k \chi_k$ is simultaneously a function of the form (2.1), by (2.3) we have

$$\|x\|_E = \left\| \left(\sum_{k=1}^{n} (c_k \chi_k)^2 \right)^{1/2} \right\|_E, \qquad \|(I - S)x\|_E = \left\| \left(\sum_{k=2}^{n} (c_k \chi_k)^2 \right)^{1/2} \right\|_E,$$

where I is the identity operator. It follows that $\|(I - S)x\|_E \leqslant \|x\|_E$. If E is separable, then the set of polynomials in the Haar system is dense in it, and hence, $\|I - S\|_{E \to E} = 1$. Applying one of the results from [10], we see that in this case $E = L_2$ (isometrically), and thus the theorem is proved in the separable case.

Given an arbitrary symmetric space E, we denote the closure of L_∞ in E by E_0. The space $P(L_\infty)$ is not symmetric (even up to equivalence of norms; see [6], Corollary 1.3.7 and the remark after it), and hence there is no loss of generality in assuming that $E \neq L_\infty$. Consequently, E_0 is a separable symmetric space, and so $E_0 = L_2$ by the assertion just proved. It follows that $E'' = (E_0)'' = L_2$, and so, since E is either separable or maximal, $E = E_0$ or $E = E''$. In both cases, $E = L_2$.

2. If $0 < \alpha_E \leqslant \beta_E < 1$, then, as mentioned in § 1, $\|Px\|_E \asymp \|x\|_E$. Turning to proving necessity, we assume that x is an arbitrary function of the form (2.1). By hypothesis, there exists $C > 0$ for which

$$C^{-1}\|Px\|_E \leqslant \|P(\varepsilon \cdot x)\|_E \leqslant C\|Px\|_E,$$

where the function $\varepsilon(t)$ is defined exactly as in the first part of the proof. Hence, taking (2.2) into account,

$$\|Px\|_E \asymp \|P(\varepsilon \cdot x)\|_E = \big\||x|\big\|_E = \|x\|_E$$

for an arbitrary Haar polynomial x. Again using Theorem 2.c.6 in [1] and Theorem 1.11 in [5] (see also [2], Theorem 2.9.6), it follows that the Boyd indices of the space E_0, and hence also of E, are nontrivial.

Remark 1. It follows from the above proof that in both assertions in the theorem necessity holds under an even weaker ideal condition on a space $P(E)$ (isometrically or up to equivalence of norms, respectively).

§ 3. The operator P in symmetric spaces

In § 1 we have already pointed out that the norms of the spaces $P(E)$ and E are equivalent if and only if $0 < \alpha_E \leqslant \beta_E < 1$. We will prove two more precise results of this kind to characterize the embeddings $P(E) \subset E$ and $E \subset P(E)$.

Theorem 2. *The following assertions hold for an arbitrary symmetric space E.*

1. *The embedding $P(E) \subset E$ holds if and only if $\alpha_E > 0$.*
2. *The embedding $E \subset P(E)$ holds if and only if $0 < \alpha_E \leqslant \beta_E < 1$.*

Proof. 1. Let $S_n x$, $n = 1, 2, \dots$, be the sequence of partial Fourier-Haar sums of a function $x \in L_1$; that is, $S_n x(t) = \sum_{k=1}^n c_k(x)\chi_k(t)$. If $\alpha_E > 0$, then in view of Theorem 6.3 in [8] (see also the more general result in [11]), $\|Px\|_E \asymp \|\bar{S}x\|_E$, where $\bar{S}x(t) := \sup_{n=1,2,\dots} |S_n x(t)|$. Since $\bar{S}x(t) \geqslant |x(t)|$ almost everywhere on $[0, 1]$, we have $\|Px\|_E \geqslant c\|x\|_E$ for some $c > 0$. Hence, $P(E) \subset E$.

To prove the converse assertion we assume the contrary: $\alpha_E = 0$. By Proposition 2.b.7 of [1], for any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist equidistributed functions $x_1, x_2, \dots, x_{2^n} \in E$, with disjoint supports, such that

$$\max_{1\leqslant i\leqslant 2^n} |a_i| \leqslant \bigg\|\sum_{i=1}^{2^n} a_i x_i\bigg\|_E \leqslant (1+\varepsilon) \max_{1\leqslant i\leqslant 2^n} |a_i| \tag{3.1}$$

for any $a_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. Since E is separable or maximal, without loss of generality we may assume that each function x_i is a finite linear combination of the characteristic functions of intervals of the form $[(l_j - 1)2^{-k}, l_j 2^{-k})$ for some fixed $k \in \mathbb{N}$ which is independent of i. Moreover (possibly after some rearrangement of values of the functions), we can suppose that each function x_i is nonzero on just one interval among the first 2^n dyadic intervals of length 2^{-k}, and takes some value (b_1, say) on this interval, which is independent of i. The same is true for the subsequent 2^n intervals of length 2^{-k} (with b_2 instead of b_1) and so on.

Consider the following 'Haar functions' constructed from the sequence $\{x_i\}_{i=1}^{2^n}$:

$$
\begin{aligned}
h_1 &:= x_1 + \cdots + x_{2^{n-1}} - x_{2^{n-1}+1} - \cdots - x_{2^n}, \\
h_2 &:= x_1 + \cdots + x_{2^{n-2}} - x_{2^{n-2}+1} - \cdots - x_{2^{n-1}} \\
&\quad + x_{2^{n-1}+1} + \cdots + x_{2^{n-1}+2^{n-2}} - x_{2^{n-1}+2^{n-2}+1} - \cdots - x_{2^n}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
h_n &:= \sum_{j=1}^{2^n} (-1)^{j+1} x_j.
\end{aligned}
$$

Note that these functions are linear combinations of the Haar functions. Namely,

$$
\begin{aligned}
h_1 &= b_1 \chi^1_{k-n} + b_2 \chi^2_{k-n} + \cdots + b_m \chi^m_{k-n}, \\
h_2 &= b_1(\chi^1_{k-n+1} + \chi^2_{k-n+1}) + b_2(\chi^3_{k-n+1} + \chi^4_{k-n+1}) + \cdots + b_m(\chi^{2m-1}_{k-n+1} + \chi^{2m}_{k-n+1}), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
h_n &= b_1(\chi^1_{k-1} + \chi^2_{k-1} + \cdots + \chi^{2^{n-1}}_{k-1}) + b_2(\chi^{2^{n-1}+1}_{k-1} + \cdots + \chi^{2^n}_{k-1}) \\
&\quad + \cdots + b_m(\chi^{(m-1)2^{n-1}+1}_{k-1} + \cdots + \chi^{m2^{n-1}}_{k-1}),
\end{aligned}
$$

where $m \leqslant 2^{k-n}$. By the definition of the Paley function,

$$
P\biggl(\sum_{j=1}^n h_j\biggr) = \sqrt{n}\biggl(\sum_{i=1}^m b_i^2 \kappa_{[(i-1)2^{n-k}, i2^{n-k})}\biggr)^{1/2} = \sqrt{n}\sum_{i=1}^m |b_i| \kappa_{[(i-1)2^{n-k}, i2^{n-k})}.
$$

The last sum is equimeasurable with the function $\sqrt{n}\sum_{i=1}^{2^n} |x_i|$, and hence, by virtue of (3.1),

$$
\biggl\| P\biggl(\sum_{j=1}^n h_j\biggr)\biggr\|_E = \sqrt{n}\biggl\|\sum_{i=1}^{2^n} |x_i|\biggr\|_E \leqslant (1+\varepsilon)\sqrt{n}.
$$

On the other hand, again applying (3.1), this gives

$$
\biggl\|\sum_{j=1}^n h_j\biggr\|_E \geqslant n\|x_1\|_E = n.
$$

Since $n \in \mathbb{N}$ is arbitrary, comparing the last estimates, it follows that the embedding $P(E) \subset E$ does not hold, which contradicts the hypothesis.

2. It suffices to show that if $E \subset P(E)$ then $0 < \alpha_E \leqslant \beta_E < 1$. First, arguing as in the proof of assertion 1 of the theorem, we claim that $\alpha_E > 0$. Assuming the contrary, given arbitrary $n \in \mathbb{N}$ and $\varepsilon > 0$, we find, as before, equidistributed functions $x_1, x_2, \dots, x_{2^n} \in E$, with disjoint support, such that (3.1) holds for any $a_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. We again assume that the x_i are 'rearranged' as in the proof

of assertion 1. Now the functions

$$\begin{gathered} g_1 := x_1 + \cdots + x_{2^{n-1}} - x_{2^{n-1}+1} - \cdots - x_{2^n}, \\ g_2 := x_1 + \cdots + x_{2^{n-2}} - x_{2^{n-2}+1} - \cdots - x_{2^{n-1}}, \\ g_3 := x_1 + \cdots + x_{2^{n-3}} - x_{2^{n-3}+1} - \cdots - x_{2^{n-2}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_n := x_1 - x_2 \end{gathered}$$

can be written as linear combinations of the Haar functions:

$$\begin{gathered} g_1 = b_1\chi_{k-n}^1 + b_2\chi_{k-n}^2 + \cdots + b_m\chi_{k-n}^m, \\ g_2 = b_1\chi_{k-n+1}^1 + b_2\chi_{k-n+1}^3 + \cdots + b_m\chi_{k-n+1}^{2(m-1)+1}, \\ g_3 = b_1\chi_{k-n+2}^1 + b_2\chi_{k-n+2}^5 + \cdots + b_m\chi_{k-n+2}^{2^2(m-1)+1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_n = b_1\chi_{k-1}^1 + b_2\chi_{k-1}^{2^{n-1}+1} + \cdots + b_m\chi_{k-1}^{(m-1)2^{n-1}+1}. \end{gathered}$$

We have

$$\begin{aligned} P\biggl(\sum_{j=1}^n (-1)^{j-1} g_j\biggr) &\geqslant \sqrt{n}\biggl(\sum_{i=1}^m b_i^2 \kappa_{[(i-1)2^{n-k},(i-1)2^{n-k}+2^{1-k})}\biggr)^{1/2} \\ &= \sqrt{n}\sum_{i=1}^m |b_i| \kappa_{[(i-1)2^{n-k},(i-1)2^{n-k}+2^{1-k})}, \end{aligned}$$

$$\biggl(\sum_{i=1}^m |b_i| \kappa_{[(i-1)2^{n-k},(i-1)2^{n-k}+2^{1-k})}\biggr)^* \geqslant x_1^*,$$

and hence, applying (3.1),

$$\biggl\|P\biggl(\sum_{j=1}^n (-1)^{j-1} g_j\biggr)\biggr\|_E \geqslant \sqrt{n}\|x_1\|_E \geqslant \sqrt{n}. \tag{3.2}$$

At the same time, it is readily checked that

$$\biggl|\sum_{j=1}^n (-1)^{j-1} g_j\biggr| \leqslant 2\sum_{i=1}^{2^n} |x_i|,$$

and so, in view of (3.1),

$$\biggl\|\sum_{j=1}^n (-1)^{j-1} g_j\biggr\|_E \leqslant 2(1+\varepsilon).$$

Since $n \in \mathbb{N}$ is arbitrary, together with (3.2) the last estimate shows that the embedding $E \subset P(E)$ does not hold. This contradicts the hypothesis, and hence $\alpha_E > 0$. Consequently, $P(E) = E$, by assertion 1, which we have already proved. Hence, $\beta_E < 1$ by Theorem 1.

Corollary 1. *Let E be a symmetric space on* $[0,1]$.

1. *If $E \subset P(E)$, then $E = P(E)$.*
2. *If $\alpha_E > 0$ and $\beta = 1$, then $P(E) \overset{\neq}{\subset} E$.*

From assertion 2 of Theorem 2 (which we have already proved) it follows that the embedding $E \subset P(E)$ is equivalent to the condition that the Haar system forms an unconditional basis in a symmetric space E. From this it is easy to obtain, in particular, a criterion for the Haar system to be an RUC-basis in a symmetric space; this result was announced in [12]. We first recall a few definitions.

Let E be a Banach space with dual space E^*. A biorthogonal system (e_k, e_k^*), $e_k \in E$, $e_k^* \in E^*$, is called an RUC-system (random unconditional convergence system) if, for any x in the closed linear hull $[e_k]$ of the vectors e_k, $k = 1, 2, \ldots$, the series $\sum_{k=1}^{\infty} r_k(s) e_k^*(x) e_k$ converges in E for almost all $s \in [0,1]$, where the r_k are the Rademacher functions. This is equivalent to saying that for any (some) $p \in [1,\infty)$ there exists a constant $K_p > 0$ such that the inequality

$$\sup_{n=1,2,\ldots} \left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(s) e_k^*(x) e_k \right\|_E^p ds \right)^{1/p} \leqslant K_p \|x\|_E$$

holds for all $x \in [e_k]$ (see [13]). A basis that, together with the basis coefficient functionals, forms an RUC-system, is called an RUC-*basis*. Clearly, each unconditional basis is an RUC-basis. Counterexamples showing that the converse statement is false, as well as other information on RUC-systems, can be found in [13].

Corollary 2. *The Haar system forms an* RUC-*basis in a separable symmetric space E if and only if* $0 < \alpha_E \leqslant \beta_E < 1$.

Proof. If the Haar system forms an RUC-basis in a separable symmetric space E, then for some K and all $n \in \mathbb{N}$,

$$\int_0^1 \left\| \sum_{k=1}^{n} r_k(s) c_k \chi_k \right\|_E ds \leqslant K \left\| \sum_{k=1}^{n} c_k \chi_k \right\|_E.$$

At the same time, by the L_1-Khinchin inequality with optimal constant (found by Szarek [14]),

$$\int_0^1 \left\| \sum_{k=1}^{n} r_k(s) c_k \chi_k \right\|_E ds \geqslant \left\| \int_0^1 \left| \sum_{k=1}^{n} r_k(s) c_k \chi_k \right| ds \right\|_E \geqslant \frac{1}{\sqrt{2}} \left\| \left(\sum_{k=1}^{n} (c_k \chi_k)^2 \right)^{1/2} \right\|_E.$$

Consequently, since E is separable, $\|Px\|_E \leqslant \sqrt{2} K \|x\|_E$ for all $x \in E$, and it now follows from Theorem 2 that $0 < \alpha_E \leqslant \beta_E < 1$. The opposite assertion being clear, the corollary follows.

Remark 2. It is well known that the sequence of partial Fourier-Haar sums $S_n x$ of an arbitrary function $x \in L_1[0,1]$ forms a martingale with respect to the natural sequence of σ-algebras generated by the dyadic intervals of $[0,1]$. From this point of view it is interesting to compare the assertions of Theorem 2 with the results in [15], which looks at the square function $P_{\mathbf{f}}$ corresponding to an arbitrary martingale

$\mathbf{f}=\{f_n\}_{n=1}^\infty$. In particular, from assertion 1 of Theorem 2 and Theorem 3.1 of [15] we have the following result: if the inequality $\|Px\|_E \geqslant c\|x\|_E$ holds in some symmetric space E, then the analogous inequality $\|P_{\mathbf{f}}x\|_E \geqslant c_{\mathbf{f}}\|x\|_E$ holds for any martingale $\mathbf{f}$ on $[0,1]$.

§ 4. The Rademacher system in the space $P(E)$

Recall that a closed subspace F of a Banach space E is called *complemented* (1-*complemented*) in E if there exists a bounded projection (a norm-one projection) from E onto F.

In view of (1.2), for any symmetric space E the set Rad of sums of almost everywhere convergent Rademacher series is a closed linear subspace of $P(E)$ which is isometric to ℓ_2.

Theorem 3. *The subspace* Rad *is* 1-*complemented in* $P(E)$ *for any symmetric space* E.

Proof. It suffices to show that the orthogonal projection

$$Tf(s):=\sum_{n=1}^\infty \int_0^1 fr_n\,dt\cdot r_n(s)$$

is bounded on $P(E)$ and that

$$\|Tf\|_{P(E)}\leqslant \|f\|_{P(E)}. \tag{4.1}$$

If $f=\displaystyle\int_0^1 f\,dt+\sum_{I\in\mathbb{D}}a_I\chi_I$, then for any $n\in\mathbb{N}$

$$\int_0^1 fr_n\,dt=\sum_{I\in\mathbb{D},\,|I|=2^{-n+1}}a_I\int_I\chi_I r_n\,dt=2^{-n+1}\sum_{I\in\mathbb{D},\,|I|=2^{-n+1}}a_I.$$

Hence,

$$\begin{aligned}Tf&=\sum_{n=1}^\infty\biggl(2^{-n+1}\sum_{I\in\mathbb{D},\,|I|=2^{-n+1}}a_I\biggr)r_n\\&=\sum_{n=1}^\infty\biggl(2^{-n+1}\sum_{I\in\mathbb{D},\,|I|=2^{-n+1}}a_I\biggr)\sum_{I\in\mathbb{D},\,|I|=2^{-n+1}}\chi_I\end{aligned}$$

(that is, the operator T averages the Fourier-Haar coefficients of the function f corresponding to each block of the Haar system). Thus, by definition

$$\|Tf\|_{P(E)}^2=\sum_{n=1}^\infty\biggl(2^{-n+1}\sum_{I\in\mathbb{D},\,|I|=2^{-n+1}}a_I\biggr)^2. \tag{4.2}$$

On the other hand, for arbitrary $n\in\mathbb{N}$ and $j=1,2,\dots,2^n$, we consider the following sequence of dyadic intervals $\{I_k^{(n,j)}\}_{k=1}^\infty$: if $1\leqslant k\leqslant n+1$ and if

$(i-1)2^{n-k+1} < j \leqslant i2^{n-k+1}$, where $i=1,\dots,2^{k-1}$, then $I_k^{(n,j)} = \Delta_{k-1}^i$ (in particular, $I_1^{(n,j)} = (0,1)$ for $j=1,2,\dots,2^n$, $I_2^{(n,j)} = (0,1/2)$ for $j=1,2,\dots,2^{n-1}$, and $I_2^{(n,j)} = (1/2,1)$ for $j=2^{n-1}+1,\dots,2^n$, and so on); if $k>n+1$, then $I_k^{(j)}$ is an arbitrary dyadic interval of length 2^{-k+1}. We note that $Pf \geqslant (\sum_{k=1}^\infty a_{I_k}^2)^{1/2}$ for each subsequence of dyadic intervals $\{I_k\}_{k=1}^\infty$ such that $|I_k| = 2^{-k+1}$ and $I_{k+1} \subset I_k$, $k=1,2,\dots$. Hence,

$$Pf \geqslant \sum_{j=1}^{2^n} \biggl(\sum_{k=1}^{n+1} a^2_{\Delta_{k-1}^{i(j)}}\biggr)^{1/2} \kappa_{\Delta_n^j}, \qquad n\in\mathbb{N},$$

where $i(j)$ is such that $(i(j)-1)2^{n-k+1} < j \leqslant i(j)2^{n-k+1}$. Consequently,

$$\|f\|_{P(E)} \geqslant \int_0^1 Pf(t)\,dt \geqslant 2^{-n} \sum_{j=1}^{2^n} \biggl(\sum_{k=1}^{n+1} a^2_{\Delta_{k-1}^{i(j)}}\biggr)^{1/2} \geqslant 2^{-n} \Biggl(\sum_{k=1}^{n+1}\biggl(\sum_{j=1}^{2^n} |a_{\Delta_{k-1}^{i(j)}}|\biggr)^2\Biggr)^{1/2}$$
$$= 2^{-n} \Biggl(\sum_{k=1}^{n+1}\biggl(2^{n-k+1}\sum_{i=1}^{2^{k-1}} |a_{\Delta_{k-1}^{i}}|\biggr)^2\Biggr)^{1/2} = \Biggl(\sum_{k=1}^{n+1}\biggl(2^{-k+1}\sum_{I\in\mathbb{D},\,|I|=2^{-k+1}} |a_I|\biggr)^2\Biggr)^{1/2}.$$

Since $n\in\mathbb{N}$ is arbitrary, letting $n\to\infty$ we obtain

$$\Biggl(\sum_{k=1}^{\infty}\biggl(2^{-k+1}\sum_{I\in\mathbb{D},\,|I|=2^{-k+1}} |a_I|\biggr)^2\Biggr)^{1/2} \leqslant \|f\|_{P(E)},$$

and in view of (4.2) this gives (4.1), which proves the theorem.

Corollary 3. *For any symmetric space E the space $P(E)$ contains a 1-complemented subspace isometric to ℓ_2.*

Remark 3. It is well known (see Theorem 2.b.4 in [1], or [16]) that the operator T in the proof of Theorem 3 is bounded in a symmetric space E if and only if $G \subset E \subset G'$, where G is the closure of L_∞ in the exponential Orlicz space $\exp L_2$, and G' is the associate space of G. Moreover, according to [17] (see also [18]), the last condition is equivalent to the condition that a separable symmetric space E which fails to contain a complemented sublattice isomorphic to ℓ_2 must contain ℓ_2 as a complemented subspace. Comparing these results with Theorem 3 indicates a crucial difference between the properties of symmetric and $P(E)$-spaces.

§ 5. The symmetric hull of Rad and $P(L_\infty)$

Let F be a set of functions defined on $[0,1]$, $F\subset L_1[0,1]$.

By definition, the *symmetric hull* of F is the symmetric space E satisfying the following properties:

1) $F \subset E$;

2) there exists a constant $\alpha>0$ such that, for an arbitrary function $x\in E$, there exists $f\in F$ such that $x^*(t) \leqslant f^*(t) + \alpha\|x\|_E$ almost everywhere on $[0,1]$.

It is easily seen that the symmetric hull of any set (up to equivalence of norms) is unique.

It is worth noting that the concept we have just introduced is slightly different from the concept of a rearrangement invariant hull of a set, which was investigated in [19] (see also Theorem 5.7.10 in [20]).

To characterize the symmetric hulls of the sets Rad and $P(L_\infty)$ we shall require some properties of the dyadic BMO space, which we will denote by BMO_d. Recall that the space $\mathrm{BMO} = \mathrm{BMO}[0,1]$ consists of all functions $f \in L_1[0,1]$ such that

$$\|f\|_{\mathrm{BMO}} := \sup_I \frac{1}{|I|} \int_I |f(s) - f_I|\, ds < \infty,$$

where the supremum is taken over all intervals $I \subseteq [0,1]$, $f_I = \dfrac{1}{|I|} \displaystyle\int_I f(s)\, ds$. If in the last definition we consider only $I \in \mathbb{D}$, we obtain the seminorm $\|f\|_d$ defining the dyadic space BMO_d. It is clear that $\|f\|_d \leqslant \|f\|'_d$, where

$$\|f\|'_d := \sup_{I\in\mathbb{D}} \left(\frac{1}{|I|} \int_I |f(s) - f_I|^2\, ds \right)^{1/2}.$$

Moreover, $\|f\|'_d \leqslant C\|f\|_d$ for some $C > 0$, by one of the corollaries to the John-Nirenberg theorem (see [21], Remark 3.9). In addition, we need the following straightforward equality: if $f = \sum_{J\in\mathbb{D}} a_J \chi_J$, then

$$\|f\|'_d = \sup_{I\in\mathbb{D}} \left(\frac{1}{|I|} \sum_{J\in\mathbb{D},\, J\subseteq I} a_J^2 |J| \right)^{1/2}.$$

Proposition 3. *The space* $P(L_\infty)$ *is contained in* BMO_d *and* $\|f\|_d \leqslant \|f\|_{P(L_\infty)}$ *for all* $f \in P(L_\infty)$.

Proof. We may assume without loss of generality that $\displaystyle\int_0^1 f\, ds = 0$, and hence, $f = \sum_{J\in\mathbb{D}} a_J \chi_J$. Given an arbitrary interval $I \in \mathbb{D}$, we set

$$P_I(f)(x) := \left(\sum_{J\in\mathbb{D},\, J\subseteq I} a_J^2 \kappa_J(x) \right)^{1/2}.$$

Hence, $P_I(f)(x) \leqslant Pf(x)$, $0 \leqslant x \leqslant 1$, and

$$\|P_I(f)\|_{L_\infty(I, dx/|I|)} \leqslant \|Pf\|_{L_\infty[0,1]} = \|f\|_{P(L_\infty)}. \tag{5.1}$$

We have

$$\begin{aligned} \|P_I(f)\|_{L_2(I, dx/|I|)} &= \left(\frac{1}{|I|} \int_I P_I(f)^2\, dx \right)^{1/2} \\ &= \left(\frac{1}{|I|} \sum_{J\in\mathbb{D},\, J\subseteq I} a_J^2 \int_I \kappa_J(x)\, dx \right)^{1/2} = \left(\frac{1}{|I|} \sum_{J\in\mathbb{D},\, J\subseteq I} a_J^2 |J| \right)^{1/2}, \end{aligned}$$

and therefore, by (5.1),

$$\left(\frac{1}{|I|} \sum_{J\in\mathbb{D},\, J\subset I} a_J^2 |J| \right)^{1/2} \leqslant \|P_I(f)\|_{L_\infty(I, dx/|I|)} \leqslant \|f\|_{P(L_\infty)}.$$

Since $I \in \mathbb{D}$ is arbitrary, in view of the remark after the definition of BMO_d, this proves both assertions of Proposition 3.

Remark 4. In [22] it was proved that

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{BMO}} \asymp \|(a_k)_{k=1}^{\infty}\|_{\ell_2} + \sup_{n\in\mathbb{N}}\Big|\sum_{k=1}^{n} a_k\Big|,$$

with constants independent of $a_k \in \mathbb{R}$. In particular, it follows that $\mathrm{Rad} \not\subset \mathrm{BMO}$. *A fortiori*, $P(E) \not\subset \mathrm{BMO}$ for an arbitrary symmetric space E, which indicates that Proposition 3 is, in a certain sense, sharp.

Theorem 4. *For any function $x \in G$, there exists a Rademacher sum*

$$f = \sum_{k=1}^{\infty} a_k r_k, \qquad a = (a_k)_{k=1}^{\infty} \in \ell_2,$$

such that $\|a\|_{\ell_2} \leqslant C_1\|x\|_G$ and

$$x^*(t) \leqslant C_2(\|a\|_{\ell_2} + f^*(t)), \qquad 0 < t \leqslant 1, \tag{5.2}$$

where $C_1 > 0$ and $C_2 > 0$ are independent of x and f.

Proof. It is well known (see the proof of Theorem 2.b.4, (i) in [1]) that the limit

$$u(t) := \lim_{n\to\infty}\bigg(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} r_k(t)\bigg)^* \tag{5.3}$$

exists almost everywhere on $[0,1]$, and $u(t)$ is equivalent to the function $v(t) := \log_2^{1/2}(2/t)$ on $[0,1/2]$. We let δ denote the corresponding equivalence constant. It follows, in particular, that

$$\int_0^t v(s)\,ds \leqslant 2\delta\int_0^t u(s)\,ds, \qquad 0 < t \leqslant 1. \tag{5.4}$$

In view of the characterization (1.1) of the space G, if $x \in G$ then

$$x^*(t) \leqslant \gamma\|x\|_G v(t), \quad 0 < t \leqslant 1, \qquad \lim_{t\to 0}\frac{x^*(t)}{v(t)} = 0 \tag{5.5}$$

for some universal constant $\gamma > 0$

We can find a sequence of points $t_1 > t_2 > \cdots$ in $[0,1]$, $t_k \downarrow 0$, such that

$$x^*(t) \leqslant \gamma\|x\|_G v(t)\cdot\sum_{k=1}^{\infty} 2^{1-k}\chi_{(t_{k+1},t_k]}(t), \qquad 0 < t \leqslant 1, \tag{5.6}$$

$$\int_{t_{k+1}}^{\tau} u(t)\,dt \geqslant \frac{1}{2}\int_0^{\tau} u(t)\,dt \quad \text{for all} \ \ \tau \in [t_k, t_{k-1}], \quad k = 2,3,\dots. \tag{5.7}$$

We take $t_1 = 1$ and choose $t_2 < 1$ so as to have

$$x^*(t) \leqslant \frac{1}{2}\gamma\|x\|_G v(t) \quad \text{for all} \ \ 0 < t \leqslant t_2. \tag{5.8}$$

For any $\tau \in [t_2, t_1]$, there exists $b(\tau) \in (0, t_2)$ such that

$$\int_{b(\tau)}^{\tau} u(t)\,dt = \frac{1}{2}\int_0^{\tau} u(t)\,dt.$$

Since the integral is absolutely continuous, the function $b(\tau)$ is continuous on $[t_2, t_1]$. Hence, $t_3' := \min_{t_2 \leqslant \tau \leqslant t_1} b(\tau) > 0$. Let $t_3 \in (0, t_3']$ be such that

$$x^*(t) \leqslant \frac{1}{2^2}\gamma\|x\|_G v(t) \quad \text{for all} \quad 0 < t \leqslant t_3.$$

Therefore, because of the choice of t_3 (see the first inequality in (5.5) and (5.8)), relations (5.6) and (5.7) are satisfied for $t \in [t_3, t_1]$ and $k = 2$, respectively. The subsequent points t_k are chosen similarly.

Now set $m_1 = 0$. Note that the convergence in (5.3) is uniform on any subinterval $[\lambda, 1/2]$, $0 < \lambda < 1/2$. Hence, we can take $m_2 > m_1$ so that

$$\left(\frac{1}{(m_2 - m_1)^{1/2}} \sum_{k=m_1+1}^{m_2} r_k(t)\right)^* \geqslant \frac{1}{2}u(t)$$

for all $t \in [t_2, t_1]$. Assume that $m_1 < m_2 < \cdots < m_i$ have already been constructed. From the elementary properties of the Rademacher functions (see, for example, [4], Proposition 1.4), one can choose $m_{i+1} > m_i$ so that

$$\left(\frac{1}{(m_{i+1} - m_i)^{1/2}} \sum_{k=m_i+1}^{m_{i+1}} r_k(t)\right)^* \geqslant \frac{1}{2}u(t) \quad \text{for all} \quad t \in [t_{i+1}, t_i]. \tag{5.9}$$

We set

$$a_k := \frac{2^{5-i}\gamma\delta\|x\|_G}{(m_{i+1} - m_i)^{1/2}}, \qquad m_i < k \leqslant m_{i+1}, \quad i \in \mathbb{N},$$
$$z_i := \sum_{k=m_i+1}^{m_{i+1}} a_k r_k, \quad i \in \mathbb{N}, \qquad f := \sum_{k=1}^{\infty} a_k r_k.$$

Since the Rademacher system is unconditional (with unconditional constant 1) in any symmetric space (and in particular, in L_1 with norm $\int_0^{\tau} x^*(s)\,ds$, $0 < \tau \leqslant 1$), for any $i = 2, 3, \ldots$ and $\tau \in [t_i, t_{i-1}]$ we have

$$\int_0^{\tau} f^*(t)\,dt \geqslant \int_0^{\tau} z_i^*(t)\,dt \geqslant \int_{t_{i+1}}^{\tau} z_i^*(t)\,dt.$$

Moreover, for all $\tau \in [t_i, t_{i-1}]$,

$$\begin{aligned}\int_{t_{i+1}}^{\tau} z_i^*(t)\,dt &\geqslant 2^{4-i}\gamma\beta\|x\|_G \int_{t_{i+1}}^{\tau} u(t)\,dt \\ &\geqslant 2^{3-i}\gamma\beta\|x\|_G \int_0^{\tau} u(t)\,dt \geqslant 2^{2-i}\gamma\|x\|_G \int_0^{\tau} v(t)\,dt,\end{aligned}$$

in view of (5.9), (5.7) and (5.4). At the same time (5.6) shows that on the interval $(0, t_{i-1}]$ $2^{2-i}\gamma\|x\|_G v(t) \geqslant x^*(t)$. Thus, since $\bigcup_{i=2}^{\infty}[t_i, t_{i-1}] = (0,1]$, it follows from the last two inequalities that

$$\int_0^\tau x^*(t)\,dt \leqslant \int_0^\tau f^*(t)\,dt, \qquad 0 < \tau \leqslant 1. \tag{5.10}$$

Furthermore,

$$\|a\|_{\ell_2} \leqslant 32\gamma\delta\|x\|_G\biggl(\sum_{i=1}^{\infty} 4^{-i}\biggr)^{1/2} = C_1\|x\|_G,$$

where $C_1 := 32\cdot 3^{-1/2}\gamma\delta$ is a universal constant.

To prove (5.2) we recall (see [19] or [21], Theorem 3.3) that

$$\frac{1}{t}\int_0^t g^*(s)\,ds - g^*(t) \leqslant 32\|g\|_d, \qquad 0 < t \leqslant \frac{1}{4},$$

for any function $g \in \mathrm{BMO}_d$ (in these references this inequality is proved only for BMO, but a simple analysis of the proof shows that an analogous result also holds in BMO_d). Hence,

$$x^*(t) \leqslant \frac{1}{t}\int_0^t x^*(s)\,ds \leqslant \frac{1}{t}\int_0^t f^*(s)\,ds \leqslant f^*(t) + 32\|f\|_{P(L_\infty)} = f^*(t) + 32\|a\|_{\ell_2},$$

using (5.10), Proposition 3, and equality (1.2) for $0 < t \leqslant 1/4$. If $1/4 < t \leqslant 1$, then again applying (5.10), we get

$$x^*(t) \leqslant \frac{1}{t}\int_0^t x^*(s)\,ds \leqslant 4\int_0^t f^*(s)\,ds \leqslant 4\|f\|_2 = 4\|a\|_{l_2}.$$

Since (5.2) is a direct consequence of these relations, this proves Theorem 4.

Now, $\mathrm{Rad} \subset G$ (see [23], Theorem 5.8.7, or [24]), and so Theorem 4 has the following corollary.

Corollary 4. *The space G is the symmetric hull of the subspace* Rad.

Next, we will find the symmetric hull of the set $P(L_\infty)$. First, note that $\mathrm{Rad} \subset P(L_\infty)$ (see (1.2)), and so Theorem 4 shows that it contains G. On the other hand, in view of Corollary 1.3.2 in [6], the following estimate for the distribution of functions from $P(L_\infty)$ is valid: if $\int_0^1 x(s)\,ds = 0$, then

$$\bigl|\{s \in [0,1] : |x(s)| > t\}\bigr| \leqslant 2\exp\biggl(-\frac{t^2}{2\|x\|^2_{P(L_\infty)}}\biggr), \qquad t > 0.$$

Thus, taking (1.1) and the obvious inequality $\left|\int_0^1 x(s)\,ds\right| \leqslant \|x\|_{P(L_\infty)}$ into account, we have $P(L_\infty) \subset \exp L_2$ and $\|x\|_{\exp L_2} \leqslant C\|x\|_{P(L_\infty)}$ for all $x \in P(L_\infty)$ and some constant $C > 0$. Thus, in order to show that the symmetric hull of $P(L_\infty)$ coincides with the space $\exp L_2$ it suffices to prove the following theorem.

Theorem 5. *For any function $x \in \exp L_2$ there exists a function $f \in P(L_\infty)$ such that*

$$\frac{1}{C_1}\|x\|_{\exp L_2} \leqslant \|f\|_{P(L_\infty)} \leqslant C_1\|x\|_{\exp L_2}, \tag{5.11}$$

$$x^*(t) \leqslant C_2(f^*(t) + \|f\|_{P(L_\infty)}), \tag{5.12}$$

where the constants $C_1 > 0$ and $C_2 > 0$ are independent of x and f.

In particular, there exists a function $f_0 \in P(L_\infty)$, $\|f_0\|_{P(L_\infty)} = 1$, for which

$$|f_0(t)| \leqslant \log_2^{1/2}\frac{2}{t}, \qquad 0 < t \leqslant 1, \tag{5.13}$$

$$f_0^*(t) \geqslant \frac{1}{4}\log_2^{1/2}\frac{2}{t} \tag{5.14}$$

for all $t \in (0, t_0]$ and some $t_0 \in (0,1)$.

Proof. We first prove the second assertion of the theorem. For each $k = 1, 2, \dots$ we define two families of dyadic intervals:

$$\Sigma_k^1 := \{\Delta_{k+j-1}^{1+2^{j-1}}, j = 2, \dots, k\}, \qquad \Sigma_k^2 := \{\Delta_{k+j-1}^{2^j}, j = 2, \dots, k\}$$

(in particular, $\Sigma_1^1 = \Sigma_1^2 = \varnothing$). Note that the intervals in both families Σ_k^1 and Σ_k^2 are nested and are contained in the dyadic interval Δ_k^2. To be more precise,

$$\Delta_k^2 \supset \Delta_{k+1}^3 \supset \dots \supset \Delta_{2k-1}^{1+2^{k-1}}, \quad \Delta_k^2 \supset \Delta_{k+1}^4 \supset \dots \supset \Delta_{2k-1}^{2^k}, \qquad k = 1, 2, \dots . \tag{5.15}$$

Moreover, from the definition of the Haar functions it follows that, for all $k = 1, 2, \dots$,

$$\chi_I(t) = \begin{cases} 1 & \text{for } \ I \in \Sigma_k^1 \cup \{\Delta_k^2\},\ t \in \Delta_{2k}^{1+2^k}, \\ -1 & \text{for } \ I \in \Sigma_k^2 \cup \{\Delta_k^2\},\ t \in \Delta_{2k}^{2^{k+1}}. \end{cases} \tag{5.16}$$

We set

$$f_0 := \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\biggl(\chi_{\Delta_k^2} + \sum_{I\in\Sigma_k^1}\chi_I + \sum_{I\in\Sigma_k^2}\chi_I\biggr).$$

Since the intervals $\Delta_k^2 = (2^{-k}, 2^{-k+1}]$, $k = 1, 2, \dots$, are pairwise disjoint and since $\bigcup_{k=1}^\infty \Delta_k^2 = (0,1]$, we have $(Pf_0)^2(t) \leqslant 1$, $0 < t \leqslant 1$. Moreover, if $E_k := \Delta_{2k}^{1+2^k} \cup \Delta_{2k}^{2^{k+1}}$, then $(Pf_0)^2(t) = 1$ for $t \in E_k$. Hence, $\|f_0\|_{P(L_\infty)} = \|Pf_0\|_\infty = 1$.

Now we will estimate the function f_0 itself. In view of (5.15) and (5.16), $|f_0(t)| \leqslant \sqrt{k}$ for $t \in \Delta_k^2$, and $|f_0(t)| = \sqrt{k}$ for $t \in E_k$, $k = 1, 2, \dots$. This gives

$$\sum_{k=1}^{\infty}\sqrt{k}\kappa_{E_k}(t) \leqslant |f_0(t)| \leqslant \sum_{k=1}^{\infty}\sqrt{k}\kappa_{(2^{-k},2^{-k+1}]}(t), \qquad 0 < t \leqslant 1. \tag{5.17}$$

We also note that the sets E_k are pairwise disjoint and

$$|E_k| = 2^{-2k+1}, \qquad k = 1, 2, \dots . \tag{5.18}$$

On the other hand,

$$\sum_{k=1}^{\infty}\sqrt{k}\kappa_{(2^{-k},2^{-k+1}]}(t)\leqslant\log_2^{1/2}\frac{2}{t}\leqslant\sum_{k=1}^{\infty}\sqrt{k+1}\kappa_{(2^{-k},2^{-k+1}]}(t),\quad 0<t\leqslant 1.\quad(5.19)$$

Now (5.13) is a direct consequence of inequalities (5.17) and (5.19). Further, the right-hand side of (5.19) equals $u(t)+v(t)$, where

$$u(t):=\sum_{i=1}^{\infty}\sqrt{2i}\kappa_{(2^{-2i+1},2^{-2i+2}]}(t),\qquad v(t):=\sum_{i=1}^{\infty}\sqrt{2i+1}\kappa_{(2^{-2i},2^{-2i+1}]}(t).$$

By (5.17) and (5.18), $u^*(t)\leqslant\sqrt{2}f_0^*(t)$ and $v^*(t)\leqslant\sqrt{3}f_0^*(t)$, $0<t\leqslant 1$. Thus, again applying (5.19) we see that

$$\begin{aligned}\log_2^{1/2}\frac{2}{t}\leqslant\frac{1}{t}\int_0^t\log_2^{1/2}\frac{2}{s}\,ds\leqslant\frac{1}{t}\int_0^t u^*(s)\,ds\\ +\frac{1}{t}\int_0^t v^*(s)\,ds\leqslant\frac{\sqrt{2}+\sqrt{3}}{t}\int_0^t f_0^*(s)\,ds.\end{aligned}$$

Now using exactly the same analysis as in the concluding part of the proof of Theorem 4 we apply Theorem 3.3 in [21] and Proposition 3 to show that

$$\begin{aligned}\log_2^{1/2}\frac{2}{t}&\leqslant(\sqrt{2}+\sqrt{3})(f_0^*(t)+32\|f_0\|_d)\leqslant(\sqrt{2}+\sqrt{3})(f_0^*(t)+32\|f_0\|_{P(L_\infty)})\\&=(\sqrt{2}+\sqrt{3})(f_0^*(t)+32)\end{aligned}$$

for $0<t\leqslant 1/4$, whence (5.14) follows.

The proof of the first assertion of the theorem is similar. First of all, by (1.1)

$$\|x\|_{\exp L_2}\asymp\sup_{k=1,2,\dots}\frac{x^*(2^{-2k})}{\sqrt{k}}$$

for $x\in\exp L_2$ with universal constants. Hence, for some $C_1>0$ and $k_0\in\mathbb{N}$

$$x^*(2^{-2k_0})\geqslant C_1^{-1}\sqrt{k_0}\|x\|_{\exp L_2},\qquad x^*(2^{-2k})\leqslant C_1\sqrt{k}\|x\|_{\exp L_2},\quad k=1,2,\dots.$$

We set

$$f:=\sum_{k=1}^{\infty}\frac{x^*(2^{-2k})}{k}\Big(\chi_{\Delta_k^2}+\sum_{I\in\Sigma_k^1}\chi_I+\sum_{I\in\Sigma_k^2}\chi_I\Big),$$

where Σ_k^1 and Σ_k^2 are the families of dyadic intervals defined above. Now, by the above inequalities,

$$\begin{aligned}\Big(\sum_{i=1}^{k_0}\Big(\frac{x^*(2^{-2k_0})}{k_0}\Big)^2\Big)^{1/2}\geqslant C_1^{-1}\|x\|_{\exp L_2},\\ \Big(\sum_{i=1}^{k}\Big(\frac{x^*(2^{-2k})}{k}\Big)^2\Big)^{1/2}\leqslant C_1\|x\|_{\exp L_2},\end{aligned}$$

for $k=1,2,\dots$. Consequently, recalling the definition of f we obtain (5.11) as in the first part of the proof. Furthermore, by the same argument,

$$|f(t)| \geqslant \sum_{k=1}^{\infty} x^*(2^{-2k})\kappa_{E_k}(t), \tag{5.20}$$

where, as before, $E_k = \Delta_{2k}^{1+2^k} \cup \Delta_{2k}^{2^{k+1}}$.

On the other hand,

$$x^*(t) \leqslant \sum_{k=1}^{\infty} x^*(2^{-k})\kappa_{(2^{-k},2^{-k+1}]}(t) = u_1(t)+v_1(t),$$

where

$$u_1(t) := \sum_{i=1}^{\infty} x^*(2^{-2i+1})\kappa_{(2^{-2i+1},2^{-2i+2}]}(t)$$

and

$$v_1(t) := \sum_{i=1}^{\infty} x^*(2^{-2i})\kappa_{(2^{-2i},2^{-2i+1}]}(t).$$

Using (5.20) and (5.18) it follows that $u_1^*(t) \leqslant f^*(t)$ and $v_1^*(t) \leqslant f^*(t)$, $0<t\leqslant 1$. Thus, as in the first part of the proof,

$$\begin{aligned} x^*(t) &\leqslant \frac{1}{t}\int_0^t x^*(s)\,ds \leqslant \frac{1}{t}\int_0^t u_1^*(s)\,ds + \frac{1}{t}\int_0^t v_1^*(s)\,ds \leqslant \frac{2}{t}\int_0^t f^*(s)\,ds \\ &\leqslant 2(f^*(t)+32\|f\|_d) \leqslant 2(f^*(t)+32\|f\|_{P(L_\infty)}), \end{aligned}$$

and now (5.12) is proved.

The results obtained above give the following.

Corollary 5. *The symmetric hull of the space $P(L_\infty)$ coincides with the exponential Orlicz space* $\exp L_2$.

§ 6. Interpolation properties of the Banach couple $(L_\infty, P(L_\infty))$

Let (X_0,X_1) be a Banach couple, that is, Banach spaces X_0 and X_1 embedded into some Hausdorff topological linear space; let $x\in X_0+X_1$ and $t>0$. The functional

$$\mathscr{K}(t,x,X_0,X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x=x_0+x_1,\ x_0\in X_0,\ x_1\in X_1\}$$

is called the *Peetre $\mathscr{K}$-functional* of the couple (X_0,X_1). Let Y_0 and Y_1 be closed subspaces of X_0 and X_1, respectively. A couple (Y_0,Y_1) is called a *K-closed subcouple* of the couple (X_0,X_1) if the functionals $\mathscr{K}(t,y,X_0,X_1)$ and $\mathscr{K}(t,y,Y_0,Y_1)$ are equivalent with constants independent of $t>0$ and $y\in Y_0+Y_1$. From a Banach couple (X_0,X_1) and a Banach lattice F on $(0,\infty)$ with measure dt/t such that $\min(1,t)\in F$ we can construct the space of the $\mathscr{K}$-method of real interpolation $(X_0,X_1)_F^{\mathscr{K}}$ with norm $\|x\| := \|\mathscr{K}(t,x,X_0,X_1)\|_F$, which is an interpolation space with respect to the Banach couple (X_0, X_1) (see [25], § 3.3). This means that an arbitrary linear operator that is bounded on X_0 and X_1 is also bounded on $(X_0, X_1)_F^{\mathscr{K}}$. In the particular case when F is a weighted $L_p(t^{-\theta}, dt/t)$-space, $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$, we have the classical spaces $(X_0, X_1)_{\theta,p}$. A detailed account of the properties of these spaces can be found in [26].

Consider the operator

$$Ra(t) = \sum_{k=1}^{\infty} a_k r_k(t), \qquad a = (a_k)_{k=1}^{\infty} \in \ell_2,$$

where, as before, $r_k(t)$ are the Rademacher functions. The operator R is an isometry from ℓ_1 into L_∞ and (in view of equality (1.2)) from ℓ_2 into $P(L_\infty)$. So, the couple $(R(\ell_1), R(\ell_2))$ may be regarded as a subcouple of the Banach couple $(L_\infty, P(L_\infty))$, where $R(\ell_1)$ is a subspace of L_∞, and $R(\ell_2)$ $(=$ Rad$)$ is a subspace of $P(L_\infty)$. The couples $(R(\ell_1), R(\ell_2))$ and (ℓ_1, ℓ_2) are isometric, so from the definition of the $\mathscr{K}$-functional it follows that

$$\mathscr{K}(t, Ra; R(\ell_1), R(\ell_2)) = \mathscr{K}(t, a; \ell_1, \ell_2). \tag{6.1}$$

Theorem 6. *The Banach couple $(R(\ell_1), R(\ell_2))$ is a K-closed subcouple of the couple $(L_\infty, P(L_\infty))$.*

Proof. In view of (6.1), it suffices to show that

$$\mathscr{K}(t, Ra; L_\infty, P(L_\infty)) \asymp \mathscr{K}(t, a; \ell_1, \ell_2)$$

with constants independent of $a = (a_k)_{k=1}^{\infty} \in \ell_2$ and $t > 0$. First of all, by the definition of the $\mathscr{K}$-functional,

$$\begin{aligned}\mathscr{K}(t, Ra; L_\infty, P(L_\infty)) &\leqslant \inf\bigl\{\|Rb\|_{L_\infty} + t\|Rc\|_{P(L_\infty)} : a = b + c,\ b \in \ell_1,\ c \in \ell_2\bigr\}\\ &= \inf\bigl\{\|b\|_{\ell_1} + t\|c\|_{\ell_2} : a = b + c,\ b \in \ell_1,\ c \in \ell_2\bigr\} = \mathscr{K}(t, a; \ell_1, \ell_2).\end{aligned}$$

On the other hand, as we have already pointed out, $P(L_\infty) \subset \exp L_2$ and $\|x\|_{\exp L_2} \leqslant C\|x\|_{P(L_\infty)}$ for all $x \in P(L_\infty)$. Hence,

$$\mathscr{K}(t, Ra; L_\infty, G) = \mathscr{K}(t, Ra; L_\infty, \exp L_2) \leqslant C\mathscr{K}(t, Ra; L_\infty, P(L_\infty)).$$

Furthermore, by Theorem 4.1 in [4]

$$\mathscr{K}(t, Ra; L_\infty, G) \asymp \mathscr{K}(t, a; \ell_1, \ell_2)$$

with constants independent of $a = (a_k)_{k=1}^{\infty} \in \ell_2$ and $t > 0$. Thus,

$$\mathscr{K}(t, a; \ell_1, \ell_2) \leqslant C'\mathscr{K}(t, Ra; L_\infty, P(L_\infty)),$$

and the theorem follows.

From this theorem, and also from the definition of spaces of the $\mathscr{K}$-method of real interpolation we have the following result.

Corollary 6. *Let F be a Banach lattice such that $\min(1, t) \in F$, and let $\mathscr{F} = (\ell_1, \ell_2)_F^{\mathscr{K}}$. If $X = (L_\infty, P(L_\infty))_F^{\mathscr{K}}$, then the norms $\|\sum_{n=1}^{\infty} a_n r_n\|_X$ and $\|(a_n)\|_{\mathscr{F}}$ are equivalent.*

Given arbitrary interpolation space $\mathscr{F}$ with respect to the couple (ℓ_1, ℓ_2), Corollary 6 makes it possible to find a space X that is an interpolation space with respect to the couple $(L_\infty, P(L_\infty))$ such that on $\mathrm{Rad}\cap X$ the norm of X is equivalent to the norm of $\mathscr{F}$. In particular, since $\ell_p = (\ell_1, \ell_2)_{\theta,p}$, where $1 < p < 2$ and $\theta = 2(p-1)/p$ (see [26], Theorem 5.2.1), we have

$$\biggl\|\sum_{n=1}^{\infty} a_n r_n\biggr\|_{X_p} \asymp \|(a_n)\|_{\ell_p}$$

for $X_p = (L_\infty, P(L_\infty))_{\theta,p}$. The relation

$$\biggl\|\sum_{n=1}^{\infty} a_n r_n\biggr\|_{\Lambda_p} \asymp \|(a_n)\|_{\ell_p}, \qquad 1 < p < 2,$$

for the Lorentz space Λ_p with the norm

$$\|x\|_{\Lambda_p} = \biggl((p-1)\int_0^1 (x^*(t))^p \ln^{-p}\frac{e}{s}\,\frac{ds}{s}\biggr)^{1/p},$$

which is close to the above, was first proved by Nikishin [27] (for different proofs of this result see [24] or [4], Example 4.1). In conclusion, we observe that $\Lambda_p = (L_\infty, G)_{\theta,p}$, where $\theta = 2(p-1)/p$.

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