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MATHEMATICS

Haar Series and Spaces Determined by the Paley Function

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1. A Banach space E of measurable functions on $[0, 1]$ is said to be symmetric, or rearrangement-invariant, if

(i) $|x(t)| \le |y(t)|$ for all $t \in [0, 1]$ and $y \in E$ imply $x \in E$ and $\|x\|_E \le \|y\|_E$;

(ii) the equimeasurability of the functions x and y, i.e., the equality

$$m\{t \in [0,1]: |y(t)| > u\} = m\{t \in [0,1]: |x(t)| > u\} \quad (u > 0),$$

where m is the Lebesgue measure, and $y \in E$, implies $x \in E$ and $\|x\|_E = \|y\|_E$.

Following [1], we assume that E is separable or dual to a separable space. We also assume that the normalization condition $\|\kappa_{[0,1]}\|_E = 1$ holds, where $\kappa_e(t)$ is the characteristic function of a measurable set $e \subset [0, 1]$. Then $L_\infty \subset E \subset L_1$, and $\|x\|_{L_1} \le \|x\|_E \le \|x\|_{L_\infty}$ for any $x \in L_\infty$. On any symmetric space E, the dilation operator

$$\sigma_\tau x(t) = \begin{cases} x\left(\dfrac{t}{\tau}\right), & 0 \le t \le \min(1, \tau), \\ 0 & \text{for the other } t \in [0,1] \end{cases}$$

boundedly acts; we have $\|\sigma_\tau\|_{E \to E} \le \max(\tau, 1)$ for all $\tau > 0$. The numbers

$$\alpha_E = \lim_{\tau \to 0} \frac{\ln\|\sigma_\tau\|_E}{\ln\tau}, \quad \beta_E = \lim_{\tau \to \infty} \frac{\ln\|\sigma_\tau\|_E}{\ln\tau}$$

are called the Boyd indices of the symmetric space E. We always have $0 \le \alpha_E \le \beta_E \le 1$. For example, $\alpha_{L_p} = \beta_{L_p} = \dfrac{1}{p}$ for all $p \in [1, \infty]$. An important class of symmetric spaces is formed by the Orlicz spaces. Let $M(u)$ be a strictly increasing convex function on $[0, \infty)$ with $M(0) = 0$. Then the Orlicz space L_M consists of all $x \in L_1$ such that

$$\int_0^1 M\left(\frac{|x(t)|}{\lambda}\right) dt \le 1$$

for sufficiently large $\lambda > 0$. For a norm on L_M the functional $\|x\|_{L_M} = \inf\lambda$ is taken, where the infimum is taken over all $\lambda > 0$ for which the above inequality holds. The Orlicz space generated by the function $M(u) = e^{u^2} - 1$ is denoted by $\exp L_2$.

The functions $\chi_0^0(t) = 1$ and

$$\chi_n^k(t) = \begin{cases} 2^{n/2}, & t \in \Delta_{n+1}^{2k-1}, \\ -2^{n/2}, & t \in \Delta_{n+1}^{2k}, \\ 0 & \text{for the other } t \in [0,1], \end{cases}$$

where $\Delta_n^k = ((k-1)2^{-n}, k2^{-n})$ for $1 \le k \le 2^n$ and $n = 0, 1, \ldots$ form the Haar system. The intervals Δ_n^k, with $1 \le k \le 2^n$ and $n = 0, 1, \ldots$, are said to be dyadic. Let Ω denote the set of indices of the Haar functions, i.e., let $\Omega := \{(0, 0), (n, k), 1 \le k \le 2^n, n = 0, 1, \ldots\}$. The formula $m = 2^n + k$ establishes a one-to-one correspondence between Ω and the set of positive integers $\mathbb{N}$, which allows us to write this system by using only one index as $\{\chi_m\}_{m \in \mathbb{N}}$. The Haar functions form a complete orthonormal system, which is a monotone basis in any separable symmetric space. The Haar system is an unconditional basis in a separable symmetric space E if and only if $0 < \alpha_E \le \beta_E < 1$. In this case, the norm $\|x\|_E$ is equivalent to $\|Px\|_E$, where

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$$x(t) = \sum_{m=1}^{\infty} c_m \chi_m(t), \quad Px(t) = \left(\sum_{m=1}^{\infty} (c_m \chi_m(t))^2\right)^{1/2}.$$

The function $Px(t)$ is called the Paley function. For more details on symmetric spaces, see books [1, 2], and on Haar and Rademacher systems (see below), books [3–6].

Given a symmetric space E, by $P(E)$ we denote the function space with norm

$$\|x\|_{P(E)} := \|Px\|_E.$$

This paper is devoted to the study of spaces $P(E)$ and, in particular, $P(L_\infty)$.

2. **Theorem 1.** *Let E be a symmetric space. Then the space $P(E)$ is symmetric if and only if $E = L_2$ and $\|x\|_E = \|x\|_{L_2}$ for all $x \in L_2$.*

Certainly, sufficiency follows at once from the Parseval equality. The answer to the same question in isomorphic setting is quite different.

Theorem 2. *Let E be a symmetric space. Then $P(E)$ is a symmetric space up to the equivalence of norms if and only if $0 < \alpha_E \le \beta_E < 1$.*

The sufficiency part of this theorem is a direct consequence of the criterion mentioned above for the Haar system in a symmetric space to be unconditional (see [1, 2.c.6] or [2, 2.9.6]).

The sequence of functions

$$r_n(t) = \operatorname{sgn} \sin 2^n \pi t, \quad n \in \mathbb{N}$$

is called the Rademacher system. If $a = (a_1, a_2, \ldots) \in l_2$, then, according to Rademacher's theorem, the series

$$\sum_{n=1}^{\infty} a_n r_n(t) \tag{1}$$

converges almost everywhere. We denote the set of functions of the form (1) by Rad. Since

$$r_{n+1}(t) = 2^{-\frac{n}{2}} \sum_{k=1}^{2^n} \chi_n^k(t), \quad n = 0, 1, 2, \ldots,$$

it follows that

$$P\left(\sum_{n=1}^{\infty} a_n r_n\right)(t) = \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}.$$

Therefore, $\mathrm{Rad} \subset P(E)$, and

$$\left\|\sum_{n=1}^{\infty} a_n r_n\right\|_{P(E)} = \|a\|_{l_2}$$

for any symmetric space E. Thus, together with Rad, the space $P(E)$ contains unbounded functions.

Theorem 3. *The subspace* Rad *is* 1*-complemented in $P(E)$ for any symmetric space E.*

Recall that the space of functions of bounded oscillation $\mathrm{BMO} = \mathrm{BMO}[0, 1]$ consists of all $x \in L_1[0, 1]$ such that

$$\|x\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|} \int_I |x(t) - x_I| dt < \infty, \tag{2}$$

where the supremum is taken over all nondegenerate intervals $I \subset [0, 1]$ and $x_I = \frac{1}{|I|}\int_I x(s)\, ds$. If the supremum in definition (2) is taken over only the set of dyadic intervals, then we obtain the dyadic version of this space, which is denoted by BMO_d. Obviously, $\mathrm{BMO} \subset \mathrm{BMO}_d$ and $\|x\|_{\mathrm{BMO}_d} \le \|x\|_{\mathrm{BMO}}$ for all $x \in \mathrm{BMO}$.

Before stating the main theorems about the space $P(L_\infty)$, note that, by virtue of Theorem 2, the norm of $P(L_\infty)$ is not equivalent to the norm of any symmetric space. It is also easy to show that $P(L_\infty)$ is not separable. Moreover, $P(L_\infty) \subset \mathrm{BMO}_d$ and $\|x\|_{\mathrm{BMO}_d} \le \|x\|_{P(L_\infty)}$ for all $x \in P(L_\infty)$.

By virtue of [5, Corollary 1.3.2], the following estimate for the distributions of functions from the space $P(L_\infty)$ holds: if $\int_0^1 f ds = 0$, then

$$m\{s \in [0, 1] : |f(s)| > t\} \le 2\exp\left(-\frac{t^2}{2\|f\|^2_{P(L_\infty)}}\right) \quad (t > 0).$$

It follows that $P(L_\infty) \subset \exp L_2$ and $\|x\|_{\exp L_2} \le C\|x\|_{P(L_\infty)}$ for all $x \in P(L_\infty)$ and some constant $C > 0$. As is known [2, 2.5.3], the space $\exp L_2$ has the following extremal property: a function $x \in L_1$ belongs to $\exp L_2$ if and only if

$$\sup_{0 < t \le 1} x^*(t) \ln^{-1/2}\frac{e}{t} < \infty,$$

where $x^*(t)$ is the nonincreasing rearrangement of the function $|x(t)|$. The following theorem indicates a close relation between the spaces $P(L_\infty)$ and $\exp L_2$.

Theorem 4. *For any function $x \in \exp L_2$, there exists a function $f \in P(L_\infty)$ such that*

$$\frac{1}{C_1}\|x\|_{\exp L_2} \le \|f\|_{P(L_\infty)} \le C_1 \|x\|_{\exp L_2},$$

$$x^*(t) \le C_2(f^*(t) + \|f\|_{P(L_\infty)}),$$

where constants C_1 and C_2 do not depend on x and f.

In particular, there exists a function $f_0 \in P(L_\infty)$ with $\|f_0\|_{P(L_\infty)} = 1$ such that $f_0^*(t) \le \ln^{1/2}\frac{e}{t}$ for all $t \in (0, 1]$ and $f_0^*(t) \ge \frac{1}{4}\ln^{1/2}\frac{e}{t}$ for all $t \in (0, t_0]$ and some $t_0 \in (0, 1)$.

3. Let (X_0, X_1) be a Banach couple; this means that X_0 and X_1 are embedded in some Hausdorff linear topological space and $x \in X_0 + X_1$, $t > 0$. The functional

$$K(t, x, X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}: \\ x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

is called the Peetre K-functional of the couple (X_0, X_1). Let Y_0 and Y_1 be closed subspaces of X_0 and X_1, respectively. A pair (Y_0, Y_1) is called a K-closed subcouple of the couple (X_0, X_1) if the functionals $K(t, x, X_0, X_1)$and $K(t, x, Y_0, Y_1)$ are equivalent with constants not depending on $t > 0$ and $x \in Y_0 + Y_1$. Given a Banach couple (X_0, X_1) and a Banach lattice F on $(0, \infty)$ with measure $\frac{dt}{t}$, we can construct the space $(X_0, X_1)_F^K$ with the norm

$$\|x\| := \|K(t, x, X_0, X_1)\|_F,$$

which is an interpolation space with respect to the Banach couple (X_0, X_1) [1, 2.g.4].

Consider the operator

$$\mathrm{Ra}(t) = \sum_{n=1}^{\infty} a_n r_n(t), \quad a = (a_n)_{n=1}^{\infty} \in l_2,$$

where r_n are the Rademacher functions. The operator R acts isometrically from l_1 to L_∞ and from l_2 to $P(L_\infty)$.

Theorem 5. *The Banach couple* $(R(l_1), R(l_2))$ *is a K-closed subcouple of the couple* $(L_\infty, P(L_\infty))$.

Using this theorem, we can prove the following assertion.

Theorem 6. *Let F be a Banach lattice, and let* $E = (l_1, l_2)_F^K$. *If* $X = (L_\infty, P(L_\infty))_F^K$, *then the norms* $\left\|\sum_{n=1}^{\infty} a_n r_n\right\|_X$ *and* $\|a\|_E$ *are equivalent.*

Theorem 6 makes it possible, given an interpolation space E in the couple (l_1, l_2), to find a space X being interpolation with respect to the couple $(L_\infty, P(L_\infty))$ and such that the norm induced by X on Rad is equivalent to the norm of E.

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REFERENCES

1. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Vol. 2: *Function Spaces* (Springer, Heidelberg, 1979).
2. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators* (Nauka, Moscow, 1978; Am. Math. Soc., Providence, RI, 1982).
3. B. S. Kashin and A. A. Saakyan, *Orthogonal Series* (Am. Math. Soc., Providence, RI, 1989; AFTs, Moscow, 1999).
4. I. Novikov and E. Semenov, *Haar Series and Linear Operators* (Kluwer, Dordrecht, 1996).
5. P. F. Müller, *Isomorphisms between H^1 Spaces* (Birkhäuser, Basel, 2005).
6. S. V. Astashkin, J. Math. Sci. **169**, 725–886 (2010).