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On the geometric properties of Cesàro spaces

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Abstract. It is proved that the Cesàro space $\mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, contains a complemented subspace isomorphic to l^q if and only if either $q=1$ or $q=p$. A class of subspaces of this space that contain complemented copies of the space l^p is distinguished.

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§ 1. Introduction and preliminaries

As is well known, the norms of many of the Banach spaces which play an important role in functional analysis and its applications are generated by L_p-norms and by positive sublinear operators. The Cesàro spaces are interesting examples of spaces of this kind (for other examples, see [1], and also the concluding part of this paper). These spaces became widely known at the end of the 1960s in connection with the problem of finding their duals; this problem was posed by the Dutch Mathematical Society [2]. In what follows, we consider the Cesàro spaces $\mathrm{Ces}_p[0,1]$ which consist of all measurable functions f on $[0,1]$ that satisfy the following conditions:

$$\|f\|_{C(p)} = \biggl[\int_0^1 \biggl(\frac{1}{t}\int_0^t |f(s)|\,ds\biggr)^p dt\biggr]^{1/p} < \infty, \qquad 1 \leqslant p < \infty, \tag{1.1}$$

$$\|f\|_{C(\infty)} = \sup_{0<t\leqslant 1} \frac{1}{t}\int_0^t |f(s)|\,ds < \infty. \tag{1.2}$$

We note that the space $\mathrm{Ces}_\infty[0,1]$, which is also known as the Korenblyum-Kreĭn-Levin space K, first arose in 1948 in connection with problems in the theory of singular integrals (see [3] and [4], Chs. 0 and 1).

One of the most important characteristics of the geometric structure of a Banach space is the existence of (complemented) l^q-copies, that is, of (complemented) subspaces isomorphic to the space l^q, $1 \leqslant q \leqslant \infty$, in the space in question (see, for example, [5], Chs. 6, 10 and 11). In the case of L_p-spaces, the following results hold, which show the difference between the geometric properties of the spaces for $1<p<2$ and $2<p<\infty$ (see [5], Theorem 6.4.19 and Theorem 6.4.21 and also Kadets-Pełczyński's classical paper [6]).

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Theorem A. *Let* $1 \leqslant q \leqslant \infty$.

(a) *If* $1 \leqslant p \leqslant 2$, *then the space* l^q *can be embedded isomorphically in* $L_p[0,1]$ *if and only if* $p \leqslant q \leqslant 2$.

(b) *If* $2 < p < \infty$, *then the space* l^q *can be embedded isomorphically in* $L_p[0,1]$ *if and only if* $q = p$ *or* $q = 2$.

(c) *If* $1 < p < \infty$, *then* $L_p[0,1]$ *contains a complemented subspace isomorphic to* l^q *if and only if* $q = p$ *or* $q = 2$.

A similar description of the set of all q for which isomorphic copies of l^q are contained in the Cesàro space $\mathrm{Ces}_p[0,1]$ was given in [7], Theorem 10.

Theorem B. (a) *If* $1 \leqslant p \leqslant 2$, *then the space* l^q *can be embedded isomorphically in* $\mathrm{Ces}_p[0,1]$ *if and only if* $q \in [1,2]$.

(b) *If* $2 < p < \infty$, *then the space* l^q *can be embedded isomorphically in* $\mathrm{Ces}_p[0,1]$ *if and only if* $q \in [1,2]$ *or* $q = p$.

The main objective of this paper is to determine the set of all q, $1 \leqslant q \leqslant \infty$, for which a Cesàro space contains complemented l^q-copies. It turns out (see Theorem 2) that the space $\mathrm{Ces}_p[0,1]$, $1 < p < \infty$, is in a sense a 'combination' of the spaces L_1 and L_p, because it contains only complemented l^1- and l^p-copies (examples of these were presented earlier in [8], Theorem 1 and [7], Theorem 6, respectively). A similar result is also obtained for the l^q-copies of a Cesàro space that are spanned by sequences of disjoint functions (Theorem 1 and Corollary 1). In § 3, a theorem of Kadets-Pełczyński type [6] on the geometric structure of the space $\mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, is also obtained, namely Theorem 3, which shows that every 'nontrivial' subspace of $\mathrm{Ces}_p[0,1]$, $p < \infty$, contains a complemented l^p-copy. The proofs of these statements rely heavily on the results of § 2 (which are also of independent interest, in our opinion). These state that the geometric properties of a Cesàro space 'near' zero and one are similar to those of the space $L_p[0,1]$. In the concluding section, § 4, we give a general construction of spaces defined by sublinear operators.

We first recall the definitions we need below. Let $L^0 = L^0[0,1]$ be the set of all real-valued functions that are finite almost everywhere and Lebesgue measurable on $[0,1]$ (to be more precise, L^0 is the set of equivalence classes of these functions). Then by a *Banach lattice* (a *Banach ideal space*) we mean a Banach space $X \subset L^0$ with the following property: if $|f| \leqslant |g|$ almost everywhere on $[0,1]$ and $g \in X$, then $f \in X$ and $\|f\| \leqslant \|g\|$. For details concerning properties of Banach lattices, see the monographs [9]–[11].

We recall that the *lattice associated* to a Banach lattice X on $[0,1]$ is the set X' of all functions $f \in L^0$ such that

$$\|f\|_{X'} := \sup_{g \in X,\, \|g\|_X \leqslant 1} \int_0^1 |f(t)g(t)|\,dt < \infty.$$

This is also a Banach lattice on $[0,1]$. Moreover, $X \subset X''$ and $\|f\|_{X''} \leqslant \|f\|_X$ for any $f \in X$, and we have $X = X''$ and $\|f\|_{X''} = \|f\|_X$ if and only if the space X is *maximal*, that is, if $0 \leqslant f_n \nearrow f$ almost everywhere on $[0,1]$ and $\sup_{n \in \mathbf{N}} \|f_n\| < \infty$, then $f \in X$ and $\|f_n\| \nearrow \|f\|$. The embedding $X' \subset X^*$ holds, where X^* stands for the space dual to X. Moreover, $X' = X^*$ if and only if X is separable.

The following Hölder-type inequality holds for an arbitrary Banach lattice X on $[0,1]$ and the associated lattice X': if $f \in X$ and $g \in X'$, then fg is integrable and

$$\int_0^1 |f(t)g(t)|\,dt \leqslant \|f\|_X\,\|g\|_{X'}.$$

The Cesàro space $\mathrm{Ces}_p[0,1]$, $1 \leqslant p \leqslant \infty$, with the norm defined in (1.1) for any finite p and in (1.2) for $p=\infty$ is obviously a maximal Banach lattice on $[0,1]$. It can readily be shown that $\mathrm{Ces}_1[0,1]$ coincides isometrically with the weighted space $L^1(\ln(1/t))$ (see [7], Theorem 1). We also note that the space $\mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, is separable, whilst $\mathrm{Ces}_\infty[0,1]$ is not. As was proved in [7], Theorem 4, for any $1 \leqslant p < \infty$ the Cesàro space is p-concave with constant 1, that is,

$$\biggl(\sum_{k=1}^n \|f_k\|_{C(p)}^p\biggr)^{1/p} \leqslant \biggl\|\biggl(\sum_{k=1}^n |f_k|^p\biggr)^{1/p}\biggr\|_{C(p)}$$

for any $f_1, f_2, \dots, f_n \in \mathrm{Ces}_p[0,1]$.

For an arbitrary measurable function f on $[0,1]$ we write

$$\widetilde{f}(t) = \operatorname*{ess\,sup}_{s\in[t,1]} |f(s)|.$$

Then $(\mathrm{Ces}_\infty[0,1])' = \widetilde{L}_1$ and $\|f\|_{C(\infty)'} \asymp \|f\|_{\widetilde{L}_1}$, where $\|f\|_{\widetilde{L}_1} = \|\widetilde{f}\|_{L_1}$, as was proved in [12]. In the case $1 < p < \infty$, the dual (and thus also the associated) space to $\mathrm{Ces}_p[0,1]$ was described in [7], Theorem 3,

$$(\mathrm{Ces}_p[0,1])^* = (\mathrm{Ces}_p[0,1])' = U(q), \qquad \|f\|_{C(p)'} \asymp \|f\|_{U(q)}, \quad \frac{1}{p}+\frac{1}{q}=1,$$

where

$$\|f\|_{U(q)} = \biggl\|\frac{\widetilde{f}(t)}{1-t}\biggr\|_{L_q} = \biggl[\int_0^1 \biggl(\frac{\widetilde{f}(t)}{1-t}\biggr)^q dt\biggr]^{1/q}. \tag{1.3}$$

Below we use the following notation. Let $I=[a,b]$ and $J=[c,d]$ be two closed intervals on the real line. Then the inequality $I<J$ means that $b \leqslant c$. If Banach spaces X and Y are isomorphic, we write $X \simeq Y$. A closed linear subspace of a Banach space is referred to below simply as a subspace. We denote the closed linear span in X of a sequence $\{x_n\}_{n=1}^\infty$ in X by $[x_n]$ and the support of a function f, that is, the set $\{t \in [0,1] : f(t) \neq 0\}$ by $\operatorname{supp} f$. Finally, the expression $F \asymp G$ means that $cF \leqslant G \leqslant CF$ for some constants $c>0$ and $C>0$ independent of all or some arguments of the functions (quasi-norms) F and G.

§ 2. Complementable l^p-copies in the space $\mathrm{Ces}_p[0,1]$ 'near' zero and one

In what follows, the following statement showing that, 'near' zero, the geometric properties of the Cesàro space $\mathrm{Ces}_p[0,1]$ are similar to properties of the space $L_p[0,1]$ plays an important role.

Proposition 1. *Let $1 < p \leqslant \infty$. Suppose that a sequence $\{x_n\} \subset \mathrm{Ces}_p[0,1]$ satisfies the condition $\operatorname{supp} x_n \subset I_n = [a_n, b_n]$, $n = 1, 2, \dots$, $I_1 > I_2 > \cdots$, and $b_n \to 0^+$. Then there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that the inequalities*

$$\biggl(\sum_{k=1}^{\infty} \|x_{n_k}\|_{C(p)}^p\biggr)^{1/p} \leqslant \biggl\|\sum_{k=1}^{\infty} x_{n_k}\biggr\|_{C(p)} \leqslant C\biggl(\sum_{k=1}^{\infty} \|x_{n_k}\|_{C(p)}^p\biggr)^{1/p}, \qquad 1 < p < \infty, \tag{2.1}$$

$$\sup_{k=1,2,\dots} \|x_{n_k}\|_{C(\infty)} \leqslant \biggl\|\sum_{k=1}^{\infty} x_{n_k}\biggr\|_{C(\infty)} \leqslant C \sup_{k=1,2,\dots} \|x_{n_k}\|_{C(\infty)}, \tag{2.2}$$

hold for some constant $C > 0$ depending on p only.

Moreover, the subspace $[x_{n_k}]$ is complemented in the space $\mathrm{Ces}_p[0,1]$.

Proof. Since the space $\mathrm{Ces}_p[0,1]$, $1 \leqslant p \leqslant \infty$, is p-concave with constant 1 (see § 1 or [7], Theorem 4), it follows that the left-hand inequalities in (2.1) and (2.2) hold for an arbitrary sequence of disjoint functions. We claim that the reverse inequality holds for some subsequence (and with some constant).

Obviously, we may assume that $x_n \geqslant 0$, $x_n \neq 0$ and $b_1 \leqslant 1/2$. We first consider the case $1 < p < \infty$.

It can readily be seen that from any given sequence of closed intervals $I_n = [a_n, b_n]$, $n = 1, 2, \dots$, one can extract a subsequence (the closed intervals in the subsequence will again be denoted by $I_n = [a_n, b_n]$) for which

$$\sum_{k=n+1}^{\infty} b_k \leqslant 2^{-n/(p-1)} a_n, \qquad n = 1, 2, \dots . \tag{2.3}$$

We claim that the corresponding subsequence of functions (which we still denote by $\{x_n\}$, $\operatorname{supp} x_n \subset I_n$) satisfies the right-hand inequality in (2.1).

For any $n \in \mathbb{N}$ and $t \in (0,1]$

$$\begin{aligned}
\frac{1}{t}\int_0^t \biggl|\sum_{k=1}^{n} x_k(s)\biggr|\,ds &= \frac{1}{t}\sum_{j=1}^{n}\sum_{i=n-j+2}^{n}\int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[a_{n-j+1},b_{n-j+1}]}(t) \\
&\quad + \frac{1}{t}\sum_{j=1}^{n}\int_{a_{n-j+1}}^{t} x_{n-j+1}(s)\,ds\,\chi_{[a_{n-j+1},b_{n-j+1}]}(t) \\
&\quad + \frac{1}{t}\sum_{j=1}^{n}\sum_{i=n-j+1}^{n}\int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[b_{n-j+1},a_{n-j}]}(t) \\
&= T_1(t) + T_2(t) + T_3(t) + T_4(t),
\end{aligned}$$

where $a_0 = 1$ and

$$T_1(t) = \frac{1}{t}\sum_{j=2}^{n}\sum_{i=n-j+2}^{n}\int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[a_{n-j+1},b_{n-j+1}]}(t),$$

$$T_2(t) = \frac{1}{t}\sum_{j=2}^{n}\sum_{i=n-j+2}^{n}\int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[b_{n-j+1},a_{n-j}]}(t),$$

$$T_3(t) = \frac{1}{t}\sum_{j=1}^{n}\int_{a_{n-j+1}}^{b_{n-j+1}} x_{n-j+1}(s)\,ds\,\chi_{[b_{n-j+1},a_{n-j}]}(t),$$

$$T_4(t) = \frac{1}{t}\sum_{j=1}^{n}\int_{a_{n-j+1}}^{t} x_{n-j+1}(s)\,ds\,\chi_{[a_{n-j+1},b_{n-j+1}]}(t).$$

Since $\|\chi_{[a,b]}\|_{C(p)'} \asymp \|\chi_{[a,b]}\|_{U(q)}$ (see § 1), it follows from (1.3) that

$$\int_{a_i}^{b_i} x_i(s)\,ds \leqslant C_1\|x_i\|_{C(p)}\cdot\|\chi_{[a_i,b_i]}\|_{U(q)} = C_1\|x_i\|_{C(p)}\left[\frac{1-(1-b_i)^{q-1}}{(1-b_i)^{q-1}}\right]^{1/q}$$
$$\leqslant C_1'(1-(1-b_i)^{q-1})^{1/q}$$

for any $i \in \mathbb{N}$. This, together with the elementary inequality

$$1-(1-t)^{q-1} \leqslant 2(q-1)t,$$

which holds for any $1 < q < \infty$ and $0 < t \leqslant 1/2$, implies that

$$\int_{a_i}^{b_i} x_i(s)\,ds \leqslant C_2\|x_i\|_{C(p)} b_i^{1/q}, \qquad i \in \mathbb{N},$$

where $C_2 > 0$ depends only on p. Thus,

$$\begin{aligned}
\|T_1\|_p^p &= \sum_{j=2}^{n}\biggl(\sum_{i=n-j+2}^{n}\int_{a_i}^{b_i} x_i(s)\,ds\biggr)^p\int_{a_{n-j+1}}^{b_{n-j+1}}\frac{dt}{t^p}\\
&\leqslant C_2^p\sum_{j=2}^{n}\biggl(\sum_{i=n-j+2}^{n}\|x_i\|_{C(p)} b_i^{1/q}\biggr)^p\frac{b_{n-j+1}^{p-1}-a_{n-j+1}^{p-1}}{(p-1)a_{n-j+1}^{p-1}b_{n-j+1}^{p-1}}\\
&\leqslant C_3^p\sum_{j=2}^{n}\sum_{i=n-j+2}^{n}\|x_i\|_{C(p)}^p\biggl(\sum_{i=n-j+2}^{n} b_i\biggr)^{p-1}a_{n-j+1}^{1-p}\\
&\leqslant C_3^p\sum_{j=2}^{n}\sum_{i=n-j+2}^{n}\|x_i\|_{C(p)}^p 2^{-n+j-1} = C_3^p\sum_{i=2}^{n}\|x_i\|_{C(p)}^p\sum_{j=n-i+2}^{n}2^{-n+j-1}\\
&\leqslant C_3^p\sum_{i=2}^{n}\|x_i\|_{C(p)}^p
\end{aligned}$$

by Hölder's inequality and by the relation (2.3), where $C_3 > 0$ depends only on p.

We can estimate $\|T_2\|_p^p$ similarly,

$$\begin{aligned}
\|T_2\|_p^p &\leqslant C_2^p \sum_{j=2}^n \biggl(\sum_{i=n-j+2}^n \|x_i\|_{C(p)} b_i^{1/q}\biggr)^p \frac{a_{n-j}^{p-1} - b_{n-j+1}^{p-1}}{(p-1)a_{n-j}^{p-1} b_{n-j+1}^{p-1}} \\
&\leqslant C_4^p \sum_{j=2}^n \sum_{i=n-j+2}^n \|x_i\|_{C(p)}^p \biggl(\sum_{i=n-j+2}^n b_i\biggr)^{p-1} b_{n-j+1}^{1-p} \\
&\leqslant C_4^p \sum_{j=2}^n \sum_{i=n-j+2}^n \|x_i\|_{C(p)}^p 2^{-n+j-1} \\
&= C_4^p \sum_{i=2}^n \|x_i\|_{C(p)}^p \sum_{j=n-i+2}^n 2^{-n+j-1} \leqslant C_4^p \sum_{i=2}^n \|x_i\|_{C(p)}^p,
\end{aligned}$$

$$\|T_3\|_p^p \leqslant C_4^p \sum_{j=1}^n \|x_{n-j+1}\|_{C(p)}^p b_{n-j+1}^{p-1} b_{n-j+1}^{1-p} = C_4^p \sum_{i=1}^n \|x_i\|_{C(p)}^p,$$

where $C_4 > 0$ depends only on p. Finally,

$$\begin{aligned}
\|T_4\|_p^p &\leqslant \sum_{j=1}^n \int_{a_{n-j+1}}^{b_{n-j+1}} \biggl(\frac{1}{t}\int_{a_{n-j+1}}^t x_{n-j+1}(s)\,ds\biggr)^p dt \\
&\leqslant \sum_{j=1}^n \int_0^1 \biggl(\frac{1}{t}\int_0^t x_{n-j+1}(s)\,ds\biggr)^p dt = \sum_{j=1}^n \|x_{n-j+1}\|_{C(p)}^p = \sum_{i=1}^n \|x_i\|_{C(p)}^p
\end{aligned}$$

by the definition of the norm in the Cesàro space.

As a result, the above relations yield

$$\biggl\|\sum_{k=1}^n x_k\biggr\|_{C(p)} \leqslant \sum_{k=1}^4 \|T_k\|_p \leqslant C\biggl(\sum_{i=1}^n \|x_i\|_{C(p)}^p\biggr)^{1/p},$$

where $C > 0$ depends only on p. Since the space $\mathrm{Ces}_p[0,1]$ is maximal, it follows that a similar inequality holds for infinite sums too. The inequality (2.1) is thus proved.

If $p = \infty$, then the proof is more or less the same; only minor modifications are needed. In particular, we replace the condition (2.3) by the condition

$$\sum_{k=n+1}^\infty b_k \leqslant a_n, \qquad n = 1, 2, \dots . \tag{2.4}$$

Since $\|\chi_{[a,b]}\|_{C(\infty)'} \asymp \|\chi_{[a,b]}\|_{\widetilde{L}_1}$ in this case (see § 1), it follows that

$$\int_{a_i}^{b_i} x_i(s)\,ds \leqslant C' \|x_i\|_{C(\infty)} \cdot \|\chi_{[a_i,b_i]}\|_{\widetilde{L}_1} = C' \|x_i\|_{C(\infty)} b_i$$

for any $i \in \mathbb{N}$, where $C' > 0$ depends only on p. Hence, applying (2.4), we obtain

$$\begin{aligned}
\|T_1\|_\infty &= \max_{j=2,\dots,n}\biggl(\sum_{i=n-j+2}^{n}\int_{a_i}^{b_i} x_i(s)\,ds\frac{1}{a_{n-j+1}}\biggr)\\
&\leqslant C'\max_{j=2,\dots,n}\biggl(\sum_{i=n-j+2}^{n}\|x_i\|_{C(\infty)}\frac{b_i}{a_{n-j+1}}\biggr)\\
&\leqslant C'\max_{j=2,\dots,n}\|x_j\|_{C(\infty)}\sum_{i=n-j+2}^{n}\frac{b_i}{a_{n-j+1}}\leqslant C'\max_{j=2,\dots,n}\|x_j\|_{C(\infty)},\\
\|T_2\|_\infty &= \max_{j=2,\dots,n}\biggl(\sum_{i=n-j+2}^{n}\int_{a_i}^{b_i} x_i(s)\,ds\frac{1}{b_{n-j+1}}\biggr)\\
&\leqslant C'\max_{j=2,\dots,n}\biggl(\sum_{i=n-j+2}^{n}\|x_i\|_{C(\infty)}\frac{b_i}{b_{n-j+1}}\biggr)\leqslant C'\max_{j=2,\dots,n}\|x_j\|_{C(\infty)},\\
\|T_3\|_\infty &= \max_{j=1,2,\dots,n}\int_{a_{n-j+1}}^{b_{n-j+1}} x_{n-j+1}(s)\,ds\frac{1}{b_{n-j+1}}\leqslant C'\max_{j=1,2,\dots,n}\|x_j\|_{C(\infty)}
\end{aligned}$$

and, finally, by the definition of the norm in the space $\mathrm{Ces}_\infty[0,1]$ we have

$$\begin{aligned}
\|T_4\|_\infty &= \max_{j=1,2,\dots,n}\sup_{a_{n-j+1}\leqslant t\leqslant b_{n-j+1}}\biggl(\frac{1}{t}\int_0^t x_{n-j+1}(s)\,ds\biggr)\\
&\leqslant \max_{j=1,2,\dots,n}\sup_{0<t\leqslant 1}\biggl(\frac{1}{t}\int_0^t x_{n-j+1}(s)\,ds\biggr)=\max_{j=1,2,\dots,n}\|x_j\|_{C(\infty)}.
\end{aligned}$$

Hence,

$$\biggl\|\sum_{k=1}^{n}x_k\biggr\|_{C(\infty)}\leqslant\sum_{k=1}^{4}\|T_k\|_\infty\leqslant 4C'\max_{j=1,2,\dots,n}\|x_j\|_{C(\infty)}.$$

Since $\mathrm{Ces}_\infty[0,1]$ is maximal, this completes the proof of the relation (2.2).

In conclusion, we claim that the subspace $[x_n]$ thus constructed is complemented in $\mathrm{Ces}_p[0,1]$. Let $\overline{x}_n = x_n/\|x_n\|_{C(p)}$ and let g_n be functions in $\mathrm{Ces}_p[0,1]'$ such that

$$\|g_n\|_{C(p)'}\asymp 1,\qquad \operatorname{supp} g_n\subset[a_n,b_n],\qquad \int_0^1 g_n(t)\overline{x}_n(t)\,dt=1,\quad n=1,2,\dots .$$

It is sufficient to prove that the projection

$$Qf(s):=\sum_{n=1}^{\infty}\int_0^1 g_n(t)f(t)\,dt\cdot\overline{x}_n(s)$$

acts boundedly on the space $\mathrm{Ces}_p[0,1]$.

If $f\in\mathrm{Ces}_p[0,1]$ is arbitrary and $f_n:=f\cdot\chi_{[a_n,b_n]}$, $n=1,2,\dots$, it follows that

$$Qf=\sum_{n=1}^{\infty}Qf_n,\quad\text{where } Qf_n(s):=\int_0^1 g_n(t)f_n(t)\,dt\cdot\overline{x}_n(s).$$

Moreover, since $\|g_n\|_{C(p)'} \leqslant C''$, $n = 1, 2, \ldots$, it follows that

$$\|Qf_n\|_{C(p)} \leqslant \|g_n\|_{C(p)'}\|f_n\|_{C(p)} \leqslant C''\|f_n\|_{C(p)}, \qquad n = 1, 2, \ldots .$$

Therefore, if $1 < p < \infty$, then

$$\begin{aligned}\|Qf\|_{C(p)} &\leqslant C_5\Big(\sum_{n=1}^{\infty}\|Qf_n\|_{C(p)}^p\Big)^{1/p} \leqslant C_6\Big(\sum_{n=1}^{\infty}\|f_n\|_{C(p)}^p\Big)^{1/p} \leqslant C_6\Big\|\sum_{n=1}^{\infty} f_n\Big\|_{C(p)} \\ &\leqslant C_6\|f\|_{C(p)}\end{aligned}$$

by the relation (2.1) proved above and the p-concave property of $\mathrm{Ces}_p[0,1]$. Since the arguments for $p = \infty$ are very similar, this completes the proof of the proposition.

If p is finite, then a similar result also holds for sequences of disjoint functions whose supports 'tend' to one.

Proposition 2. *Let* $1 < p < \infty$. *Suppose that a sequence* $\{x_n\} \subset \mathrm{Ces}_p[0,1]$ *satisfies the condition* $\operatorname{supp} x_n \subset I_n = [a_n, b_n]$, $n = 1, 2, \ldots$, $I_1 < I_2 < \cdots$, *and* $a_n \to 1^-$.

Then there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ *such that the relation* (2.1) *holds for some constant* $C > 0$ *depending on* p *only, and the corresponding subspace* $[x_{n_k}]$ *is also complemented in the space* $\mathrm{Ces}_p[0,1]$.

Proof. Since the space $\mathrm{Ces}_p[0,1]$ is p-concave (see § 1), we only need prove the right-hand inequality in (2.1). We can also assume that $x_n \geqslant 0$, $x_n \neq 0$, and $a_1 \geqslant 1/2$.

We choose a subsequence of closed intervals (which we again denote by $I_m = [a_m, b_m]$) in such a way that the following inequalities hold:

$$\Big(\sum_{i=1}^{m-1}(1-b_i)^{-1/(p-1)}\Big)^{p-1} \cdot (1 - a_m^{p-1}) \leqslant 2^{-m}, \qquad m = 1, 2, \ldots . \tag{2.5}$$

We claim that the corresponding subsequence of functions, which we again denote by $\{x_m\}$, $\operatorname{supp} x_m \subset I_m$, satisfies the desired properties.

For arbitrary $n \in \mathbb{N}$ and $t \in (0, 1]$ we have

$$\begin{aligned}\frac{1}{t}\int_0^t\Big|\sum_{k=1}^{n} x_k(s)\Big|\,ds &= \frac{1}{t}\sum_{m=1}^{n}\Big(\sum_{i=1}^{m-1}\int_{a_i}^{b_i} x_i(s)\,ds + \int_{a_m}^{t} x_m(s)\,ds\Big)\chi_{[a_m,b_m]}(t) \\ &\quad + \frac{1}{t}\sum_{m=1}^{n}\sum_{i=1}^{m}\int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[b_m,a_{m+1}]}(t) \\ &= S_1(t) + S_2(t) + S_3(t) + S_4(t),\end{aligned}$$

where $a_{n+1}=1$ and

$$S_1(t)=\frac{1}{t}\sum_{m=2}^{n}\sum_{i=1}^{m-1}\int_{a_i}^{b_i}x_i(s)\,ds\,\chi_{[a_m,b_m]}(t),$$

$$S_2(t)=\frac{1}{t}\sum_{m=2}^{n}\sum_{i=1}^{m-1}\int_{a_i}^{b_i}x_i(s)\,ds\,\chi_{[b_m,a_{m+1}]}(t),$$

$$S_3(t)=\frac{1}{t}\sum_{m=1}^{n}\int_{a_m}^{b_m}x_m(s)\,ds\,\chi_{[b_m,a_{m+1}]}(t),$$

$$S_4(t)=\frac{1}{t}\sum_{m=1}^{n}\int_{a_m}^{t}x_m(s)\,ds\,\chi_{[a_m,b_m]}(t).$$

Using the description of the space dual to $\mathrm{Ces}_p[0,1]$ again (see (1.3)), we can write

$$\begin{aligned}\int_{a_i}^{b_i}x_i(s)\,ds &\leqslant C_1\|x_i\|_{C(p)}\cdot\|\chi_{[a_i,b_i]}\|_{U(q)}=C_1\|x_i\|_{C(p)}\cdot\left(\int_0^{b_i}\frac{dt}{(1-t)^q}\right)^{1/q}\\ &=C_1\|x_i\|_{C(p)}\cdot(p-1)^{1/q}\left[\frac{1}{(1-b_i)^{q-1}}-1\right]^{1/q}\leqslant\frac{C_2\|x_i\|_{C(p)}}{(1-b_i)^{1/p}},\end{aligned}$$

where $C_2>0$ depends only on p. Therefore, by (2.5),

$$\begin{aligned}\|S_1\|_p^p &=\sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}\int_{a_i}^{b_i}x_i(s)\,ds\right)^p\int_{a_m}^{b_m}\frac{dt}{t^p}\\ &\leqslant C_2^p\sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}(1-b_i)^{-1/p}\|x_i\|_{C(p)}\right)^p\frac{b_m^{p-1}-a_m^{p-1}}{(p-1)a_m^{p-1}b_m^{p-1}}\\ &\leqslant C_3\sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}(1-b_i)^{-1/(p-1)}\right)^{p-1}\sum_{i=1}^{m-1}\|x_i\|_{C(p)}^p(1-a_m^{p-1})\\ &\leqslant C_3\sum_{m=2}^{n}2^{-m}\sum_{i=1}^{m-1}\|x_i\|_{C(p)}^p=C_3\sum_{i=1}^{n-1}\|x_i\|_{C(p)}^p\sum_{m=i+1}^{n}2^{-m}\leqslant C_3\sum_{i=1}^{n-1}\|x_i\|_{C(p)}^p,\end{aligned}$$

where $C_3>0$ also depends only on p. Similarly,

$$\begin{aligned}\|S_2\|_p^p &\leqslant C_2^p\sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}(1-b_i)^{-1/p}\|x_i\|_{C(p)}\right)^p\frac{a_{m+1}^{p-1}-b_m^{p-1}}{(p-1)a_{m+1}^{p-1}b_m^{p-1}}\\ &\leqslant C_3\sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}(1-b_i)^{-1/(p-1)}\right)^{p-1}\sum_{i=1}^{m-1}\|x_i\|_{C(p)}^p(1-b_m^{p-1})\\ &\leqslant C_3\sum_{m=2}^{n}2^{-m}\sum_{i=1}^{m-1}\|x_i\|_{C(p)}^p=C_3\sum_{i=1}^{n-1}\|x_i\|_{C(p)}^p\sum_{m=i+1}^{n}2^{-m}\leqslant C_3\sum_{i=1}^{n-1}\|x_i\|_{C(p)}^p.\end{aligned}$$

Moreover, by the definition of the norm in the space $\mathrm{Ces}_p[0,1]$,

$$\begin{aligned}
\|S_3\|_p^p &= \sum_{m=1}^{n} \int_{b_m}^{a_{m+1}} \frac{1}{t^p}\,dt\biggl(\int_{a_m}^{b_m} x_m(s)\,ds\biggr)^p \\
&\leqslant \sum_{m=1}^{n} \int_0^1 \biggl(\frac{1}{t}\int_0^t x_m(s)\,ds\biggr)^p dt = \sum_{m=1}^{n} \|x_m\|_{C(p)}^p, \\
\|S_4\|_p^p &= \sum_{m=1}^{n} \int_{a_m}^{b_m} \biggl(\frac{1}{t}\int_{a_m}^t x_m(s)\,ds\biggr)^p dt \\
&\leqslant \sum_{m=1}^{n} \int_0^1 \biggl(\frac{1}{t}\int_0^t x_m(s)\,ds\biggr)^p dt = \sum_{m=1}^{n} \|x_m\|_{C(p)}^p.
\end{aligned}$$

Finally,

$$\biggl\|\sum_{k=1}^{n} x_k\biggr\|_{C(p)} \leqslant \sum_{k=1}^{4} \|S_k\|_p \leqslant C\biggl(\sum_{m=1}^{n} \|x_m\|_{C(p)}^p\biggr)^{1/p},$$

where $C>0$ depends only on p. As above, this implies (2.1). That $[x_n]$ is complemented can be proved in just the same way as in the previous proposition.

A simple analysis of the proofs of Propositions 1 and 2 shows that the following 'two-sided' version of these propositions also holds.

Proposition 3. *Let $1<p<\infty$ and let the sequence $\{x_n\}\subset \mathrm{Ces}_p[0,1]$ satisfy the condition $\operatorname{supp} x_n \subset I_n' \cup I_n''$, $n=1,2,\dots$, where $I_n'=[a_n',b_n']$ and $I_n''=[a_n'',b_n'']$ are closed intervals on $[0,1]$ such that $I_1'>I_2'>\cdots$, $b_n'\to 0^+$, $I_1''<I_2''<\cdots$, and $a_n''\to 1^-$.*

Then there is a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that the inequalities (2.1) hold for some constant $C>0$ depending only on p. Moreover, the subspace $[x_{n_k}]$ is complemented in the space $\mathrm{Ces}_p[0,1]$.

Remark 1. As we noted in § 1, we have $\mathrm{Ces}_1[0,1]=L_1(\ln(1/t))$ isometrically (see also [7], Theorem 1). Therefore, for an arbitrary sequence of disjoint functions $\{x_n\}\subset \mathrm{Ces}_1[0,1]$ we have

$$\biggl\|\sum_{n=1}^{\infty} x_n\biggr\|_{C(1)} = \sum_{n=1}^{\infty} \|x_n\|_{C(1)}.$$

Remark 2. The statements of Propositions 2 and 3 do not hold for $p=\infty$. Indeed, if $h\in(0,1)$ and f is a measurable function on $[0,1]$ such that $\operatorname{supp} f\subset[h,1]$, then

$$\|f\|_{L_1[h,1]} \leqslant \|f\|_{C(\infty)} = \sup_{h\leqslant t\leqslant 1} \frac{1}{t}\int_h^t |f(s)|\,ds \leqslant \frac{1}{h}\|f\|_{L_1[h,1]}.$$

Therefore, for any $h\in(0,1)$ $\mathrm{Ces}_\infty[0,1]|_{[h,1]} = L_1[h,1]$, and the norms are equivalent. Thus, if a sequence $\{x_n\}\subset \mathrm{Ces}_\infty[0,1]$ consists of disjoint functions whose supports are contained in the closed interval $[h,1]$ for some $h\in(0,1)$, then

$$h\sum_{n=1}^{\infty} \|x_n\|_{C(\infty)} \leqslant \biggl\|\sum_{n=1}^{\infty} x_n\biggr\|_{C(\infty)} \leqslant \frac{1}{h}\sum_{n=1}^{\infty} \|x_n\|_{C(\infty)}.$$

§ 3. The main results

The results in § 2 let us characterize the set of q for which the Cesàro space contains a complemented copy of the l^q-space. However, first we solve a similar problem for l^q-copies generated by sequences of disjoint functions.

As is well known (see, for example, [5], Proposition 6.4.1), every sequence of disjoint functions $\{f_n\}_{n=1}^\infty \subset L_p[0,1]$, $\|f_n\|_{L_p} = 1$, $n = 1, 2, \dots$, is equivalent to the canonical basis of l^p. The following theorem shows that the Cesàro space has a property close to this.

Theorem 1. *If a basic sequence* $\{f_n\}_{n=1}^\infty \subset \mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, *consisting of disjoint functions is equivalent to each of its subsequences, then it is equivalent to the canonical basis of one of the two spaces* l^1 *and* l^p.

Proof. First of all, by Remark 1, the statement is obvious for $p = 1$. Suppose now that $1 < p < \infty$. We can and will assume that $\|f_n\|_{C(p)} = 1$, $n = 1, 2, \dots$. There are two cases.

(a) Let there be numbers $h_0 \in (0, 1/2)$ and $\varepsilon_0 > 0$ such that $\|f_n \chi_{[h_0, 1-h_0]}\|_{C(p)} \geqslant \varepsilon_0$ for all sufficiently large n. In this case, if $g_n := f_n \chi_{[h_0,1-h_0]}$, then $\operatorname{supp} g_n \subset [h_0, 1-h_0]$. Since $\mathrm{Ces}_p[0,1]|_{[h_0,1-h_0]} = L_1[h_0, 1-h_0]$ with an equivalence of norms [8], it follows that $\|g_n\|_{L_1} \asymp \|g_n\|_{C(p)} \asymp 1$, $n = 1, 2, \dots$, and, since g_n are disjoint, we have

$$\Bigl\|\sum_{n=1}^\infty a_n g_n\Bigr\|_{C(p)} \asymp \Bigl\|\sum_{n=1}^\infty a_n g_n\Bigr\|_{L_1} \asymp \|(a_n)\|_{l^1}.$$

At the same time, it is clear that

$$\Bigl\|\sum_{n=1}^\infty a_n g_n\Bigr\|_{C(p)} \leqslant \Bigl\|\sum_{n=1}^\infty a_n f_n\Bigr\|_{C(p)},$$

and thus the previous relation, together with $\|f_n\|_{C(p)} = 1$, $n = 1, 2, \dots$, yields

$$\Bigl\|\sum_{n=1}^\infty a_n f_n\Bigr\|_{C(p)} \asymp \|(a_n)\|_{l^1}.$$

(b) Suppose now that the condition in the previous part of the proof fails to hold. In this case, for arbitrary $h \in (0, 1/2)$ and $\varepsilon > 0$ there is an n such that $\|f_n \chi_{[h,1-h]}\|_{C(p)} \leqslant \varepsilon$. Hence, there exist a subsequence $\{f_{n_j}\} \subset \{f_n\}$ and two sequences of segments $I_n' = [a_n', b_n']$ and $I_n'' = [a_n'', b_n'']$, $n = 1, 2, \dots$, such that $I_1' > I_2' > \cdots$, $b_n' \to 0^+$, $I_1'' < I_2'' < \cdots$, $a_n'' \to 1^-$, and

$$2K \sum_{j=1}^\infty \|f_{n_j} \chi_{I_j' \cup I_j''} - f_{n_j}\|_{C(p)} < 1, \tag{3.1}$$

where K is the basis constant of the sequence $\{f_n\}$. We write $h_j := f_{n_j} \chi_{I_j' \cup I_j''}$, $j = 1, 2, \dots$. Then $\operatorname{supp} h_j \subset I_j' \cup I_j''$ and $\|h_j\|_{C(p)} \asymp 1$, $j = 1, 2, \dots$. Applying Proposition 3, we find a subsequence $\{h_{j_k}\} \subset \{h_j\}$ such that

$$\Bigl\|\sum_{k=1}^\infty a_k h_{j_k}\Bigr\|_{C(p)} \asymp \|(a_k)\|_{l^p}.$$

for any $a_k \in \mathbb{R}$. On the other hand, using (3.1), the stability of the basis [5], Theorem 1.3.9, and also the hypothesis of the theorem we obtain

$$\left\|\sum_{n=1}^{\infty} a_n f_n\right\|_{C(p)} \asymp \left\|\sum_{k=1}^{\infty} a_k f_{n_{j_k}}\right\|_{C(p)} \asymp \left\|\sum_{k=1}^{\infty} a_k h_{j_k}\right\|_{C(p)}.$$

This, together with the previous relation, implies that the sequence $\{f_n\}_{n=1}^{\infty}$ is equivalent to the canonical basis of the space l^p. The theorem is thus proved.

Corollary 1. *Let $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset \mathrm{Ces}_p[0,1]$ is a sequence of disjoint functions for which $[f_n] \simeq l^q$. Then $q = 1$ or $q = p$.*

In [13], the authors studied the subspace $[r_n]$ of the Cesàro space spanned by the Rademacher functions $r_n(t) = \mathrm{sign}(\sin 2^n \pi t)$, $n \in \mathbb{N}$. Together with other results, they proved there that the sequence $\{r_n\}_{n=1}^{\infty}$ in $\mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, is equivalent to the canonical basis in l^2 and $[r_n]$ is not complemented in this space. The following theorem, which characterizes the complemented copies of l^q-spaces, shows in particular that $\mathrm{Ces}_p[0,1]$ contains a complemented subspace $X \simeq l^2$ if and only if $p = 2$.

Theorem 2. *Let $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$. The Cesàro space $\mathrm{Ces}_p[0,1]$ contains a complemented subspace isomorphic to l^q if and only if $q = 1$ or $q = p$.*

Proof. First of all, as we mentioned in § 1, the space $\mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, contains complemented subspaces isomorphic to l^1 (see [8], Theorem 1) and to l^p (see Proposition 1, and also [7], Theorem 6). Now we prove the converse statement.

Since the space $\mathrm{Ces}_1[0,1] = L_1(\ln(1/t))$ contains no complemented subspaces isomorphic to reflexive spaces ([5], Proposition 5.6.1), it follows that the statement of the theorem is obvious for $p = 1$. Therefore, let $1 < p < \infty$.

Suppose that a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathrm{Ces}_p[0,1]$, $\|f_n\|_{\mathrm{Ces}_p} = 1$ is equivalent to the canonical basis in l^q and the subspace $[f_n]$ is complemented in $\mathrm{Ces}_p[0,1]$. Moreover, let $q \neq 1$. To prove the theorem, it suffices to show that $q = p$.

By assumption, there is a bounded projection $P\colon \mathrm{Ces}_p[0,1] \to [f_n]$, which (as is easily seen) can be represented in the form

$$Px(s) = \sum_{k=1}^{\infty} \varphi_k(x) f_k(s), \qquad 0 \leqslant s \leqslant 1,$$

where $\varphi_k \in (\mathrm{Ces}_p[0,1])^*$, $\varphi_k(f_k) = 1$, $k = 1, 2, \dots$, and $\varphi_k(f_n) = 0$ if $k \neq n$. Since $\mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, is a separable space, it follows that $(\mathrm{Ces}_p[0,1])^* = (\mathrm{Ces}_p[0,1])'$. Therefore,

$$\varphi_k(x) = \int_0^1 g_k(t) x(t)\, dt, \qquad x = x(t) \in \mathrm{Ces}_p[0,1],$$

for some $g_k \in (\mathrm{Ces}_p[0,1])'$, $k = 1, 2, \dots$, and thus

$$Px(s) = \sum_{k=1}^{\infty} \int_0^1 g_k(t) x(t)\, dt \cdot f_k(s), \qquad 0 \leqslant s \leqslant 1, \tag{3.2}$$

where

$$\int_0^1 g_k(t)f_k(t)\,dt=1,\quad k=1,2,\dots,\qquad \int_0^1 g_k(t)f_n(t)\,dt=0,\quad k\ne n.$$

Moreover, since $\{f_n\}_{n=1}^\infty$ is equivalent to the canonical basis in l^q, it follows that

$$\|Px\|_{C(p)}^q\asymp\sum_{k=1}^\infty\biggl(\int_0^1 g_k(t)x(t)\,dt\biggr)^q,\qquad x=x(t)\in\mathrm{Ces}_p[0,1]. \tag{3.3}$$

In particular, this implies that $g_k\to 0$ weak-star in $(\mathrm{Ces}_p[0,1])^*$ and

$$\|g_k\|_{C(p)'}\leqslant M,\qquad k=1,2,\dots, \tag{3.4}$$

for some $M>0$.

We note that $\mathrm{Ces}_p[0,1]\subset\mathrm{Ces}_1[0,1]=L_1(\ln(1/t))$ for every $1\leqslant p<\infty$. Since $q\ne 1$ by assumption, it follows that $\{f_n\}$ contains no sequences equivalent to the canonical basis in l^1. Hence, by the Dunford-Pettis criterion (see, for example, [5], Theorem 5.2.9), $\{f_n\}$ is a relatively weakly compact set in $L_1(\ln(1/t))$. Passing to a subsequence, we see that $f_n\to f$ weakly in $L_1(\ln(1/t))$. We recall (see § 1) that $(\mathrm{Ces}_p[0,1])^*=(\mathrm{Ces}_p[0,1])'=U(q)$, where $1/q+1/p=1$, with the norm defined by the relation (1.3). It can readily be seen that the space $L_\infty(\ln^{-1}(1/t))=(L_1(\ln(1/t)))^*$ is dense in $U(q)$. This, together with the relations $\|f_n\|_{C(p)}=1$, $n=1,2,\dots$, implies that $f_n\to f$ weakly in $\mathrm{Ces}_p[0,1]$. In this case, in particular,

$$\int_0^1 g_k(t)f_n(t)\,dt\to\int_0^1 g_k(t)f(t)\,dt\quad\text{as } n\to\infty$$

for any $k=1,2,\dots$. Hence,

$$\int_0^1 g_k(t)f(t)\,dt=0,\qquad k=1,2,\dots.$$

Since $f\in[f_n]$, it follows that $f=0$, and thus $f_n\to 0$ weakly in $\mathrm{Ces}_p[0,1]$.

Suppose that a subsequence $\{f_{n_k}\}\subset\{f_n\}$ and an element $h\in(0,1/2)$ exist for which

$$\biggl|\int_h^{1-h} g_{n_k}(t)f_{n_k}(t)\,dt\biggr|\geqslant\frac12,\qquad k=1,2,\dots. \tag{3.5}$$

We introduce the operator $P_hx:=P(x\chi_{[h,1-h]})$. Since $\mathrm{Ces}_p[0,1]|_{[h,1-h]}=L_1[h,1-h]$ (see [8], Lemma 1) and P is bounded in $\mathrm{Ces}_p[0,1]$, it follows that P_h is bounded as an operator from $L_1[h,1-h]$ to $\mathrm{Ces}_p[0,1]$. Moreover, since $f_n\chi_{[h,1-h]}\to 0$ weakly in $\mathrm{Ces}_p[0,1]$, it follows that $f_n\chi_{[h,1-h]}\to 0$ both weakly and in $L_1[h,1-h]$.

As is well known ([5], Theorem 5.4.5), the space $L_1[h,1-h]$ has the Dunford-Pettis property. Therefore, the operator P_h takes sequences which converge weakly to zero to sequences which converge to zero in the norm. Hence,

$$\|P_hf_n\|_{C(p)}\to 0\quad\text{as } n\to\infty.$$

On the other hand, by (3.3) and (3.5), for any $k = 1, 2, \dots$ we have

$$\|P_h f_{n_k}\|_{C(p)} \asymp \left(\sum_{i=1}^{\infty}\left|\int_h^{1-h} g_i(t) f_{n_k}(t)\,dt\right|^q\right)^{1/q} \geqslant \left|\int_h^{1-h} g_{n_k}(t) f_{n_k}(t)\,dt\right| \geqslant \frac{1}{2},$$

which contradicts the previous relation.

Thus, (3.5) does not hold, and therefore one of the two relations,

$$\left|\int_0^h g_n(t) f_n(t)\,dt\right| > \frac{1}{4} \quad \text{or} \quad \left|\int_{1-h}^1 g_n(t) f_n(t)\,dt\right| > \frac{1}{4},$$

holds for any $h \in (0, 1/2)$ and for all sufficiently large $n \in \mathbb{N}$. Therefore, one can single out a subsequence (which we shall still denote by$\{f_n\}$), which satisfies one of the following two conditions:

$$\left|\int_0^h g_n(t) f_n(t)\,dt\right| > \frac{1}{4}, \qquad n \in \mathbb{N}, \tag{3.6}$$

or

$$\left|\int_{1-h}^1 g_n(t) f_n(t)\,dt\right| > \frac{1}{4}, \qquad n \in \mathbb{N}. \tag{3.7}$$

Suppose, for example, that (3.6) holds. Then, setting $h_0 = 1/2$, we can find an $n_1 \in \mathbb{N}$ and an $h_1 \in (0, h_0)$ such that

$$\left|\int_{h_1}^{h_0} g_{n_1}(t) f_{n_1}(t)\,dt\right| > \frac{1}{4}.$$

We write $u_{n_1} := g_{n_1}\chi_{(h_1,h_0)}$ and $f'_{n_1} := f_{n_1}\chi_{(h_1,h_0)}$. As we noted above, $f_n \to 0$ weakly in $(\mathrm{Ces}_p[0,1])^*$ and $g_n \to 0$ weak-star in $(\mathrm{Ces}_p[0,1])^*$. Therefore, and also by (3.6), there is an $n_2 > n_1$ for which

$$\max\left(\left|\int_0^{h_1} g_{n_1}(t) f_{n_2}(t)\,dt\right|, \left|\int_0^1 u_{n_1}(t) f_{n_2}(t)\,dt\right|\right) \leqslant \frac{1}{2^3},$$
$$\max\left(\left|\int_0^{h_1} g_{n_2}(t) f_{n_1}(t)\,dt\right|, \left|\int_0^1 g_{n_2}(t) f'_{n_1}(t)\,dt\right|\right) \leqslant \frac{1}{2^3},$$
$$\left|\int_0^{h_1} g_{n_2}(t) f_{n_2}(t)\,dt\right| > \frac{1}{4}.$$

But then $h_2 \in (0, h_1)$ exists such that the functions

$$f'_{n_2} := f_{n_2}\chi_{(h_2,h_1)} \quad \text{and} \quad u_{n_2} := g_{n_2}\chi_{(h_2,h_1)}$$

satisfy the relations

$$\left|\int_0^1 u_{n_j}(t) f_{n_i}(t)\,dt\right| \leqslant \frac{1}{2^3}, \qquad 1 \leqslant i \neq j \leqslant 2,$$
$$\left|\int_0^1 g_{n_j}(t) f'_{n_i}(t)\,dt\right| \leqslant \frac{1}{2^3}, \qquad 1 \leqslant i \neq j \leqslant 2,$$
$$\left|\int_0^1 u_{n_2}(t) f'_{n_2}(t)\,dt\right| > \frac{1}{4}.$$

Suppose that indices $n_1 < n_2 < \cdots < n_k$ and numbers $1/2 = h_0 > h_1 > \cdots > h_k > 0$ are chosen for some $k \in \mathbb{N}$ in such a way that the functions f_{n_i}, g_{n_i}, $f'_{n_i} := f_{n_i}\chi_{(h_i,h_{i-1})}$, and $u_{n_i} = g_{n_i}\chi_{(h_i,h_{i-1})}$, $i = 1,2,\dots,k$, satisfy the conditions

$$\left|\int_0^1 u_{n_j}(t) f_{n_i}(t)\,dt\right| \leqslant \frac{1}{2^{i+j}}, \qquad 1 \leqslant i \neq j \leqslant k, \tag{3.8}$$

$$\left|\int_0^1 g_{n_j}(t) f'_{n_i}(t)\,dt\right| \leqslant \frac{1}{2^{i+j}}, \qquad 1 \leqslant i \neq j \leqslant k, \tag{3.9}$$

$$\left|\int_0^1 u_{n_i}(t) f'_{n_i}(t)\,dt\right| > \frac{1}{4}, \qquad i = 1,2,\dots,k. \tag{3.10}$$

Again using the fact that $f_n \to 0$ weakly in $\mathrm{Ces}_p[0,1]$ and $g_n \to 0$ weak-star in $(\mathrm{Ces}_p[0,1])^*$ and also the relation (3.6), we find an index $n_{k+1} > n_k$ for which

$$\max\left(\left|\int_0^{h_k} g_{n_i}(t) f_{n_{k+1}}(t)\,dt\right|, \left|\int_0^1 u_{n_i}(t) f_{n_{k+1}}(t)\,dt\right|\right) \leqslant \frac{1}{2^{i+k+1}}, \quad i = 1,2,\dots,k,$$

$$\max\left(\left|\int_0^{h_k} g_{n_{k+1}}(t) f_{n_i}(t)\,dt\right|, \left|\int_0^1 g_{n_{k+1}}(t) f'_{n_i}(t)\,dt\right|\right) \leqslant \frac{1}{2^{i+k+1}}, \quad i = 1,2,\dots,k,$$

$$\left|\int_0^{h_k} g_{n_{k+1}}(t) f_{n_{k+1}}(t)\,dt\right| > \frac{1}{4}.$$

Obviously, $h_{k+1} \in (0, h_k)$ exists such that (3.8) and (3.9) hold for

$$f'_{n_{k+1}} := f_{n_{k+1}}\chi_{(h_{k+1},h_k)} \quad \text{and} \quad u_{n_{k+1}} := g_{n_{k+1}}\chi_{(h_{k+1},h_k)}$$

for any $1 \leqslant i \neq j \leqslant k+1$ and the relations (3.10) hold for any $i = 1,2,\dots,k+1$.

The above arguments let us single out sequences of indices $n_1 < n_2 < \cdots < n_i < \cdots$ and numbers $1/2 = h_0 > h_1 > \cdots > h_i > \cdots > 0$ for which the functions f_{n_i}, g_{n_i}, $f'_{n_i} := f_{n_i}\chi_{(h_i,h_{i-1})}$, and $u_{n_i} = g_{n_i}\chi_{(h_i,h_{i-1})}$, $i = 1,2,\dots$, satisfy the following conditions:

$$\left|\int_0^1 u_{n_j}(t) f_{n_i}(t)\,dt\right| \leqslant \frac{1}{2^{i+j}}, \qquad 1 \leqslant i \neq j < \infty, \tag{3.11}$$

$$\left|\int_0^1 g_{n_j}(t) f'_{n_i}(t)\,dt\right| \leqslant \frac{1}{2^{i+j}}, \qquad 1 \leqslant i \neq j < \infty, \tag{3.12}$$

$$\left|\int_0^1 u_{n_i}(t) f'_{n_i}(t)\,dt\right| > \frac{1}{4}, \qquad i = 1,2,\dots. \tag{3.13}$$

By Proposition 1, we may assume that the sequence $\{f'_{n_i}\}$ is equivalent to the canonical basis in l^p and that the subspace $[f'_{n_i}]$ is complemented in $\mathrm{Ces}_p[0,1]$. Moreover, since $\|f'_{n_i}\|_{C(p)} \leqslant \|f_{n_i}\|_{C(p)} = 1$, it follows from the relations (3.4) and (3.13) that

$$\frac{1}{4} \leqslant \|u_{n_i}\|_{C(p)'} \leqslant \|g_{n_i}\|_{C(p)'} \leqslant M.$$

Therefore (see the proof of Proposition 1), the projection

$$Qf(t) := \sum_{i=1}^{\infty} \int_0^1 \overline{u}_{n_i}(s) f(s)\,ds \cdot f'_{n_i}(t), \qquad \overline{u}_{n_i} := \left(\int_0^1 u_{n_i}(t) f'_{n_i}(t)\,dt\right)^{-1} u_{n_i},$$

is bounded on $\mathrm{Ces}_p[0,1]$. This, together with our assumption, implies that

$$\Big\|Q\Big(\sum_{i=1}^{m} f_{n_i}\Big)\Big\|_{C(p)} \leqslant \|Q\| \, \Big\|\sum_{i=1}^{m} f_{n_i}\Big\|_{C(p)} \leqslant C\|Q\|m^{1/q}, \qquad m \in \mathbb{N}. \tag{3.14}$$

On the other hand, using (3.13) again, together with the inequality $\|f'_{n_i}\|_{C(p)} \leqslant 1$, we obtain

$$\begin{aligned}\Big\|Q\Big(\sum_{i=1}^{m} f_{n_i}\Big)\Big\|_{C(p)} &= \Big\|\sum_{j=1}^{\infty}\sum_{i=1}^{m}\int_0^1 \overline{u}_{n_j}(s)f_{n_i}(s)\,ds\cdot f'_{n_j}\Big\|_{C(p)} \\ &\geqslant \Big\|\sum_{i=1}^{m}\int_0^1 \overline{u}_{n_i}(s)f_{n_i}(s)\,ds\cdot f'_{n_i}\Big\|_{C(p)} - 4\sum_{i,j=1,\, i\neq j}^{\infty}\Big|\int_0^1 u_{n_j}(s)f_{n_i}(s)\,ds\Big|.\end{aligned}$$

We note that $\|f'_{n_i}\|_{C(p)} \geqslant (4M)^{-1}$, $i = 1, 2, \dots$, by (3.4) and (3.13). Therefore, using the last inequality, formulae (3.11) and (3.13), and also the p-concavity of the space $\mathrm{Ces}_p[0,1]$, we obtain

$$\Big\|Q\Big(\sum_{i=1}^{m} f_{n_i}\Big)\Big\|_{C(p)} \geqslant \Big(\sum_{i=1}^{m}\Big|\int_0^1 \overline{u}_{n_i}(s)f_{n_i}(s)\,ds\Big|^p\cdot\|f'_{n_i}\|_{C(p)}^p\Big)^{1/p} - 4 \geqslant \frac{m^{1/p}}{4M} - 4.$$

This, together with (3.14), implies that $q \leqslant p$.

It is clear that the sequence $\{f_{n_i}\}$ has the properties similar to those of $\{f_n\}$, that is, the sequence $\{f_{n_i}\}$ is equivalent to the canonical basis in l^q, and the subspace $[f_{n_i}]$ is complemented in $\mathrm{Ces}_p[0,1]$. Moreover, it can readily be seen that the projection P_1, whose definition is similar to that of the projection P (see (3.2)) with f_k replaced by f_{n_i} and g_k by g_{n_i}, is bounded on the space $\mathrm{Ces}_p[0,1]$. Hence, since $\{f'_{n_i}\}$ is equivalent to the canonical basis in l^p, it follows that

$$\Big\|P_1\Big(\sum_{i=1}^{m} f'_{n_i}\Big)\Big\|_{C(p)} \leqslant \|P_1\| \, \Big\|\sum_{i=1}^{m} f'_{n_i}\Big\|_{C(p)} \leqslant C\|P_1\|\Big(\sum_{i=1}^{m}\|f'_{n_i}\|_{C(p)}^p\Big)^{1/p} \leqslant C\|P_1\|m^{1/p} \tag{3.15}$$

for any $m \in \mathbb{N}$. At the same time, by (3.12) and (3.13),

$$\begin{aligned}\Big\|P_1\Big(\sum_{i=1}^{m} f'_{n_i}\Big)\Big\|_{C(p)} &= \Big\|\sum_{j=1}^{\infty}\sum_{i=1}^{m}\int_0^1 g_{n_j}(s)f'_{n_i}(s)\,ds\cdot f_{n_j}\Big\|_{C(p)} \\ &\geqslant \Big\|\sum_{i=1}^{m}\int_0^1 g_{n_i}(s)f'_{n_i}(s)\,ds\cdot f_{n_i}\Big\|_{C(p)} - \sum_{i,j=1,\, i\neq j}^{\infty}\Big|\int_0^1 g_{n_j}(s)f'_{n_i}(s)\,ds\Big| \\ &\geqslant \Big\|\sum_{i=1}^{m}\int_0^1 u_{n_i}(s)f'_{n_i}(s)\,ds\cdot f_{n_i}\Big\|_{C(p)} - 1 \\ &\geqslant c\Big(\sum_{i=1}^{m}\Big|\int_0^1 u_{n_i}(s)f'_{n_i}(s)\,ds\Big|^q\Big)^{1/q} - 1 \geqslant \frac{cm^{1/q}}{4} - 1.\end{aligned}$$

This, together with (3.15), implies that $p \leqslant q$, and if condition (3.6) holds, then the theorem is proved. The arguments remain the same if condition (3.7) holds; we only need apply Proposition 2 instead of Proposition 1.

The following statement resulting from Theorem 2 and from Corollary 1 shows that there is a substantial difference between the geometric properties of L_p-spaces and Cesàro spaces (cf. Theorem A).

Corollary 2. *Let $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$. The Cesàro space $\mathrm{Ces}_p[0,1]$ contains a complemented subspace $X \simeq l^q$ if and only if there is a sequence of disjoint functions $\{f_n\}_{n=1}^{\infty} \subset \mathrm{Ces}_p[0,1]$ such that $[f_n] \simeq l^q$.*

As was proved in [7], Theorem 7, the space $\mathrm{Ces}_p[0,1]$, $1 < p \leqslant \infty$, is not isomorphic to $L_q[0,1]$ for any $1 \leqslant q \leqslant \infty$. Using Theorem 2, together with the fact that the space $L_q[0,1]$, $1 < q < \infty$, contains a complemented subspace isomorphic to l^2 (see, for example, Theorem A), we obtain a sharpening of Corollary 2.

Corollary 3. *Let $1 < p < \infty$ and $p \neq 2$. Then the Cesàro space $\mathrm{Ces}_p[0,1]$ contains no complemented copies of the space $L_q[0,1]$ for any $q \in (1,\infty]$.*

Remark 3. As follows from Proposition 1 (see also [7], Theorem 6), the space $\mathrm{Ces}_2[0,1]$ contains a complemented copy of l^2 and hence of $L_2[0,1]$. Moreover, as we have often mentioned above, the space $\mathrm{Ces}_p[0,1]$ contains a complemented copy of $L_1[0,1]$ for any p, $1 \leqslant p \leqslant \infty$. Thus, the result of the last corollary does not hold if $p = 2$ or $q = 1$.

We claim that the space $\mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, is in a sense 'saturated' by complemented copies of l^p. We let $\mathscr{L}$ denote the vector space of all functions $x = x(t)$ measurable on $[0,1]$ and such that

$$\int_0^u |x(t)|\,dt < \infty \quad \text{for any } 0 < u < 1.$$

We equip $\mathscr{L}$ with a countable system of seminorms

$$p_n(x) := \int_0^{1-1/n} |x(t)|\,dt, \qquad n = 2, 3, \dots, \tag{3.16}$$

which equips $\mathscr{L}$ with the structure of an F-space (that is, a complete metrizable linear topological space). We denote the topology defined on $\mathscr{L}$ by the system of seminorms (3.16) by τ. It is clear that for any p, $1 \leqslant p \leqslant \infty$, there is a continuous embedding $\mathrm{Ces}_p[0,1] \subset \mathscr{L}$.

Theorem 3. *Let X be an arbitrary subspace of $\mathrm{Ces}_p[0,1]$, $1 \leqslant p < \infty$, which is not closed with respect to the topology τ. Then X contains a subspace $Y \simeq l_p$ complemented in $\mathrm{Ces}_p[0,1]$.*

Proof. Since by assumption X is not closed with respect to the topology τ, there is a sequence $\{x_n\} \subset X$ such that $\|x_n\|_{C(p)} = 1$, $n = 1, 2, \dots$, and

$$\int_h^{1-h} |x_n(t)|\,dt \to 0 \quad \text{as } n \to \infty \tag{3.17}$$

for any $h \in (0, 1/2)$. Hence, using the equation $\mathrm{Ces}_p[0,1]|_{[h,1-h]} = L_1[h, 1-h]$, we have $\lim_{n\to\infty} \|x_n \chi_{[h,1-h]}\|_{C(p)} = 0$ for every $h \in (0, 1/2)$, with an equivalence of the norms (see [8]). However, it can readily be seen in this case that there is a subsequence $\{x'_n\} \subset \{x_n\}$ and a sequence of closed intervals $I'_n = [a'_n, b'_n]$ and $I''_n = [a''_n, b''_n]$, $n = 1, 2, \dots$, such that $I'_1 > I'_2 > \cdots$, $b'_n \to 0^+$, $I''_1 < I''_2 < \cdots$, $a''_n \to 1^-$, and

$$\sum_{n=1}^{\infty} \|x'_n \chi_{I'_n \cup I''_n} - x'_n\|_{C(p)} < 1. \tag{3.18}$$

We write $z_n := x'_n \chi_{I'_n \cup I''_n}$, $n = 1, 2, \dots$. Applying Proposition 3, we can find a subsequence $\{z_{n_i}\} \subset \{z_n\}$ equivalent to the canonical basis of l^p and such that the subspace $[z_{n_i}]$ is complemented in $\mathrm{Ces}_p[0,1]$. Passing to a subsequence if necessary, we can assume by (3.18) that

$$2K \sum_{i=1}^{\infty} \frac{\|z_{n_i} - x'_{n_i}\|_{C(p)}}{\|z_{n_i}\|_{C(p)}} < 1,$$

where K is the basis constant of the sequence $\{z_{n_i}\}$. Applying [5], Theorem 1.3.9 again, we conclude that the subspace $Y := [x'_{n_i}] \simeq l^p$ is complemented in $\mathrm{Ces}_p[0,1]$, together with $[z_{n_i}]$. Since $\{x'_{n_i}\} \subset X$, this completes the proof of the theorem.

Remark 4. The condition that the subspace X is not closed with respect to the topology τ is essential. If $1 < p < \infty$, then it is sufficient to consider the subspace $\mathrm{Ces}_p[0,1]|_{[h,1-h]} = L_1[h, 1-h]$ for an arbitrary $h \in (0, 1/2)$, and, if $p = 1$, to consider the subspace $[r_n]$ spanned by the Rademacher system. Since $[r_n] \simeq l^2$ (see [13]), this subspace contains no subspaces isomorphic to l^1.

§ 4. The general construction and related questions

The following general method of constructing Banach lattices using a sublinear operator was suggested in [1].

Let (T, Σ, μ) be a space with σ-finite measure μ defined on a σ-algebra Σ of subsets of the set T. The family of all measurable almost everywhere finite real-valued functions on T with the natural algebraic operators and the topology of convergence in the measure μ on the sets of finite measure is a linear topological space, which we denote by $L^0(T, \mu)$. Moreover, let $\mathscr{U}$ be a real F-space and $S \colon \mathscr{U} \to L^0(T, \mu)$ an injective positive sublinear operator, so that the following properties hold for any $x, y \in \mathscr{U}$ and $\lambda \in \mathbb{R}$:

(a) $Sx \geqslant 0$,

(b) $S(\lambda x) = |\lambda| S(x)$,

(c) $S(x + y) \leqslant Sx + Sy$.

In this case, we can define a linear normed space

$$D_E(S) := \{x \in \mathscr{U} : Sx \in E\}$$

with norm

$$\|x\|_{D_E(S)} = \|Sx\|_E, \qquad x \in D_E(S),$$

for an arbitrary Banach lattice E of Σ-measurable real-valued functions defined on T. In particular, if $E = L_p(T, \mu)$ with $1 \leqslant p \leqslant \infty$, then we denote this normed space by $D_p(S)$. In this case, if the closed interval $[0,1]$ with the Lebesgue measure is taken for the measure space, $\mathscr{L}$ for the space $\mathscr{U}$ (see § 3), and

$$Sx(t) := \frac{1}{t}\int_0^t |x(s)|\,ds, \qquad 0 \leqslant t \leqslant 1,$$

then we obtain the Cesàro space $\mathrm{Ces}_p[0,1]$.

Another, more important example of spaces of this type is given by spaces obtained using the real interpolation method (other interesting examples can be found in [1]). A pair (X_0, X_1) of two Banach spaces X_0 and X_1 is said to be a *Banach pair* if both the spaces are embedded linearly and continuously in the same separable topological linear space. We can define the *intersection* $X_0 \cap X_1$ and *sum* $X_0 + X_1$ of a Banach pair (X_0, X_1) as Banach spaces with the norms $\|x\|_{X_0\cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$ and $\|x\|_{X_0+X_1} = \mathscr{K}(1, x; X_0, X_1)$, where

$$\mathscr{K}(t, x; X_0, X_1) = \inf\bigl\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_i \in X_i\bigr\}$$

for any $x \in X_0 + X_1$ and $t > 0$.

Let (X_0, X_1) be an arbitrary Banach pair and $\theta \in (0,1)$. We take the set of integers $\mathbb{Z}$ for the measure space, with measure $\mu(A) := \operatorname{card} A$, where A is an arbitrary subset of $\mathbb{Z}$, and we take the sum $X_0 + X_1$ with the topology generated by the norm of the sum for the space $\mathscr{U}$. We define a positive sublinear operator $S\colon X_0 + X_1 \to L_0(\mathbb{Z}, \mu)$ as follows:

$$Sx := \bigl\{2^{-n\theta}\mathscr{K}(2^n, x; X_0, X_1)\bigr\}_{n=-\infty}^{\infty}, \qquad x \in X_0 + X_1.$$

In interpolation theory, the corresponding spaces $D_p(S)$ are the well known Lions-Peetre spaces, which are denoted by $(X_0, X_1)_{\theta,p}$ (see [14]).

The following result (similar to Theorem 3 proved above for Cesàro spaces) holds for the spaces $(X_0, X_1)_{\theta,p}$: if the intersection $X_0 \cap X_1$ is not closed in the sum $X_0 + X_1$, then, by the Mireille Levy theorem [15], every subspace of $(X_0, X_1)_{\theta,p}$ which is not closed in $X_0 + X_1$ contains a subspace isomorphic to l^p and complemented in $(X_0, X_1)_{\theta,p}$. Thus, the following question (see [1]) seems to be quite natural.

Suppose that there is a continuous embedding $D_p(S) \subset \mathscr{U}$ and that $D_p(S)$ is not closed in $\mathscr{U}$. Under what conditions on the operator S does every subspace of $D_p(S)$ which is not closed in $\mathscr{U}$ contain a complemented subspace isomorphic to l^p?

The author has just published a paper [16] in which conditions of this kind are obtained.

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