Positivity (2013) 17:223–234
DOI 10.1007/s11117-012-0159-7

Positivity

Geometrical properties of Banach spaces generated by sublinear operators

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Received: 22 July 2011 / Accepted: 27 January 2012 / Published online: 12 February 2012

Abstract We solve a problem posed by Mastylo (Math Japon 36(1), 85–92, 1991) proving that every "non-trivial" subspace of a Banach space X generated by some positive sublinear operator and an L_p-space with $1 \leq p < \infty$ contains, for any $\varepsilon > 0$, an $(1+\varepsilon)$-copy of l_p which is $(1+\varepsilon)$-complemented in X.

Keywords Sublinear operator · Complemented l_p-copy · Real interpolation space · Extrapolation space · Cesàro space

Mathematics Subject Classification (2000) Primary 46B20 · 46E30; Secondary 46B42 · 46B70

1 Introduction and preliminaries

Many Banach spaces which play an important role in functional analysis and its applications are obtained in a special way: the norms of these spaces are generated by positive sublinear operators. The well-known examples of such spaces are real interpolation spaces (see [1,2]), extrapolation spaces (see [3–6]), Triebel spaces $B^s_{p,q}$ and $F^s_{p,q}$ [7], "tent" spaces [8].

Mastylo [9] investigated under some conditions Banach spaces of the above type. In this paper, we weaken in a natural way the conditions imposed there and give a solution of a problem of existing complemented l_p-copies in such spaces raised in [9].

Research was partially supported by RFBR Grant no. 10-01-00077.

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Let (Ω, Σ, μ) be a complete σ-finite measure space and let $L^0 = L^0(\Omega, \Sigma, \mu)$ denote the space of all (equivalence classes of) μ-measurable real-valued functions, equipped with the topology of convergence in measure on μ-finite sets.

Throughout this paper by $\mathcal{X} = (\mathcal{X}, \tau)$ we will denote an $\mathcal{F}$-space, i.e., complete topological vector space with a topology τ generated by a translation-invariant metric. Moreover, let $S\colon \mathcal{X} \to L^0(\Omega, \Sigma, \mu)$ be a positive sublinear operator vanishing only at zero. This means that (1) $Sx \geq 0$, (2) $S(\lambda x) = |\lambda| S(x) (\lambda \in \mathbb{R})$, (3) $S(x+y) \leq Sx+Sy$ for all $x, y \in \mathcal{X}$ and (4) $Sx = 0$ if and only if $x = 0$. For every Banach function lattice $F \subset L^0(\Omega, \Sigma, \mu)$ we define the linear space

$$D_F(S) := \{x \in \mathcal{X} : Sx \in F\}$$

with the norm

$$\|x\|_{D_F(S)} = \|Sx\|_F, \quad x \in D_F(S).$$

We will be interested, mainly, in the case when $F = L_p := L_p(\Omega, \Sigma, \mu)$ $(1 \leq p < \infty)$ and will write $D_p(S)$ (or shortly D_p) instead of $D_{L_p}(S)$.

Throughout we will assume that the following hold:

(a) The space $D_F(S)$ is continuously embedded into $(\mathcal{X}, \tau)$;

(b) For every $A \in \Sigma(\mu) := \{B \in \Sigma : \mu(B) < \infty\}$ and any $\varepsilon > 0$ there is a set $A_\varepsilon \in \Sigma$ such that $A_\varepsilon \subset A$, $\mu(A_\varepsilon) \geq \mu(A) - \varepsilon$ and the mapping $P_{A_\varepsilon} S\colon (D_F(S), \tau) \to F$ is continuous at zero.

Here and in the sequel by P_B it will be denoted the multiplication operator by the characteristic function χ_B, where $B \in \Sigma$, i.e., $P_B x := x \cdot \chi_B (x \in L^0)$.

Clearly, in the case when $\mu(\Omega) < \infty$ condition (b) is equivalent to the following one:

(c) For any $\varepsilon > 0$ there is a set $\Omega_\varepsilon \in \Sigma$ such that $\mu(\Omega_\varepsilon) \geq \mu(\Omega) - \varepsilon$ and the mapping $P_{\Omega_\varepsilon} S\colon (D_F(S), \tau) \to F$ is continuous at zero.

Note that Mastylo [9] considered instead of (b) a stronger condition:

(b′) For every $A \in \Sigma(\mu)$ the mapping $P_A S\colon (D_F(S), \tau) \to F$ is continuous at zero.

An important advantage of condition (b) by comparison with (b′) is that in this case the general construction described above includes various classes of spaces corresponding to the case when $\mu(\Omega) < \infty$, for example, extrapolation spaces and Cesàro function spaces on [0, 1] (see Examples 2, 3).

Let us show first that conditions (a) and (b) guarantee that $D_F(S)$ is a Banach space. In fact, let $\{x_n\} \subset D_F(S)$ be a Cauchy sequence, i.e., $\|x_n - x_m\|_{D_F(S)} := \|S(x_n - x_m)\|_F \to 0$ as $n, m \to \infty$. Since $Sx_n \leq S(x_n - x_m) + Sx_m$, then $\|Sx_n - Sx_m\|_F \to 0$ as $n, m \to \infty$. Therefore, $\|Sx_n - y\|_F \to 0$ as $n \to \infty$ for some $y \in F$.

On the other hand, by (a), $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, then there is a $x \in \mathcal{X}$ such that $x_n \xrightarrow{\tau} x$. Using condition (b), for every $A \in \Sigma(\mu)$ and for any $\varepsilon > 0$, we can find a set $A_\varepsilon \in \Sigma$ such that $A_\varepsilon \subset A$, $\mu(A_\varepsilon) \geq \mu(A) - \varepsilon$ and $\|\chi_{A_\varepsilon} S(x_n - x)\|_F \to 0$. This and sublinearity of S imply that $\|\chi_{A_\varepsilon}(Sx_n - Sx)\|_F \to 0$.

Since $A \in \Sigma(\mu)$ and $\varepsilon > 0$ are arbitrary, then $Sx_n \to Sx$ in $L^0 = L^0(\Omega, \Sigma, \mu)$. Thus, $y = Sx$ and $\|x_n - x\|_{D_F(S)} \to 0$.

In what follows by a subspace of a topological vector space $\mathcal{X}$ we will mean an infinite-dimensional closed subspace. By $[x_n]$ we denote, as usual, the subspace generated by a sequence $\{x_n\}_{n=1}^{\infty}$ from $\mathcal{X}$. Let $\varepsilon > 0$. We will say that Banach spaces X and Y are $(1+\varepsilon)$-*isomorphic* if there is an isomorphism $H : X \to Y$ such that $(1-\varepsilon)\|x\|_X \leq \|Hx\|_Y \leq (1+\varepsilon)\|x\|_X$ for all $x \in X$. A subspace N of a Banach space X will be called $(1+\varepsilon)$-*complemented* in X if there exists a bounded linear projection $P : X \to X$ whose image is N and such that $\|P\| \leq 1+\varepsilon$. By η_p (or shortly by η) it will be denoted the norm topology of $D_p = D_p(S)$ and by $|\cdot|$ an $\mathcal{F}$-norm, which induces a topology τ on an $\mathcal{F}$-space $\mathcal{X}$. Moreover, if $(\mathcal{X}, \tau)$ is a topological vector space and N is a subspace of $\mathcal{X}$, then $\tau_{|N}$ is the induced topology on N.

Mastylo [9, Theorem 6] proved that under condition (b′) for any $\varepsilon > 0$ every non-closed in $\mathcal{X}$ subspace of D_p contains $(1+\varepsilon)$-isomorphic copy of l_p, which is not closed in $\mathcal{X}$. Also he asked under which conditions on S *any* non-closed in $\mathcal{X}$ subspace of D_p contains a complemented in D_p subspace which is isomorphic to l_p? Here, we solve this problem in the affirmative, even under a weaker condition than (b′), namely (b).

The paper is organized as follows. In Sect. 2 we consider some examples of Banach spaces of the type $D_p(S)$. Section 3 contains the main result, Theorem 1, showing that under conditions (a) and (b) for any $\varepsilon > 0$ every non-closed in $\mathcal{X}$ subspace X of the space $D_p(S)$ contains a subspace Y_ε, which is $(1+\varepsilon)$-isomorphic to l_p and $(1+\varepsilon)$-complemented in $D_p(S)$. In the last Sect. 4 we collect some applications of obtained results to real interpolation spaces, extrapolation spaces and Cesàro spaces.

The author thanks the referee for suggestions which helped to improve the presentation of the results of this article.

2 Examples of spaces of the type $D_p(S)$

Example 1 (Real interpolation spaces) Recall that a pair (X_0, X_1) of Banach spaces is called a *Banach couple* if X_0 and X_1 are both linearly and continuously embedded in some Hausdorff linear topological vector space. For a Banach couple (X_0, X_1) we can define *the intersection* $X_0 \cap X_1$ and *the sum* $X_0 + X_1$ as the Banach spaces with the natural norms: $\|x\|_{X_0\cap X_1} = \max\left\{\|x\|_{X_0}, \|x\|_{X_1}\right\}$ and $\|x\|_{X_0+X_1} = \mathcal{K}(1, x; X_0, X_1)$, where $\mathcal{K}(t, x; X_0, X_1)$ is the *Peetre* $\mathcal{K}$ *-functional*, i.e.,

$$\mathcal{K}(t, x; X_0, X_1) = \inf\{||x_0||_{X_0} + t||x_1||_{X_1} : x = x_0 + x_1, x_i \in X_i, i = 0, 1\}$$

for any $x \in X_0 + X_1$ and $t > 0$.

Let (X_0, X_1) be an arbitrary Banach couple, $\mathcal{X} = X_0 + X_1$ with the topology generated by the norm of $X_0 + X_1$, and let $\psi : [0, \infty) \to [0, \infty)$ be a continuous concave function with $\psi(0) = 0$ such that

$$\lim_{t\to 0+} \psi(t) = \lim_{t\to+\infty} \frac{\psi(t)}{t} = 0. \tag{1}$$

Then, define the positive sublinear operator $S : \mathcal{X} \to L^0(\mathbb{Z}, 2^{\mathbb{Z}}, \mu)$, where μ is the counting measure, by

$$Sx := \left\{\mathcal{K}(2^n, x; X_0, X_1)/\psi(2^n)\right\}_{n=-\infty}^{\infty} \quad (x \in \mathcal{X}).$$

It is not hard to check that $D_p(S)$ satisfies properties (a) (see, for example, [2]) and (b). In the interpolation theory of operators these spaces are well known as the *real interpolation spaces* [2] and they are denoted by $\mathcal{K}_{L_p^\psi}(X_0, X_1)$. In particular, if $\psi(t) = t^\theta$ $(0 < \theta < 1)$, we obtain classical *Lions–Peetre interpolation spaces* $(X_0, X_1)_{\theta,p}$ [1].

Example 2 (Extrapolation spaces) Let $\mathcal{X} = \cap_{1\le p<\infty} L_p[0,1]$ with the topology τ generated by the family of usual L_p-norms $\|x\|_p$, $1 \le p < \infty$ (i.e., $x_n \xrightarrow{\tau} 0$ means $\|x_n\|_p \to 0$ for every $1 \le p < \infty$). Take for a measure space the interval $[1,\infty)$ with the measure $d\lambda := 2^{-s}\varphi'(2^{-s})\,ds$, where $\varphi : [0,1] \to [0,1]$ is a continuous increasing concave function, $\varphi(0) = 0$. It is easy to see that the operator

$$Sx(t) := \|x\|_t \quad (1 \le t < \infty)$$

satisfies conditions (1)–(4) and property (a) is fulfilled. Moreover, for every $\beta \ge 1$

$$\lambda([\beta,\infty)) = \int_\beta^\infty d\lambda = \int_\beta^\infty 2^{-s}\varphi'(2^{-s})\,ds = \frac{1}{\log 2}\int_0^{2^{-\beta}} \varphi'(t)\,dt = \frac{1}{\log 2}\varphi(2^{-\beta}). \tag{2}$$

In particular, we see that $\lambda([1,\infty)) = \varphi(1/2)/\log 2 < \infty$, and therefore for proving (b) it is sufficiently to check (c).

Let $\varepsilon > 0$ be arbitrary. Since $\lim_{u\to 0+}\varphi(u) = 0$, then taking into account (2), we can fix $\beta > 1$ so that $\lambda([\beta,\infty)) < \varepsilon$. Suppose that $x_n \in D_p(S)$ and $x_n \xrightarrow{\tau} 0$. Let us show that

$$P_{[1,\beta]}Sx_n(t) := \chi_{[1,\beta]}(t)\|x_n\|_t \to 0 \ \text{ in } \ L_p(\lambda). \tag{3}$$

In fact,

$$\begin{aligned}\|P_{[1,\beta]}Sx_n\|_{L_p(\lambda)}^p &= \int_1^\beta \|x_n\|_t^p 2^{-t}\varphi'(2^{-t})\,dt \\ &= \frac{1}{\log 2}\int_{2^{-\beta}}^{1/2} \|x_n\|_{\log_2(1/u)}^p \varphi'(u)\,du \le \frac{\varphi(1/2)}{\log 2}\|x_n\|_\beta^p.\end{aligned}$$

By assumption, the right-hand side of the last inequality tends to zero as $n \to \infty$, and therefore (3) is proved.

The spaces $D_p(S)$ are called *the extrapolation spaces* and originally they appeared in connection with the well-known Yano theorem [10]. These spaces can be identified with the appropriate rearrangement invariant spaces (see, for example, [11,12]). In particular, as it is proved in [5, Theorem 4], $D_p(S)$ coincides with the Lorentz space $\Lambda_p(\varphi)$ if and only if $\varphi \in \Delta^2$. Recall that the *Lorentz space* $\Lambda_p(\varphi)$ consists of all measurable on [0, 1] functions $x(t)$ such that

$$\|x\|_{\Lambda_p(\varphi)} := \left(\int_0^1 (x^*(t))^p \, d\varphi(t)\right)^{1/p} < \infty$$

($x^*(t)$ is the nonincreasing rearrangement of $|x(t)|$) and the condition $\varphi \in \Delta^2$ means that there exists a $C > 0$ such that $\varphi(t) \leq C\varphi(t^2)$ for all $0 < t \leq 1$.

Example 3 (Cesàro function spaces) Let $\mathcal{X} = \cap_{0<h<1} L_1[0, h]$ with the topology τ generated by the family of semi-norms $\|x\|_{L_1[0,h]}$ $(0 < h < 1)$. Taking for a measure space the segment [0, 1] with the usual Lebesgue measure and for an operator S the well-known Hardy operator:

$$Sx(t) := \frac{1}{t} \int_0^t |x(s)| \, ds \quad (0 \leq t < 1),$$

we obtain *the Cesàro function space* $D_p(S) := Ces_p[0, 1]$. In particular, $Ces_1[0, 1] = L^1_w$ with the weight $w(t) = \ln(1/t)$ and $\{x(t) \in Ces_p[0, 1] : x(t) = 0 \text{ if } t > a\} \subset L^1[0, a]$ continuously for any $a \in (0, 1)$ but not for $a = 1$ [13, Theorem 1(a,d)]. Therefore, for every $1 \leq p < \infty$, the continuous embedding $Ces_p[0, 1] \subset \mathcal{X}$ holds, i.e., we have (a). Moreover, it is not hard to check (see [14]) that $\{x(t) \in Ces_p[0, 1] : x(t) = 0 \text{ if } t \in [0, h) \cup (h, 1]\} = L_1[h, 1 - h]$ (with equivalence of norms) for every $h \in (0, 1/2)$, which implies condition (c).

Similarly, if $\mathcal{X} = \cap_{0<h<\infty} L_1[0, h]$ with the topology τ generated by the family of semi-norms $\|x\|_{L_1[0,h]}$ $(0 < h < \infty)$ and

$$Sx(t) := \frac{1}{t} \int_0^t |x(s)| \, ds \quad (0 \leq t < \infty),$$

we obtain *the Cesàro function space* $D_p(S) := Ces_p[0, \infty)$.

3 Main results

The proof of the following result only slightly differs from the one of Theorem 5 in [9] (in view of the fact that we use condition (b) instead of (b′)) and therefore it is omitted.

Proposition 1 *Let $1 \le p < \infty$ and K be a compact subset (in $D_p = D_p(S)$) of the unit sphere of D_p, and let X be a subspace of D_p such that $\tau_{|X}$ is strictly weaker than $\eta_{|X}$. Then, for every $\alpha > 0$ and $\varepsilon \in (0, 1)$, there exists $y \in D_p$ with $\|y\|_{D_p} = 1$, $|y| < \alpha$ and such that*

$$(1-\varepsilon)(|\lambda_1|^p + |\lambda_2|^p)^{1/p} \le \|\lambda_1 x + \lambda_2 y\|_{D_p} \le (1+\varepsilon)(|\lambda_1|^p + |\lambda_2|^p)^{1/p} \quad (4)$$

for each $x \in K$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

The following theorem resolves a problem posed by Mastylo [9, see p. 90].

Theorem 1 *Let $1 \le p < \infty$ and let X be a subspace of D_p such that $\tau_{|X}$ is strictly weaker than $\eta_{|X}$. Then, for any $\varepsilon > 0$, there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset X$ with $\|x_k\|_{D_p} = 1$ $(k = 1, 2, \dots)$ and satisfying the following properties:*

(1) *for any $\lambda_k \in \mathbb{R}$ $(k = 1, 2, \dots)$ we have that*

$$(1-\varepsilon)\left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p} \le \left\| \sum_{k=1}^{\infty} \lambda_k x_k \right\|_{D_p} \le (1+\varepsilon)\left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p}; \quad (5)$$

(2) *the subspace $X_0 := [x_k]$ is $(1+\varepsilon)$-complemented in D_p;*
(3) *the topology $\tau_{|X_0}$ is strictly weaker than $\eta_{|X_0}$.*

Proof For a given $\varepsilon > 0$, let $\delta \in (0, 1), \theta \in (0, 1)$, and $\theta_n \in (0, 1)$ be such that $(1+\delta)^3 < 1+\varepsilon$, $(1+\theta)/(1-\theta) < 1+\delta$, and

$$1 - \delta < \prod_{n=1}^{\infty}(1-\theta_n) < \prod_{n=1}^{\infty}(1+\theta_n) < 1+\delta. \quad (6)$$

Moreover, let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive numbers tending to zero.

We define inductively a sequence $\{y_j\}_{j=1}^{\infty} \subset X$ such that $\|y_j\|_{D_p} = 1, |y_j| < \alpha_j (j \in \mathbb{N})$ and for every $m \in \mathbb{N}$ and for all $\lambda_1, \dots, \lambda_m \in \mathbb{R}$

$$\prod_{j=1}^{m}(1-\theta_j)\left(\sum_{j=1}^{m} |\lambda_j|^p\right)^{1/p} \le \left\| \sum_{j=1}^{m} \lambda_j y_j \right\|_{D_p} \le \prod_{j=1}^{m}(1+\theta_j)\left(\sum_{j=1}^{m} |\lambda_j|^p\right)^{1/p}. \quad (7)$$

At first, since $\tau_{|X}$ is strictly weaker than $\eta_{|X}$, there is a $y_1 \in X$ such that $\|y_1\|_{D_p} = 1$ and $|y_1| < \alpha_1$. By Proposition 1, we can find $y_2 \in X$ such that $\|y_2\|_{D_p} = 1$, $|y_2| < \alpha_2$ and

$$(1-\theta_1)(|\lambda_1|^p + |\lambda_2|^p)^{1/p} \le \|\lambda_1 y_1 + \lambda_2 y_2\|_{D_p} \le (1+\theta_1)(|\lambda_1|^p + |\lambda_2|^p)^{1/p}$$

for every $\lambda_1, \lambda_2 \in \mathbb{R}$.

Assume that we already have chosen elements $y_1, \dots, y_{n-1} \in X$ with $\|y_j\|_{D_p} = 1$, $|y_j| < \alpha_j$ $(j = 1, \dots, n-1)$ and satisfying inequality (7) for $m = n-1$ and for all

$\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R}$. It is clear that the set $K := \{x \in \operatorname{span}\{y_1, \ldots, y_{n-1}\} : \|x\|_{D_p} = 1\}$ is compact in D_p. Therefore, by Proposition 1, there exists a $y_n \in X$ such that $\|y_n\|_{D_p} = 1$, $|y_n| < \alpha_n$ and

$$(1-\theta_n)(1+|\gamma|^p)^{1/p} \leq \|x+\gamma y_n\|_{D_p} \leq (1+\theta_n)(1+|\gamma|^p)^{1/p} \tag{8}$$

for each $x \in K$ and $\gamma \in \mathbb{R}$. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be arbitrary. Clearly, we may (and will) assume that $\|\sum_{j=1}^{n-1} \lambda_j y_j\|_{D_p} > 0$. Setting $\gamma := \lambda_n \cdot \|\sum_{j=1}^{n-1} \lambda_j y_j\|_{D_p}^{-1}$ and $x := (\sum_{j=1}^{n-1} \lambda_j y_j) \cdot \|\sum_{j=1}^{n-1} \lambda_j y_j\|_{D_p}^{-1}$ (note that $x \in K$) and using (8), we infer that

$$\begin{aligned}(1-\theta_n)\left(\left\|\sum_{j=1}^{n-1}\lambda_j y_j\right\|_{D_p}^p + |\lambda_n|^p\right)^{1/p} &\leq \left\|\sum_{j=1}^{n}\lambda_j y_j\right\|_{D_p} \\ &\leq (1+\theta_n)\left(\left\|\sum_{j=1}^{n-1}\lambda_j y_j\right\|_{D_p}^p + |\lambda_n|^p\right)^{1/p}.\end{aligned}$$

Hence, from (7) for $m = n-1$ it follows that

$$\prod_{j=1}^{n}(1-\theta_j)\left(\sum_{j=1}^{n}|\lambda_j|^p\right)^{1/p} \leq \left\|\sum_{j=1}^{n}\lambda_j y_j\right\|_{D_p} \leq \prod_{j=1}^{n}(1+\theta_j)\left(\sum_{j=1}^{n}|\lambda_j|^p\right)^{1/p},$$

which means that (7) holds for $m = n$. Therefore, we can choose a sequence $\{y_j\}_{j=1}^{\infty} \subset X$ with required properties, whence by (6) we obtain that

$$(1-\delta)\left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{1/p} \leq \left\|\sum_{j=1}^{\infty}\lambda_j y_j\right\|_{D_p} \leq (1+\delta)\left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{1/p} \tag{9}$$

for any $\lambda_j \in \mathbb{R} (j = 1, 2, \ldots)$. Moreover, since $\|y_j\|_{D_p} = 1$ and $|y_j| < \alpha_j (j \in \mathbb{N})$, where $\alpha_j \to 0$, then $\tau_{|[y_j]}$ is strictly weaker than $\eta_{|[y_j]}$. Let us show that there exists a subsequence $\{x_k\} \subset \{y_j\}_{j=1}^{\infty}$, which spans an $(1+\varepsilon)$-complemented subspace in D_p.

Firstly, using a procedure similar to the Kadec–Pełczyński disjointness method [15], we prove that, for any positive $\delta_k (k = 1, 2, \ldots)$ with $\sum_{i=k+1}^{\infty} \delta_i < \delta_k$, there exist pairwise disjoint sets $A_k \in \Sigma(\mu)$ and a subsequence $\{y_{n_k}\} \subset \{y_n\}_{n=1}^{\infty}$ such that the functions $z_k := Sy_{n_k}$ satisfy the inequalities:

$$\|z_k - P_{A_k} z_k\|_p < \delta_k \quad (k = 1, 2, \ldots). \tag{10}$$

Let $B_1 \in \Sigma(\mu)$ be arbitrary. Since $|y_n| \to 0$, then by (b), there is a set $C_1 \subset B_1$ such that $\|P_{C_1} S y_n\|_p \to 0$. Hence,

$$\|P_{C_1} z_1\|_p = \|P_{C_1} S y_{n_1}\|_p \le \frac{\delta_1}{3}$$

for some $n_1 \in \mathbb{N}$. In addition, there exists a set $B_2 \in \Sigma(\mu)$ such that $B_2 \supset B_1$ and

$$\|P_{B_2'} z_1\|_p = \|P_{B_2'} S y_{n_1}\|_p \le \frac{\delta_1}{3},$$

where $E' := \Omega \backslash E (E \subset \Omega)$. Furthermore, there is a $h_1 > 0$ such that from $\mu(E) < h_1$ it follows: $\|P_E z_1\|_p \le \delta_2/3$. Applying (b) once more, we can find $C_2 \subset B_2$ such that $\mu(B_2 \backslash C_2) < h_1$ and $\|P_{C_2} S y_n\|_p \to 0$. Taking into account the choice of h_1, we have that

$$\|P_{B_2 \backslash C_2} z_1\|_p = \|P_{B_2 \backslash C_2} S y_{n_1}\|_p \le \frac{\delta_2}{3}.$$

Suppose that for some $k \in \mathbb{N}$ we already have chosen two sequences of sets $\{B_i\}_{i=1}^k$ and $\{C_i\}_{i=1}^k$, $C_i \subset B_i (i = 1, 2, \dots, k)$ and a subsequence $\{y_{n_i}\}_{i=1}^{k-1} \subset \{y_n\}_{n=1}^\infty$ so that $\|P_{C_k} S y_n\|_p \to 0$ as $n \to \infty$ and the functions $z_i = S y_{n_i}$ satisfy the inequalities:

$$\|P_{C_i} z_i\|_p \le \frac{\delta_i}{3}, \tag{11}$$

$$\|P_{B_{i+1}'} z_i\|_p \le \frac{\delta_i}{3}, \tag{12}$$

and

$$\|P_{B_{i+1} \backslash C_{i+1}} z_j\|_p \le \frac{\delta_{i+1}}{3} \quad (j = 1, 2, \dots, i) \tag{13}$$

for any $i = 1, 2, \dots, k-1$. Then, it is clear that

$$\|P_{C_k} z_k\|_p = \|P_{C_k} S y_{n_k}\|_p \le \frac{\delta_k}{3}$$

for some $n_k \in \mathbb{N}$ and that there exists $B_{k+1} \in \Sigma(\mu)$ with $B_{k+1} \supset B_k$ and

$$\|P_{B_{k+1}'} z_k\|_p = \|P_{B_{k+1}'} S y_{n_k}\|_p \le \frac{\delta_k}{3}.$$

Moreover, as above, we can find a $h_k > 0$ such that from $\mu(E) < h_k$ it follows: $\|P_E z_j\|_p \le \delta_{k+1}/3$ $(j = 1, \dots, k)$. Then, by (b), there is $C_{k+1} \subset B_{k+1}$ such that

$\mu(B_{k+1}\backslash C_{k+1}) < h_k$ and $\|P_{C_{k+1}} Sy_n\|_p \to 0$. In particular, we obtain that

$$\|P_{B_{k+1}\backslash C_{k+1}} z_j\|_p \le \frac{\delta_{k+1}}{3} \quad (j = 1, \dots, k).$$

Thus, we have constructed sequences of sets $\{B_i\}_{i=1}^{k+1}$ and $\{C_i\}_{i=1}^{k+1}$, $C_i \subset B_i (i = 1, 2, \dots, k+1)$ and a subsequence $\{y_{n_i}\}_{i=1}^{k} \subset \{y_n\}_{n=1}^{\infty}$ so that $\|P_{C_{k+1}} Sy_n\|_p \to 0$ as $n \to \infty$ and the functions $z_i = Sy_{n_i}$ satisfy inequalities (11), (12) and (13) for any $i = 1, 2, \dots, k$. Continuing in just the same way we obtain two sequences of infinite sets $\{B_i\}_{i=1}^{\infty}$ and $\{C_i\}_{i=1}^{\infty}$, $C_i \subset B_i (i = 1, 2, \dots)$ and a subsequence $\{y_{n_i}\}_{i=1}^{\infty} \subset \{y_n\}_{n=1}^{\infty}$ such that inequalities (11), (12) and (13) hold for all $i \in \mathbb{N}$.

Set

$$U_k := B_k \backslash C_k \text{ and } A_k := (U_k \cup (B_{k+1}\backslash B_k)) \setminus \left(\cup_{i=k+1}^{\infty} U_i\right) \quad (k = 1, 2, \dots).$$

It is not hard to check that A_k are pairwise disjoint sets. In fact, let $k < j$ and $\omega \in A_k$. Clearly, $\omega \notin U_j$. Therefore, assuming that $\omega \in A_j$, we obtain: $\omega \in B_{j+1}\backslash B_j$, whence $\omega \notin B_j$. Since $B_k \subset B_{k+1} \subset B_j$, we obtain that $\omega \notin A_k$, which contradicts our assumption.

Futhermore, by (11), (12) and (13), we have that

$$\begin{aligned} \|P_{A_k} z_k - z_k\|_p &\le \|P_{B'_{k+1}} z_k\|_p + \|P_{C_k} z_k\|_p + \sum_{i=k+1}^{\infty} \|P_{D_i} z_k\|_p \\ &\le \frac{\delta_k}{3} + \frac{\delta_k}{3} + \sum_{i=k+1}^{\infty} \frac{\delta_i}{3} < \delta_k, \end{aligned}$$

and (10) is proved. Combining (10) with the equality $\|z_k\|_p = \|Sy_{n_k}\|_p = 1$, we obtain that

$$\|P_{A_k} z_k\|_p > 1 - \delta_k \quad (k = 1, 2, \dots). \tag{14}$$

Setting $x_k := y_{n_k} (k \in \mathbb{N})$, by [16, Theorem 1.25], for every $k \in \mathbb{N}$ we can find a linear operator $L_k : D_p \to L_p$ such that $L_k x_k = Sx_k = z_k$ and $|L_k x| \le Sx$ for all $x \in D_p$. Define on D_p the following linear operator

$$Lx := \sum_{i=1}^{\infty} P_{A_i} L_i x.$$

Since the sets A_i are pairwise disjoint, then

$$|Lx| \le \sup_{i \in \mathbb{N}} |L_i x| \le Sx \quad (x \in D_p). \tag{15}$$

Furthermore, denote $u_k := Lx_k$ $(k = 1, 2, \dots)$. Since $P_{A_k}u_k = P_{A_k}L_kx_k = P_{A_k}z_k$, then by (15) and (10)

$$\|u_k - P_{A_k}z_k\|_p \le \|z_k - P_{A_k}z_k\|_p < \delta_k \ \ (k = 1, 2, \dots).$$

Therefore, from (14) it follows that

$$\sum_{k=1}^{\infty} \frac{\|u_k - P_{A_k}z_k\|_p}{\|P_{A_k}z_k\|_p} \le \sum_{k=1}^{\infty} \frac{\delta_k}{1-\delta_k}. \tag{16}$$

Since the sets A_k are pairwise disjoint, then $\{P_{A_k}z_k/\|P_{A_k}z_k\|_p\}_{k=1}^{\infty}$ is a basic sequence which is 1-complemented in L_p and isometrically equivalent to the canonical basis of l_p (see, for example, [17, Proposition 6.4.1]). Therefore, by the well-known principle of small perturbations [18, Proposition 1.a.9], choosing $\delta_k > 0$ in an appropriate way, we obtain that the subspace $[u_k]$ generated by the sequence $\{u_k\}_{k=1}^{\infty}$ will be both $(1+\delta)$-isomorphic to l_p and $(1+\delta)$-complemented in L_p. Moreover, the corresponding projection $Q : L_p \to [u_k]$ can be represented in the following form:

$$Qx := \sum_{k=1}^{\infty}(x, v_k)u_k,$$

where $v_k \in L_q (1/p + 1/q = 1)$, $(u_k, v_k) := \int_\Omega u_k(\omega)v_k(\omega)\,d\mu = 1$ and $(u_i, v_k) = 0$ if $i \ne k$.

Let us show that the linear operator

$$Tx := \sum_{k=1}^{\infty}(Lx, v_k)x_k$$

is a bounded projection in D_p with norm not exceeding $1 + \varepsilon$. In fact, firstly, taking into account the definition of x_k, properties of $\{u_k\}$ and Q and inequalities (9) and (15), we obtain that for any $x \in D_p$

$$\begin{aligned}\|Tx\|_{D_p} &\le (1+\delta)\left(\sum_{k=1}^{\infty}|(Lx, v_k)|^p\right)^{1/p} \le (1+\delta)^2\|Q(Lx)\|_p \\ &\le (1+\delta)^3\|Lx\|_p \le (1+\delta)^3\|Sx\|_p \le (1+\varepsilon)\|x\|_{D_p}.\end{aligned}$$

Moreover, for every $i \in \mathbb{N}$

$$Tx_i = \sum_{k=1}^{\infty}(Lx_i, v_k)x_k = \sum_{k=1}^{\infty}(u_i, v_k)x_k = x_i.$$

Therefore, $X_0 := [x_i]$ as the image of T is an $(1+\varepsilon)$-complemented subspace in D_p. Since $\{x_i\} \subset \{y_n\}_{n=1}^{\infty}$, then inequality (5) is an immediate consequence of (9). Therefore, the proof is complete.

Corollary 1 *For any $\varepsilon > 0$ every non-closed in $\mathcal{X}$ subspace of D_p contains $(1+\varepsilon)$-isomorphic copy of l_p, which is $(1+\varepsilon)$-complemented in D_p and not closed in $\mathcal{X}$.*

Arguing in the same way as in the proof of Corollary 8 from [9], we obtain the following result.

Theorem 2 *Let $1 \le p_0 \ne p_1 < \infty$. If $\tau_{|D_{p_i}}$ is strictly weaker than $\eta_{|D_{p_i}}$ for $i = 0$ and $i = 1$, then $D_{p_0} \ne D_{p_1}$.*

4 Some applications

Applying Theorems 1 and 2 to various classes of spaces of the type D_p, we get a number of results. Let us confine ourselves to the cases considered in Sect. 2.

Recall that a Banach couple (X_0, X_1) is called *nondegenerate* if $X_0 \cap X_1$ is a non-closed subspace of $X_0 + X_1$. If (X_0, X_1) is a nondegenerate Banach couple and $\psi : [0,\infty) \to [0,\infty)$ is a continuous concave function satisfying condition (1), then the restrictions of the norms $\|\cdot\|_{X_0+X_1}$ and $\|\cdot\|_{\mathcal{K}_{L_p^{\psi}}(X_0,X_1)}$ to $X_0 \cap X_1$ are not equivalent [2, Proposition 4.6.26]. Therefore, from Theorem 1 we infer the following result.

Theorem 3 [2, Theorem 4.6.25] *Let $1 \le p < \infty$ and let $\psi : [0,\infty) \to [0,\infty)$ be a continuous concave function with $\psi(0) = 0$ satisfying (1).*

(a) *If X is a non-closed in $X_0 + X_1$ subspace of the space $\mathcal{K}_{L_p^{\psi}}(X_0, X_1)$, then for every $\varepsilon > 0$ X contains a subspace Y_ε, which is $(1+\varepsilon)$-isomorphic to l_p and $(1+\varepsilon)$-complemented in $\mathcal{K}_{L_p^{\psi}}(X_0, X_1)$.*
(b) *The same holds for $X := \mathcal{K}_{L_p^{\psi}}(X_0, X_1)$ under the assumption that the couple (X_0, X_1) is nondegenerate.*

In particular, when $\psi(t) = t^{\theta}(0 < \theta < 1)$, we obtain the well-known result of M. Levy for the Lions–Peetre interpolation spaces $(X_0, X_1)_{\theta,p}$ [19].

Theorem 4 [20] *If a Banach couple (X_0, X_1) is nondegenerate, then the equality $(X_0, X_1)_{\theta_0,p_0} = (X_0, X_1)_{\theta_1,p_1}$ (with equivalence of norms) implies that $\theta_0 = \theta_1$ and $p_0 = p_1$.*

Let us proceed with extrapolation spaces (see Example 2).

Theorem 5 *Let $1 \le p < \infty$ and let $\varphi : [0,1] \to [0,1]$ is a continuous increasing concave function with $\varphi(0) = 0$ such that $\varphi \in \Delta^2$. If X is a subspace of the Lorentz space $\Lambda_p(\varphi)$, which is not closed with respect to the topology τ generated by the family of norms $\|x\|_p$, $1 \le p < \infty$, then for every $\varepsilon > 0$ X contains a subspace Y_ε that is $(1+\varepsilon)$-isomorphic to l_p and $(1+\varepsilon)$-complemented in $\Lambda_p(\varphi)$.*

And finally, we consider Cesàro function spaces (see Example 3). In the case $I = [0, 1]$, the following result is proved in [21, Theorem 3].

Theorem 6 *Let $1 \leq p < \infty$ and let $I = (0, \infty)$ or $I = [0, 1]$. Suppose that X is a subspace of the Cesàro space $Ces_p(I)$, which is not closed with respect to the topology τ generated by the family of semi-norms $\|x\|_{L_1[0,h]}$, where $0 < h < 1$ if $I = [0, 1]$ and $0 < h < \infty$ if $I = (0, \infty)$. Then for every $\varepsilon > 0$ X contains a subspace Y_ε that is $(1 + \varepsilon)$-isomorphic to l_p and $(1 + \varepsilon)$-complemented in $Ces_p(I)$.*

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