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Rosenthal Type Inequalities for Martingales in Symmetric Spaces

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Abstract—We establish inequalities similar to classical Rosenthal ones for sequences of martingale differences in general symmetric spaces. A central role is played by a predictable square variation of a martingale.

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1. INTRODUCTION

In 1970 H. Rosenthal proved remarkable inequalities [1], which imply that for an arbitrary sequence of independent zero mean functions from $L_p[0,1]$, $p \geq 2$, the map $f_k \to \overline{f}_k$, where $\overline{f}_k(t) := f_k(t-k+1)\chi_{[k-1,k)}(t)$ $(t>0)$, extends up to isomorphism between closed linear hulls of $[f_k]_{k=1}^{\infty}$ (in $L_p[0,1]$) and $[\overline{f}_k]_{k=1}^{\infty}$ (in $L_p[0,\infty) \cap L_2[0,\infty)$). Later this result was generalized by W. B. Johnson and G. Schechtman [2] for a class of symmetric spaces (s. s.). Following [3] (P. 8) (see also [4], 2.f), for each s. s. X on $[0,1]$ we define an s. s. Z_X^2 as the set of all functions f measurable on the semiaxis $[0,\infty)$ such that $\|f\|_{Z_X^2} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_{L_2[1,\infty)} < \infty$, where f^* is a non-increasing rearrangement of $|f|$. In accordance with theorem 1 in [2], for any s. s. X on $[0,1]$ containing L_p with some $p<\infty$, there exists $C>0$ such that any sequence of independent functions $\{f_k\}_{k=1}^{\infty} \subset X$, $\int\limits_0^1 f_k(s)\,ds = 0$ $(k=1,2,\dots)$, satisfies the inequality

$$\Big\|\sum_{k=1}^{\infty} f_k\Big\|_X \leq C\Big\|\sum_{k=1}^{\infty} \overline{f_k}\Big\|_{Z_X^2} \tag{1}$$

(the opposite inequality is valid for each s. s. X).

Later in papers [5, 6] (see also the review [7]) one has generalized results obtained by W. B. Johnson and G. Schechtman for s. s. with the so-called Kruglov property introduced earlier by M. Sh. Braverman ([8], P. 11) on the base of some probabilistic constructions by V. M. Kruglov [9]. Let us mention two important facts. First, the latter property is less restrictive with respect to the condition $X \supset L_p$, where $p<\infty$. For example, it is intrinsic for the exponential Orlicz space $\mathrm{Exp}(L_q)$, if $0<q\leq 1$ (see [9] or [8], P. 42); at the same time, $\mathrm{Exp}(L_q) \not\supset L_p$, $(0<q,p<\infty)$. Second, the converse assertion is also true: If an arbitrary sequence $\{f_k\}_{k=1}^{\infty}$ of independent zero mean functions satisfies (1), then X has the Kruglov property ([7], theorem 22).

The goal of this paper is to prove Rosenthal type inequalities for a more general class of martingale differences. To this end we apply a technique for estimating distributions developed earlier when obtaining analogous martingale inequalities of the modular type [10–12] (see also § 11 in [13]). A central

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role here is played by the so-called predictable quadratic variation (or characteristic) of a martingale. Another approach based on the use (as in case of independent functions) of a disjunctive sum was demonstrated in the recently published paper [14], where one has obtained martingale inequalities analogous to (1). These inequalities are used there for comparing martingale differences and nonnegative functions with their independent copies.

2. DEFINITIONS, DENOTATIONS, AND PRELIMINARY INFORMATION

Let $J = [0,1]$ or $(0,\infty)$. Two measurable on J functions x and y are called *equimeasurable*, if $\lambda\{t \in J : |x(t)| > \tau\} = \lambda\{t \in J : |y(t)| > \tau\}$ $(\tau > 0)$, where λ is the Lebesgue measure. In particular, $|x|$ is equimeasurable with its non-increasing left continuous *rearrangement* $x^*(t) := \inf\{\tau \geq 0 : n_x(\tau) < t\}$ $(t \in J)$.

A Banach functional space X is called a *symmetric* space (s. s.), if

1) the fact that $|y(t)| \leq |x(t)|$ is an s. s. and $x \in X$ implies that $y \in X$ and $\|y\|_X \leq \|x\|_X$;

2) the fact that functions $x(t)$ and $y(t)$ are equimeasurable and $x \in X$ implies that $y \in X$ and $\|y\|_X = \|x\|_X$.

If X is an s. s. on J, then the *dual* space X' consists of all measurable y such that $\|y\|_{X'} := \sup\Big\{ \int\limits_J x(t)y(t)\,dt : \|x\|_X \leq 1\Big\} < \infty$. The space X' is also symmetric and isometrically embedded in the conjugate space X^*; moreover, $X' = X^*$ if and only if X is separable. Any s. s. X is continuously embedded in its second dual one X''. An s. s. X is called *maximal*, if the fact that $x_n \in X$ $(n = 1, 2, \dots)$, $\sup\limits_{n=1,2,\dots} \|x_n\|_X < \infty$, and $x_n \to x$ a. a. implies that $x \in X$ and $\|x\|_X \leq \liminf\limits_{n\to\infty} \|x_n\|_X$. This is equivalent to the fact that $X'' = X$. *Furthermore, following* [4] (P. 118), *we assume that all s. s. are separable or maximal.* Note that so are all important in applications s. s., and each such space is isometrically embedded in its second dual one.

In each s. s. X on $[0,1]$ the *expansion operator* $\sigma_\tau x(t) := x(t/\tau) \times \chi_{[0,\min(1,\tau)]}(t)$ $(\tau > 0)$ is continuous. In addition, $\|\sigma_\tau\|_{X\to X} \leq \max(1,\tau)$ ([15], theorem 2.4.5). Since the function $\|\sigma_\tau\|_{X\to X}$ is semi-multiplicative, we can determine the *lower and upper Boyd indices* of the space X:

$$\alpha_X = \lim_{\tau\to 0+} \frac{\ln \|\sigma_\tau\|_{X\to X}}{\ln \tau} \quad \text{and} \quad \beta_X = \lim_{\tau\to +\infty} \frac{\ln \|\sigma_\tau\|_{X\to X}}{\ln \tau}.$$

We always have $0 \leq \alpha_X \leq \beta_X \leq 1$ ([15], § 2.1, P. 138).

An s. s. X is called *p-convex* $(1 \leq p < \infty)$, if there exists a constant $C_X > 0$ such that $\Big\|\Big(\sum\limits_{k=1}^n |x_k|^p\Big)^{1/p}\Big\|_X \leq C_X\Big(\sum\limits_{k=1}^n \|x_k\|_X^p\Big)^{1/p}$ for arbitrary $x_1, x_2, \dots, x_n$ from X.

We also need the notion of s. s. of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Namely, if X is an s. s. on $[0,1]$, then the corresponding space $X(\Omega)$ consists of r. v. $f : \Omega \to \mathbb{R}$ such that $f^* \in X$ and $\|f\|_{X(\Omega)} := \|f^*\|_X$. In this case, $f^*(t) := \inf\{\tau \geq 0 : \mathbb{P}\{\omega \in \Omega : |f(\omega)| > \tau\} < t\}$ $(0 < t \leq 1)$. See [15] and [4] for details on s. s.

Everywhere in what follows, $\{\mathcal{A}_k\}_{k=1}^\infty$ is a flow of σ-algebras on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, i.e., $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_k \subset \cdots \subset \mathcal{A}$. For an $\mathcal{A}$-measurable r. v. f and a σ-subalgebra $\mathcal{B} \subset \mathcal{A}$, the symbol $\mathbb{E}(f|\mathcal{B})$ stands, as usual, for the *conditional mean value* of f with respect to $\mathcal{B}$. This means that $\mathbb{E}(f|\mathcal{B})$ is $\mathcal{B}$-measurable and for each $B \in \mathcal{B}$ we have $\int\limits_B \mathbb{E}(f|\mathcal{B})\,d\mathbb{P} = \int\limits_B f\,d\mathbb{P}$.

A sequence $\{f_k\}_{k=1}^\infty$ of $\mathcal{A}_k$-measurable r. v. is called a *martingale*, if $\mathbb{E}(f_k|\mathcal{A}_{k-1}) = f_{k-1}$ a. a. for all $k = 2, 3, \dots$. Then the r. v. $d_k := f_k - f_{k-1}$ $(k = 1, 2, \dots)$, where $f_0 = 0$, are called *martingale differences*. An important example of martingale differences is a sequence of independent r. v. $\{d_k\}_{k=1}^\infty \subset L_1(\Omega)$ such that $\int\limits_\Omega d_k\,d\mathbb{P} = 0$ $(k = 1, 2, \dots)$ with respect to the flow of σ-algebras $\mathcal{A}_k = \sigma(\{f_i\}_{i=1}^k)$ generated by collections $\{d_i\}_{i=1}^k$. Another example is the sequence of Haar functions $\{\chi_k\}_{k=1}^\infty$ (see, for example, [16], § 3.1). See [17, 18, 19] for properties of conditional mean values, martingales, and martingale differences.

If $\{f_k\}_{k=1}^{\infty}$ is a sequence of $\mathcal{A}$-measurable r. v., then $M(f) := \sup\limits_{k=1,2,\dots} |f_k|$, $d_k := f_k - f_{k-1}$ $(f_0 = 0)$, $S(f) := \Big(\sum\limits_{k=1}^{\infty} d_k^2\Big)^{1/2}$, and $s(f) := \Big(\sum\limits_{k=1}^{\infty} \mathbb{E}(d_k^2|\mathcal{A}_{k-1})\Big)^{1/2}$. In the case, when $\{f_k\}_{k=1}^{\infty}$ is a martingale, values $S(f)^2$ and $s(f)^2$ are called, respectively, its *quadratic* and *predictable quadratic variation*. Finally, if f and g are two r. v., then $f \vee g := \max(f, g)$.

3. MAIN RESULTS

Theorem 1. *Let X be an s. s. on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The following conditions are equivalent*:

(i) *There exists a constant $C > 0$, depending only on the space X, such that for an arbitrary martingale $\{f_k\}_{k=1}^{\infty}$ it holds*

$$\|M(f)\|_X \le C(\|s(f)\|_X + \|M(d)\|_X); \tag{2}$$

(ii) $\alpha_X > 0$.

Scheme of the proof of implication (ii) $\Rightarrow$ (i)**.** Any $\beta > 1$ and $0 < \delta < \beta - 1$ satisfy the bound

$$\mathbb{P}\{M(f) > \beta x,\ s(f) \vee M(d) \le \delta x\} \le \frac{\delta^2}{(\beta - \delta - 1)^2}\mathbb{P}\{M(f) > x\}\ \ (x > 0). \tag{3}$$

Another component of the proof is the next assertion; it allows one to compare r. v. in symmetric spaces with a positive lower Boyd index.

Proposition. *Let X be an s. s. on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\alpha_X > 0$. Assume that r. v. $g \ge 0$ and $h \ge 0$ are measurable on Ω, $h \in X$, and numbers $\beta > 1$, $\delta > 0$, $\varepsilon > 0$, and $\gamma \in (0, \alpha_X)$ are such that $\|\sigma_\varepsilon\|_{X\to X} \le \varepsilon^\gamma$, $\beta\varepsilon^\gamma < 1$, and*

$$\mathbb{P}\{g > \beta\tau,\ h \le \delta\tau\} \le \varepsilon\mathbb{P}\{g > \tau\}\ \ (\tau > 0).$$

Then $g \in X$ and

$$\|g\|_X \le \frac{\beta}{\delta(1 - \beta\varepsilon^\gamma)}\|h\|_X.$$

Taking into account the above Proposition and bound (3), we obtain (2). Really, let $\beta > 1$ and $\gamma \in (0, \alpha_X)$ be arbitrary. Choose $\delta \in (0, \beta - 1)$ so small as to make the number $\varepsilon := 2\delta^2/(\beta - \delta - 1)^2$ satisfy the conditions $\|\sigma_\varepsilon\|_{X\to X} \le \varepsilon^\gamma$ and $\beta\varepsilon^\gamma < 1$. Since inequality (3) is valid, in particular, for chosen β and δ, applying the mentioned Proposition, we obtain

$$\|M(f)\|_X \le \frac{\beta}{\delta(1 - \beta\varepsilon^\gamma)}\|s(f) \vee M(d)\|_X \le \frac{\beta}{\delta(1 - \beta\varepsilon^\gamma)}(\|s(f)\|_X + \|M(d)\|_X). \qquad \square$$

Theorem 1 immediately implies the Rosenthal inequality for sequences of independent zero mean r. v. in the case, when the lower Boyd index of the s. s. X is positive and $X \subset L_2[0,1]$ (see eqrefeq0). Recall that for each sequence $\{d_k\}_{k=1}^{\infty}$ of functions given on $[0,1]$ we set $\overline{d}_k(t) = d_k(t - k + 1)\chi_{[k-1,k)}(t)$ $(t > 0)$. In addition, let $\overline{d} := \sum\limits_{k=1}^{\infty} \overline{d}_k$.

Corollary 1. Let an s. s. X be such that $X \subset L_2[0,1]$ and $\alpha_X > 0$. Then there exists a constant $A > 0$ such that any sequence of independent r. v. $\{d_k\}_{k=1}^{\infty}$, $\mathbb{E}d_k = 0$ $(k = 1, 2, \dots)$, satisfies the condition

$$\Big\|\sum_{k=1}^{\infty} d_k\Big\|_X \le A\|\overline{d}\|_{Z_X^2}. \tag{4}$$

In particular, we obtain the classical Rosenthal inequality: For any $p \geq 2$ there exists a constant $A_p > 0$ such that

$$\mathbb{E}\Big|\sum_{k=1}^{\infty} d_k\Big|^p \leq A_p\Big(\Big(\sum_{k=1}^{\infty}\mathbb{E}d_k^2\Big)^{p/2} + \sum_{k=1}^{\infty}\mathbb{E}|d_k|^p\Big). \tag{5}$$

Proof. R. v. $f_n := \sum_{k=1}^{n} d_k$ $(n = 1, 2, \dots)$ form a martingale with respect to the sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$, where $\mathcal{A}_n = \sigma(\{d_k\}_{k=1}^{n})$. Since d_k^2 are independent with respect to the σ-algebra $\mathcal{A}_{k-1}$, we have $\mathbb{E}(d_k^2|\mathcal{A}_{k-1}) = \mathbb{E}d_k^2$. Consequently, since $X \subset L_2[0,1]$, we have

$$\|s(f)\|_X = \Big(\sum_{k=1}^{\infty}\mathbb{E}d_k^2\Big)^{1/2} = \|\overline{d}\|_{L_2(0,\infty)} \leq \|\overline{d}^*\chi_{[0,1]}\|_{L_2[0,1]} + \|\overline{d}^*\chi_{[1,\infty]}\|_{L_2[1,\infty)} \leq C\|\overline{d}\|_{Z_X^2}.$$

In addition, (see [20], proposition 1 or [2], lemma 3) we have

$$\frac{1}{2}\lambda\{\overline{d}^*\chi_{[0,1]} > \tau\} \leq \lambda\{M(d) > \tau\} \leq \lambda\{\overline{d}^*\chi_{[0,1]} > \tau\} \ \ (\tau > 0), \tag{6}$$

whence

$$\|M(d)\|_X \leq \|\overline{d}^*\chi_{[0,1]}\|_X \leq \|\overline{d}\|_{Z_X^2}.$$

Thus, applying Theorem 1 and taking into account that X is separable or maximal, we obtain inequality (4).

The proof of (5) is based on the property

$$\|M(d)\|_{L_p} \leq \Big\|\Big(\sum_{k=1}^{\infty}|d_k|^p\Big)^{1/p}\Big\|_{L_p} = \Big(\sum_{k=1}^{\infty}\mathbb{E}|d_k|^p\Big)^{1/p},$$

whence in view of (6) and corollary 2.4.2 in [15], we get

$$\|\overline{d}\|_{Z_{L_p}^2} \leq \|\overline{d}^*\chi_{[0,1]}\|_{L_p[0,1]} + \|\overline{d}\|_{L_2(0,\infty)} \leq 2\|M(d)\|_{L_p[0,1]} + \Big(\sum_{k=1}^{\infty}\mathbb{E}d_k^2\Big)^{1/2}$$
$$\leq 2\Big(\sum_{k=1}^{\infty}\mathbb{E}|d_k|^p\Big)^{1/p} + \Big(\sum_{k=1}^{\infty}\mathbb{E}d_k^2\Big)^{1/2}.$$

Therefore, inequality (5) immediately follows from (4). □

For s. s. that lie "to one side" of L_2 the inequality opposite to (2) is also valid.

Theorem 2. *Let X be a 2-convex s. s. on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\alpha_X > 0$. Then there exists a constant $C > 0$, depending only on the space X, such that an arbitrary martingale $\{f_k\}_{k=1}^{\infty}$ satisfies the inequality*

$$\|s(f)\|_X + \|M(d)\|_X \leq C\|M(f)\|_X. \tag{7}$$

Remark. The condition of the 2-convexity of the space in the latter theorem is essential. The next example was given by M. Kikuchi; we consider it here by his authority.

Let an s. s. $X \not\subset L_2$ and let x and y be two nonnegative r. v. with one and the same distribution and various supports such that $x, y \in X \setminus L_2$. For each $n \in \mathbb{N}$ we define a sequence $f^{(n)} = \{f_k^{(n)}\}_{k=0}^{\infty}$ as follows:

$$f_0^{(n)} = 0 \ \text{ and } \ f_k^{(n)} = (x \wedge n) - (y \wedge n), \ \text{ if } \ k \geq 1.$$

Then $f^{(n)}$ is a martingale with respect to the flow of σ-algebras $\{\mathcal{A}_k\}$: $\mathcal{A}_0 = \{\emptyset, \Omega\}$, $\mathcal{A}_k = \sigma(\{x, y\})$ for $k \geq 1$. Clearly,

$$M(f^{(n)}) = (x \wedge n) + (y \wedge n) \text{ and } s(f^{(n)}) = (\mathbb{E}[(x \wedge n)^2 + (y \wedge n)^2])^{1/2}.$$

Consequently, on one hand, $\|M(f^{(n)})\|_X \leq 2\|x\|_X < \infty$ ($n \in \mathbb{N}$), and on the other one, with $n \to \infty$ we obtain $\|s(f^{(n)})\|_X \to \infty$.

Nevertheless, for some class of martingales inequality (7) appears to be valid for all s. s. with a positive lower Boyd index.

Theorem 3. *Let X be an s. s. on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\alpha_X > 0$. Assume that a martingale $\{f_k\}_{k=1}^{\infty}$ satisfies the condition: Some $\mathcal{A}_{k-1}$-measurable r. v. w_k is such that $|d_k| \leq w_k$ $(k = 1, 2, \dots)$, while $M(w) := \sup\limits_{k=1,2,\dots} w_k \in X$. Then there exists a constant $C > 0$, depending only on the space X, such that*

$$\|s(f)\|_X \leq C(\|M(f)\|_X + \|M(w)\|_X). \tag{8}$$

Corollary 2. Let χ_k be Haar functions normed in L_∞, $c_k \in \mathbb{R}$ $(k = 0, 1, \dots)$, and let $h = \{h_n\}_{n=0}^{\infty}$ be a martingale with respect to the flow of σ-algebras $\mathcal{A}_n = \sigma(\chi_0, \chi_1, \dots, \chi_n)$ formed by the sequence of Haar sums $h_n = \sum\limits_{k=0}^{n} c_k \chi_k$. Then for any s. s. X such that $\alpha_X > 0$ there exists a constant $C > 0$, depending only on X, such that $\|s(h)\|_X \leq C\|M(h)\|_X$.

Proof. Note that the corresponding martingale differences $d_k := h_k - h_{k-1}$ $(h_{-1} = 0)$ satisfy the inequalities $|d_k| = |c_k \chi_k| \leq |c_k| \chi_{\Delta_k}$, where Δ_k is a binary interval representing a support of the Haar function χ_k $(k = 0, 1, \dots)$. Since χ_{Δ_k} is an $\mathcal{A}_{k-1}$-measurable r. v. and $\sup\limits_{k=0,1,\dots} |c_k| \chi_{\Delta_k} \leq 2M(h)$, the desired inequality follows from the previous theorem. $\square$

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