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# On Certain Properties of Rademacher Chaos

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**Abstract**—We study the properties of Rademacher chaos related to its "lacunarity." It is proved that Rademacher chaos of arbitrary order $d$ is a $2^{-d}$-uniqueness system as well as a system of strict convergence in the wide (not narrow) sense.

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## 1. INTRODUCTION: STATEMENT OF THE PROBLEM AND MAIN DEFINITIONS

Rademacher functions are usually defined as follows:

$$\text{if} \quad t \in \mathbb{R}, \quad \text{then} \quad r_0(t) = 1 \quad \text{and} \quad r_n(t) = \operatorname{sign}(\sin(2^n \pi t)), \quad n = 1, 2, \dots .$$

It is easy to verify that, for arbitrary $n, m \in \mathbb{N}$ and $t \in \mathbb{R}$, the relation $r_n(t) = r_m(2^{n-m}t)$ holds.

**Definition 1.** By the *Rademacher chaos of order* $d \in \mathbb{N}$ we mean the set of all functions of the form $r_{i_1,\dots,i_d}(t) := r_{i_1}(t) \cdots r_{i_d}(t)$, where $1 \le i_1 < \dots < i_d < \infty$.

In particular, the Rademacher system $\{r_n\}_{n=1}^{\infty}$ is the chaos of first order. In addition, if the function $r_0$ is added to the union of Rademacher chaoses of all orders, then we obtain the system of Walsh functions $\{w_n\}_{n=0}^{\infty}$. Let us recall its definition $w_0(t) = r_0(t) = 1$, and if $n \in \mathbb{N}$ can be expressed as

$$n = \sum_{i=0}^{k} \varepsilon_i 2^i, \qquad \text{where} \quad \varepsilon_k = 1, \quad \varepsilon_i = 0 \text{ or } 1 \quad \text{for } i = 0, 1, \dots, k-1,$$

then, following the Paley numbering,

$$w_n(t) = \prod_{i=0}^{k} (r_{i+1}(t))^{\varepsilon_i}.$$

It is well known (see, for example, [1, Statements 1.1.5 and 2.6.3]) that the sequence $\{w_n\}_{n=0}^{\infty}$ is a complete orthonormal system on the closed interval $[0, 1]$.

In this paper, we show that, while Rademacher chaos is not a lacunary sequence of Walsh functions in the ordinary sense if its order is greater than or equal to 2, it nevertheless possesses several properties resembling those of such sequences. In this connection, we shall consider properties related to the convergence a.e. of the series

$$\sum_{1 \le i_1 < \dots < i_d < \infty} a_{i_1,\dots,i_d} r_{i_1,\dots,i_d}(t), \tag{1}$$

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regarded as the convergence a.e. of the corresponding rectangular sums

$$S_n(t) := \sum_{1\le i_1<\dots<i_d\le n} a_{i_1,\dots,i_d} r_{i_1,\dots,i_d}(t).$$

For "lacunary" properties of Rademacher chaos related to integrability and the behavior in symmetric function spaces; see [2], [3]. Detailed information about different chaoses constructed from general systems of independent functions can be obtained, for example, in the monograph [4].

Most definitions and the notation used in what follows are taken from Gaposhkin's survey [5], which is concerned with the properties of lacunary systems of functions.

## (a) $\varepsilon$-UNIQUENESS SYSTEMS

**Definition 2.** Suppose that $\varepsilon > 0$. We say that a sequence $\{f_n\}_{n=1}^{\infty}$ of functions defined on $[0,1]$ is an *$\varepsilon$-uniqueness system* if the convergence of the series

$$\sum_{n=1}^{\infty} c_n f_n(t)$$

to zero on an arbitrary set $E \subset [0,1]$ whose Lebesgue measure satisfies the condition $m(E) > 1-\varepsilon$ implies that $c_n = 0$ for all $n = 1, 2, \dots$ .

In [6], Stechkin and Ul'yanov obtained the following result.

**Theorem A.** *Suppose that $\varphi(t)$ is an arbitrary finite function with period 1 satisfying the condition*

$$\varphi\left(t+\frac{1}{2}\right) - \varphi(t) \neq 0 \qquad \text{for almost all} \quad t \in [0,1]. \tag{2}$$

*Then if the series*

$$\sum_{n=1}^{\infty} a_n \varphi(2^n t)$$

*converges to a constant $C$ on a set $E \subset [0,1]$, $m(E) > 1/2$, then $C = 0$ and $a_n = 0$, $n = 1, 2, \dots$ .*

Hence, setting $\varphi(t) = r_1(t)$, we obtain, in particular, that the sequence of Rademacher functions $\{r_n\}_{n=1}^{\infty}$ is a $1/2$-uniqueness system. Moreover, the trivial example $r_1(t) + r_2(t) = 0$, where $t \in [1/4, 3/4]$, shows that it is not an $\varepsilon$-uniqueness system for an arbitrary $\varepsilon > 1/2$. Similar results were later proved for general lacunary sequences of Walsh functions.

Recall that the sequence $\{w_{n_k}\}_{k=1}^{\infty}$ is said to be *lacunary* (*in the sense of Hadamard*) if we have $n_{k+1}/n_k \ge q$ for some $q > 1$ and all $k = 1, 2, \dots$ . Here it is said that its *degree of lacunarity* is not less than $q$. It is well known (see, for example, [7, Sec. 11.1, p. 681]) that, for any given $q > 1$, an arbitrary lacunary system $\{w_{n_k}\}$ can be expressed as the finite union of pairwise disjoint subsystems, each having degree of lacunarity not less than $q$. The least number of such subsystems with degree of lacunarity not less than 2 is called the *2-lacunarity index* of the sequence $\{w_{n_k}\}$.

The following theorem was proved in [8].

**Theorem B.** *Suppose that $\{w_{n_k}\}$ is a lacunary sequence of Walsh functions whose 2-lacunarity index is $M$. Then if the series*

$$\sum_{k=1}^{\infty} a_k w_{n_k}(t)$$

*converges to zero* (*or to a constant*) *on a set $E \subset [0,1]$, $m(E) > 1 - 2^{-M}$, then $a_k = 0$ for all $k = 1, 2, \dots$ .*

In particular, it follows from Theorem B that the sequence $\{w_{n_k}\}$ that satisfies its assumptions is a $2^{-M}$-uniqueness system.

It is easy to verify that a Rademacher chaos of order $\geq 2$ cannot be expressed as a finite union of pairwise disjoint subsystems, each having degree of lacunarity less than some $q > 1$. Let us show this, for example, for the chaos of second order. Indeed, since in the Paley numbering

$$w_{2^n+2^k} = r_{n+1}r_{k+1}, \qquad n = 1, 2, \dots, \quad k = 0, 1, \dots, n-1$$

it contains arbitrarily long closed intervals of adjacent functions from the Walsh system and, therefore, cannot be expressed in this way. Nevertheless, for the Rademacher chaos we will prove an analog of Theorem B with the 2-lacunarity index replaced by the order of the chaos. The examples given below show that this result is sharp. In fact, in this direction, an even more general statement similar to Theorem A will be obtained.

## (b) SYSTEMS OF STRICT CONVERGENCE

We begin with the following definitions from [5, Sec. 1.2].

**Definition 3.** A system of functions $\{f_k\}_{k=1}^{\infty}$ defined on $[0, 1]$ is called a *system of convergence* if any series

$$\sum_{k=1}^{\infty} c_k f_k \tag{3}$$

converges a.e. on $[0, 1]$ provided that $(c_k) \in l_2$. It is said that a system $\{f_k\}_{k=1}^{\infty}$ is a *system of semistrict convergence in the wide* (*narrow*) *sense* if the convergence of the series (3) a.e. on $[0, 1]$ (respectively, on a set $E \subset [0, 1]$, $m(E) > 0$) implies the inclusion $(c_k) \in l_2$. If $\{f_k\}_{k=1}^{\infty}$ is both a system of convergence and a system of semistrict convergence in the wide (narrow) sense, then it is called a *system of strict convergence in the wide* (*narrow*) *sense*.

In view of the classical theorems due to Rademacher and Kolmogorov (see, for example, [9, Sec. 4.5]), the sequence $\{r_n\}_{n=1}^{\infty}$ is a system of strict convergence in the narrow sense. The question about what lacunary Walsh subsystems are systems of semistrict convergence in the wide (narrow) sense is considered in Sec. 2.2 of the survey [5]. In order to state the result given there, let us define certain classes of lacunary sequences of positive integers which are the natural generalization of the notion of lacunarity in the sense of Hadamard. First, in studying the Walsh system, the following operation of the addition in the group of nonnegative integers is often used: if

$$n = \sum_{k=0}^{l_1} a_k 2^k, \qquad m = \sum_{k=0}^{l_2} b_k 2^k,$$

where the coefficients $a_k$ and $b_k$ are either 0 or 1 and $a_{l_1} = b_{l_2} = 1$, then

$$n \uplus m := \sum_{k=0}^{\max(l_1,l_2)} |a_k - b_k| 2^k.$$

Let $D_2^{\uplus}$ denote the class of all subsequences $\{n_j\}_{j=1}^{\infty}$ of the positive integers such that there exists an integer $s \geq 1$, a number $A > 0$, and a decomposition of $\{n_j\}_{j=1}^{\infty}$ into groups $\Delta_i$ such that

a) for each $i = 1, 2, \dots$, $\Delta_i$ contains at most $s$ terms;

b) if $n_k \in \Delta_i$, $n_j \in \Delta_l$, and $i < l$, then $n_k < n_j$;

c) for any integer $\nu$, the number of solutions of the equation

$$n_k \uplus n_j = \nu, \qquad n_k \in \Delta_i, \quad n_j \in \Delta_l, \quad i \neq l, \quad i, l = 1, 2, \dots,$$

does not exceed $A$.

We also define the class $G_2^{\uplus}$ of all subsequences $\{n_j\}_{j=1}^{\infty}$ such that

$$\lim_{k\neq j,k,j\to\infty} n_k \uplus n_j = \infty.$$

The following result was stated in [5] (see Theorem 2.2.2).

**Theorem C.** *If $\{n_j\}_{j=1}^{\infty} \in D_2^{\uplus}$, then the Walsh subsystem $\{w_{n_i}\}_{i=1}^{\infty}$ is a system of semistrict convergence in the narrow sense if and only if $\{n_j\}_{j=1}^{\infty} \in G_2^{\uplus}$.*

Let us show that Rademacher chaos of second order does not belong to the class $D_2^{\uplus}$ and, therefore, the last theorem is not applicable to it.

Since again

$$w_{2^n+2^k} = r_{n+1}r_{k+1}, \qquad n = 1, 2, \dots, \quad k = 0, 1, \dots, n-1,$$

it suffices to verify that, for an arbitrary choice of the decomposition of the sequence

$$\{2^n + 2^k\}_{n=1,2,\dots;\, k=0,1,\dots,n-1}$$

(numbered in increasing order) into groups $\Delta_i$ satisfying conditions a) and b), but not c). Indeed, suppose that we are given an integer $s \geq 1$, a number $A > 0$, and a decomposition of this sequence satisfying conditions a) and b). Suppose that $\nu = 2^n + 2^{n+1}$, where $n > \max(s, A)$. Then the numbers

$$2^n + 2^k, \quad k = 0, 1, \dots, n-1, \qquad \text{and} \qquad 2^{n+1} + 2^k, \quad k = 0, 1, \dots, n-1,$$

lie, obviously, in different groups $\Delta_i$ and, in addition, for each $k = 0, 1, \dots, n-1$, we have

$$(2^n + 2^k) \uplus (2^{n+1} + 2^k) = \nu.$$

Therefore, the equation from condition c) has at least $n > A$ solutions, and hence this condition does not hold. It is also easy to verify that the sequence $\{2^n + 2^k\}_{n=1,2,\dots;\, k=0,1,\dots,n-1}$ does not belong to the class $G_2^{\uplus}$.

Nevertheless, further, we shall show that the Rademacher chaos of any fixed order is a system of strict convergence in the wide sense. At the same time, if its order is $\geq 2$, then it is not a system of strict convergence in the narrow sense (in contrast to the Rademacher sequence).

## 2. RADEMACHER CHAOS AS AN $\varepsilon$-UNIQUENESS SYSTEM

The following statement generalizes Theorem A, which was proved in [6].

**Theorem 1.** *Suppose that $\varphi(t)$ is an arbitrary finite function with period 1 satisfying condition (2). If the series*

$$\sum_{0\leq i_1<\dots<i_d<\infty} a_{i_1,\dots,i_d}\varphi(2^{i_1}t)\cdots\varphi(2^{i_d}t) \tag{4}$$

*converges to a constant $C$ on a set $E \subset [0, 1]$, $m(E) > 1 - 2^{-d}$, then $C = 0$ and $a_{i_1,\dots,i_d} = 0$ for arbitrary $0 \leq i_1 < \dots < i_d$.*

**Proof.** The proof is carried out by induction on $d$. If $d = 1$, then the statement coincides with Theorem A. Suppose that the theorem is proved for those chaoses whose order is less than $d$; let us prove it for the chaos of order $d$.

If we assume that the statement is not valid, then there exists an index $i_1^0$ possessing the following property: in the appropriate hyperrow, there is at least one nonzero coefficient, while, in all the previous hyperrows, all the coefficients are 0. In other words,

$$a_{i_1^0,i_2,\dots,i_d} \neq 0 \quad \text{for some } 0 \leq i_1^0 < i_2 < \dots < i_d, \qquad a_{i_1,\dots,i_d} = 0 \quad \text{for all } 0 \leq i_1 < i_2 < \dots < i_d,$$

provided that $0 \leq i_1 < i_1^0$.

We define the operation $x \oplus y = x + y \,(\mathrm{mod}\, 1)$, $x, y \in [0,1]$. Since $\varphi$ is periodic, we have

$$\varphi(t_1 \oplus t_2) = \varphi(t_1 + t_2), \qquad t_1, t_2 \in [0,1].$$

Consider the set $\widetilde{E}$ of all points $t \in E$ such that the point $t \oplus 2^{-i_1^0-1}$ also belongs to the set $E$. It follows from the condition that its measure is

$$m(\widetilde{E}) = m(E \cap (E \oplus 2^{-i_1^0-1})) > 1 - 2^{-d+1}.$$

Suppose that the point $t \in \widetilde{E}$ is arbitrary. Then if $f(t)$ is the sum of the series (4), then, on the one hand, by assumption,

$$f(t \oplus 2^{-i_1^0-1}) - f(t) = C - C = 0, \tag{5}$$

while, on the other,

$$\begin{aligned} &f(t \oplus 2^{-i_1^0-1}) - f(t) \\ &\quad= \sum_{i_1^0 \le i_1 < \dots < i_d} a_{i_1,\dots,i_d} \varphi(2^{i_1}(t \oplus 2^{-i_1^0-1})) \cdots \varphi(2^{i_d}(t \oplus 2^{-i_1^0-1})) \\ &\qquad\quad - \sum_{i_1^0 \le i_1 < \dots < i_d} a_{i_1,\dots,i_d} \varphi(2^{i_1}t) \cdots \varphi(2^{i_d}t) \\ &\quad= \sum_{i_1^0 \le i_1 < \dots < i_d} a_{i_1,\dots,i_d} (\varphi(2^{i_1}(t \oplus 2^{-i_1^0-1})) \cdots \varphi(2^{i_d}(t \oplus 2^{-i_1^0-1})) - \varphi(2^{i_1}t) \cdots \varphi(2^{i_d}t)). \end{aligned} \tag{6}$$

For $i > i_1^0$, we have

$$\varphi(2^i(t \oplus 2^{-i_1^0-1})) = \varphi(2^i t \oplus 2^{i-i_1^0-1}) = \varphi(2^i t).$$

Therefore, in the sum on the right-hand side of (6), nonzero summands may appear only in the hyperrow with index $i_1^0$, and hence, in view of (5) and (6),

$$\sum_{i_1^0 < i_2 < \dots < i_d} a_{i_1^0, i_2, \dots, i_d} \varphi(2^{i_2}t) \cdots \varphi(2^{i_d}t) \cdot (\varphi(2^{i_1^0}t + 1/2) - \varphi(2^{i_1^0}t)) = 0.$$

Combining this with (2), we see that the equality

$$\sum_{i_1^0 < i_2 < \dots < i_d} a_{i_1^0, i_2, \dots, i_d} \varphi(2^{i_2}t) \cdots \varphi(2^{i_d}t) = 0$$

holds a.e. for $t \in \widetilde{E}$. Since the left-hand side of this equality contains the series corresponding to the chaos of order $d-1$, it follows by the induction hypothesis that all of its coefficients $a_{i_1^0, i_2, \dots, i_d}$ must be zero. This contradicts the choice of the index $i_1^0$, and hence the theorem is proved. $\square$

Setting $\varphi(t) = r_1(t)$, we obtain, as a particular case, the following result for Rademacher chaoses.

**Corollary 1.** *If the series* (1) *converges to a constant $C$ on a set $E \subset [0,1]$, $m(E) > 1 - 2^{-d}$, then $C = 0$ and $a_{i_1,\dots,i_d} = 0$ for all $0 \le i_1 < i_2 < \dots < i_d$. In particular, the Rademacher chaos of order $d$ is a system of $2^{-d}$-uniqueness.*

The following natural question arises: Does there exist a series with respect to the Rademacher chaos of order $d$ whose coefficients are not all zero and which converges to zero on a set $E \subset [0,1]$ with measure $m(E) = 1 - 2^{-d}$? The answer to this question is positive; let us present an example.

**Example 1.** For an arbitrary $d \in \mathbb{N}$, consider the product

$$f_d(t) = \prod_{i=1}^{d}(r_{2i-1}(t) + r_{2i}(t)). \tag{7}$$

First, obviously, after removing the parentheses, we obtain a series with respect to the Rademacher chaos of order $d$. Further, since the functions $r_i(t)$ are independent on $[0,1]$ with respect to Lebesgue measure and

$$m\{t \in [0,1] : r_{2i-1}(t) + r_{2i}(t) \neq 0\} = \frac{1}{2}, \qquad i = 1, 2, \dots,$$

it follows that

$$m\{t \in [0,1] : f_d(t) \neq 0\} = \prod_{i=1}^{d} m\{t \in [0,1] : r_{2i-1}(t) + r_{2i}(t) \neq 0\} = 2^{-d},$$

whence we have

$$m\{t \in [0,1] : f_d(t) = 0\} = 1 - 2^{-d}.$$

Theorem 1 and Corollary 1 can be generalized to the case of any system which is the union of Rademacher chaoses of all orders not exceeding some $d \in \mathbb{N}$.

Suppose that a function $\varphi(t)$ satisfies the conditions of Theorem 1. Let us consider series of the form

$$\sum_{0 \le i_1 < \dots < i_d < \infty} (a_{i_1}\varphi(2^{i_1}t) + a_{i_1,i_2}\varphi(2^{i_1}t)\varphi(2^{i_2}t) + \dots + a_{i_1,\dots,i_d}\varphi(2^{i_1}t)\cdots\varphi(2^{i_d}t)), \tag{8}$$

whose convergence is again regarded as the convergence of the corresponding rectangular sums.

The proof of the following theorem is practically the same as the proof of Theorem 1 and, therefore, it is not given here.

**Theorem 2.** *If the series* (8) *converges to a constant $C$ on a set $E \subset [0,1]$ with measure $m(E) > 1 - 2^{-d}$, then $C = 0$ and*

$$a_{i_1,\dots,i_k} = 0 \qquad \textit{for all} \quad 0 \le i_1 < i_2 < \dots < i_k, \quad 1 \le k \le d.$$

**Corollary 2.** *If a series*

$$\sum_{1 \le i_1 < \dots < i_d < \infty} (a_{i_1}r_{i_1}(t) + a_{i_1,i_2}r_{i_1,i_2}(t) + \dots + a_{i_1,\dots,i_d}r_{i_1,\dots,i_d}(t)) \tag{9}$$

*converges to a constant $C$ on a set $E \subset [0,1]$ with measure $m(E) > 1 - 2^{-d}$, then $C = 0$ and $a_{i_1,\dots,i_k} = 0$ for all $0 \le i_1 < i_2 < \dots < i_k$, $1 \le k \le d$. In particular, the union of Rademacher chaoses of all orders not exceeding some $d \in \mathbb{N}$ is a system of $2^{-d}$-uniqueness.*

Of course, Example 1 already shows the validity of Corollary 2. Let us present another example involving functions of all chaoses whose order does not exceed a fixed number $d \in \mathbb{N}$.

**Example 2.** For an arbitrary $d \in \mathbb{N}$, consider the function

$$g_d(t) = \prod_{i=1}^{d}(1 + r_i(t)) - 1.$$

It is easy to see that, after removing the parentheses, we obtain a linear combination of functions from Rademacher chaoses of order $\le d$. In addition, in view of the independence of Rademacher functions,

$$m\{t \in [0,1] : g_d(t) \neq -1\} = \prod_{i=1}^{d} m\{t \in [0,1] : 1 + r_i(t) \neq 0\} = 2^{-d},$$

whence it follows that

$$m\{t \in [0,1] : g_d(t) = -1\} = 1 - 2^{-d}.$$

**Remark 1.** Similar results are also valid for series of the form

$$\sum_{k=1}^{d} \sum_{0\le i_1<\dots<i_k<\infty} a_{i_1,\dots,i_k}\varphi(2^{i_1}t)\cdots\varphi(2^{i_k}t),$$

$$\sum_{k=1}^{d} \sum_{1\le i_1<\dots<i_k<\infty} a_{i_1,\dots,i_k} r_{i_1,\dots,i_k}(t),$$

respectively. In contrast to the series (8) and (9), we assume here that not only the whole sum, but also the sums corresponding to chaoses of all orders are convergent.

**Remark 2.** As was already noted in the Introduction, Corollaries 1 and 2 do not follow from Theorem B proved in [8], because the Rademacher chaos of any order $\ge 2$ is not a lacunary (in the sense of Hadamard) sequence of Walsh functions (and cannot even be expressed as a finite union of such sequences).

## 3. RADEMACHER CHAOS AS A SYSTEM OF STRICT CONVERGENCE

Let us show that Rademacher chaos is a system of strict convergence in the wide sense and is not a system of strict convergence in the narrow sense. The first of these assertions is a direct consequence of following theorem.

**Theorem 3.** *If the series* (1) *converges in measure on* $[0,1]$*, then*

$$\sum_{1\le i_1<\dots<i_d<\infty} a_{i_1,\dots,i_d}^2 < \infty. \tag{10}$$

**Proof.** First, we apply the decoupling idea, passing to the so-called "decomposed" chaos. Namely, in view of [4, Theorems 6.9.2 and 6.3.1], there exist positive constants $k_1$, $k_2$, $K_1$, and $K_2$ depending only on the order $d$ of the chaos such that, for any $z>0$ and any positive integers $n<m$, the following inequalities hold:

$$\begin{aligned}
K_1 m\Biggl\{t\in[0,1]:\Biggl|\sum_{i_d=n}^{m}\sum_{1\le i_1<i_2<\dots<i_{d-1}<i_d} a_{i_1,\dots,i_d} r_{i_1,\dots,i_d}(t)\Biggr| > k_1 z\Biggr\}\\
\le m_d\Biggl\{(t_1,\dots,t_d)\in[0,1]^d:\Biggl|\sum_{i_d=n}^{m}\sum_{1\le i_1<\dots<i_{d-1}<i_d} a_{i_1,\dots,i_d} r_{i_1}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\Biggr| > z\Biggr\}\\
\le K_2 m\Biggl\{t\in[0,1]:\Biggl|\sum_{i_d=n}^{m}\sum_{1\le i_1<i_2<\dots<i_{d-1}<i_d} a_{i_1,\dots,i_d} r_{i_1,\dots,i_d}(t)\Biggr| > k_2 z\Biggr\},
\end{aligned}$$

where $m_d$ is the $d$-dimensional Lebesgue measure. Since, by assumption, the series (1) converges in measure on $[0,1]$, this implies that the series

$$\sum_{1\le i_1<\dots<i_{d-1}<i_d<\infty} a_{i_1,\dots,i_d} r_{i_1}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d) \tag{11}$$

converges in measure on the cube $[0,1]^d$. Let us show that inequality (10) holds. To this end, we again apply induction on the order $d$ of the chaos. If $d=1$, then, in view of the independence of Rademacher functions, the corresponding series converges a.e. on $[0,1]$ [10, Sec. 4.2] and, therefore, the required result immediately follows from the well-known Kolmogorov theorem [9, Theorem 4.5.3]. Suppose that the convergence in measure on the cube $[0,1]^{d-1}$ of each series of the form (11) (with $d$ replaced by $d-1$) implies the quadratic summability of the sequence of its coefficients. Let us prove a similar assertion for the chaos of order $d$.

First, note that, by Fubini's theorem, the convergence of the series (11) in measure on the cube $[0,1]^d$ is equivalent to the fact that, for any $\alpha > 0$,

$$\lim_{\substack{n,m\to\infty\\ m>n}} \int_0^1 m_{d-1}\Big\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \Big|\sum_{i_d=n}^{m} \sum_{1\le i_1<\dots<i_{d-1}<i_d} a_{i_1,\dots,i_d} r_{i_1}(t_1)\dots r_{i_d}(t_d)\Big| > \alpha\Big\}\,dt_1 = 0. \tag{12}$$

Let us prove that then, for almost all $t_1 \in [0,1]$, the series (11) converges in measure on $[0,1]^{d-1}$.

Suppose that this is not the case. Then, denoting

$$v_{i_2,\dots,i_d}(t_1) := \sum_{i_1=1}^{i_2-1} a_{i_1,\dots,i_d} r_{i_1}(t_1), \qquad 1 < i_2 < \dots < i_d,$$

we find a set $E \subset [0,1]$, $m(E) > 0$, such that, for each $t_1 \in E$ and some $p = p(t_1) \in \mathbb{N}$ and $q = q(t_1) \in \mathbb{N}$, the following inequality holds:

$$\limsup_{\substack{n,m\to\infty\\ m>n}} m_{d-1}\Big\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \Big|\sum_{i_d=n}^{m} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\Big| > \frac{1}{p}\Big\} > \frac{1}{q}.$$

Standard arguments show that, in this case, there exists a set $F \subset [0,1]$, $m(F) > 0$, and constants $\beta > 0$, $\varepsilon > 0$ such that, for all $t_1 \in F$,

$$\limsup_{\substack{n,m\to\infty\\ m>n}} m_{d-1}\Big\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \Big|\sum_{i_d=n}^{m} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\Big| > \beta\Big\} > \varepsilon. \tag{13}$$

Then, for any $t_1 \in F$, there exists a $N_1(t_1) \in \mathbb{N}$ for which

$$m_{d-1}\Big\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \Big|\sum_{i_d=1}^{N_1(t_1)} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\Big| > \frac{\beta}{2}\Big\} > \frac{\varepsilon}{2}. \tag{14}$$

Indeed, in view of (13), there exists an $n_1 < m_1$ such that

$$m_{d-1}\Big\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \Big|\sum_{i_d=n_1}^{m_1} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\Big| > \beta\Big\} > \varepsilon.$$

Since

$$\begin{aligned}
m_{d-1}\Big\{(t_2,\dots,t_d) \in [0,1]^{d-1} &: \Big|\sum_{i_d=n_1}^{m_1} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\Big| > \beta\Big\} \\
&\le m_{d-1}\Big\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \Big|\sum_{i_d=1}^{m_1} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\Big| > \frac{\beta}{2}\Big\} \\
&\quad + m_{d-1}\Big\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \Big|\sum_{i_d=1}^{n_1} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\Big| > \frac{\beta}{2}\Big\},
\end{aligned}$$

it follows that at least one of the summands on the right-hand side is greater than $\varepsilon/2$. Choosing $N_1(t_1)$ in an appropriate way, we obtain (14).

Further, suppose that $N_1 \in \mathbb{N}$ satisfies

$$m\{t_1 \in F : N_1(t_1) > N_1\} \le m(F)2^{-2}.$$

Then $F_1 := \{t_1 \in F : N_1(t_1) \le N_1\}$ implies $m(F_1) \ge (3/2^2)m(F)$ and, for each $t_1 \in F_1$ and some $N_1(t_1) \le N_1$, inequality (14) holds. Similarly, again using (13), we can show that, for any $t_1 \in F_1$ and some $N_2(t_1) \ge N_1$,

$$m_{d-1}\biggl\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \biggl|\sum_{i_d=N_1}^{N_2(t_1)} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\biggr| > \frac{\beta}{2}\biggr\} > \frac{\varepsilon}{2}. \quad (15)$$

Choose $N_2 > N_1$ so that

$$m\{t_1 \in F_1 : N_2(t_1) > N_2\} \le m(F)2^{-3}.$$

Now if

$$F_2 := \{t_1 \in F_1 : N_2(t_1) \le N_2\},$$

then $m(F_2) \ge (5/2^3)m(F)$ and, for each $t_1 \in F_2$ and some $1 \le N_1(t_1) \le N_1 \le N_2(t_1) \le N_2$, relations (14) and (15) hold. Using similar arguments, we find the set

$$F_\infty := \bigcap_{i=1}^{\infty} F_i \qquad \text{for which} \quad m(F_\infty) \ge m(F)/2 > 0$$

and find numbers $N_0 := 1 < N_1 < N_2 < N_3 < \cdots$ such that, for any $t_1 \in F_\infty$, there exists a sequence $\{N_i(t_1)\}_{i=1}^{\infty}$ satisfying the conditions $N_{i-1} \le N_i(t_1) \le N_i$, $i = 1, 2, \dots$, and

$$m_{d-1}\biggl\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \biggl|\sum_{i_d=N_i}^{N_{i+1}(t_1)} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\biggr| > \frac{\beta}{2}\biggr\} > \frac{\varepsilon}{2} \quad (16)$$

for $i = 0, 1, 2, \dots$ .

Further, using Fubini's theorem as well as the well-known Kwapień–Rychlik inequality [11, Theorem 5.4.4], for arbitrary $b_{i_2,\dots,i_d} \in \mathbb{R}$, $z > 0$, and natural numbers $n < m < p$, we obtain

$$\begin{aligned}
&m_{d-1}\biggl\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \biggl|\sum_{i_d=n}^{m} \sum_{1<i_2<\dots<i_{d-1}<i_d} b_{i_2,\dots,i_d} r_{i_2}(t_2)\dots r_{i_d}(t_d)\biggr| > z\biggr\}\\
&\quad= \int_0^1 \dots \int_0^1 m\biggl\{t_d \in [0,1] : \biggl|\sum_{i_d=n}^{m} \sum_{1<i_2<\dots<i_{d-1}<i_d} b_{i_2,\dots,i_d} r_{i_2}(t_2)\dots r_{i_d}(t_d)\biggr| > z\biggr\}\, dt_2\dots dt_{d-1}\\
&\quad\le 2\int_0^1 \dots \int_0^1 m\biggl\{t_d \in [0,1] : \biggl|\sum_{i_d=n}^{p} \sum_{1<i_2<\dots<i_{d-1}<i_d} b_{i_2,\dots,i_d} r_{i_2}(t_2)\dots r_{i_d}(t_d)\biggr| > z\biggr\}\, dt_2\dots dt_{d-1}\\
&\quad= 2m_{d-1}\biggl\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \biggl|\sum_{i_d=n}^{p} \sum_{1<i_2<\dots<i_{d-1}<i_d} b_{i_2,\dots,i_d} r_{i_2}(t_2)\dots r_{i_d}(t_d)\biggr| > z\biggr\}.
\end{aligned}$$

Therefore, since $N_i(t_1) \le N_i$, $i = 1, 2, \dots$, in view of (16), it follows that, for each $t_1 \in F_\infty$,

$$m_{d-1}\biggl\{(t_2,\dots,t_d) \in [0,1]^{d-1} : \biggl|\sum_{i_d=N_i}^{N_{i+1}} \sum_{1<i_2<\dots<i_{d-1}<i_d} v_{i_2,\dots,i_d}(t_1) r_{i_2}(t_2)\dots r_{i_d}(t_d)\biggr| > \frac{\beta}{2}\biggr\} > \frac{\varepsilon}{4},$$

where $i = 0, 1, 2, \dots$, whence

$$\int_0^1 m_{d-1}\Big\{(t_2,\dots,t_d)\in[0,1]^{d-1} : \Big|\sum_{i_d=N_i}^{N_{i+1}} \sum_{1\le i_1<\dots<i_{d-1}<i_d} a_{i_1,\dots,i_d} r_{i_1}(t_1)\dots r_{i_d}(t_d)\Big| > \frac{\beta}{2}\Big\}\,dt_1$$
$$\ge \int_{F_\infty} m_{d-1}\Big\{(t_2,\dots,t_d)\in[0,1]^{d-1} : \Big|\sum_{i_d=N_i}^{N_{i+1}} \sum_{1\le i_1<\dots<i_{d-1}<i_d} a_{i_1,\dots,i_d} r_{i_1}(t_1)\dots r_{i_d}(t_d)\Big| > \frac{\beta}{2}\Big\}\,dt_1$$
$$\ge \frac{\varepsilon}{4} m(F_\infty) \ge \frac{\varepsilon}{8} m(F) > 0, \qquad i = 0, 1, 2, \dots .$$

Since this contradicts (12), it follows that our initial assumption is false, and hence, for almost all $t_1 \in [0,1]$, the series (11) converges in measure on $[0,1]^{d-1}$.

Note that, for each fixed $t_1 \in [0,1]$, the series (11) is constructed with respect to the Rademacher chaos of order $d-1$. Therefore, by the induction hypothesis, for $t_1 \in [0,1]$ a.e.,

$$\sum_{1<i_2<\dots<i_{d-1}<i_d<\infty} \Big(\sum_{i_1=1}^{i_2-1} a_{i_1,\dots,i_d} r_{i_1}(t_1)\Big)^2 < \infty.$$

Thus, for an arbitrarily small $\delta > 0$, there exists a set $G \subset [0,1]$, $m(G) \ge 1-\delta$, such that, for all $t_1 \in G$ and some $K > 0$,

$$\sum_{1<i_2<\dots<i_{d-1}<i_d<\infty} \Big(\sum_{i_1=1}^{i_2-1} a_{i_1,\dots,i_d} r_{i_1}(t_1)\Big)^2 \le K,$$

which after integration over $G$, we obtain

$$\sum_{1<i_2<\dots<i_{d-1}<i_d<\infty} \int_G \Big(\sum_{i_1=1}^{i_2-1} a_{i_1,\dots,i_d} r_{i_1}(t_1)\Big)^2 dt_1 \le K m(G).$$

At the same time, it is well known (see, for example, [12, Lemma 2.3.1]) that the number $\delta > 0$ can be chosen small so that, for an absolute constant $c_0 > 0$ and all $(b_i)_{i=1}^\infty \in l_2$, the following inequality holds:

$$\int_G \Big(\sum_{i=1}^{\infty} b_i r_i(s)\Big)^2 ds \ge c_0 \|(b_i)\|_{l_2}^2.$$

As a result, the last two relations imply

$$\sum_{1\le i_1<\dots<i_d<\infty} a_{i_1,\dots,i_d}^2 = \sum_{1\le i_1<i_2<\dots<i_{d-1}<i_d} \sum_{i_1=1}^{i_2-1} a_{i_1,\dots,i_d}^2 \le K c_0^{-1} m(G) < \infty,$$

and the theorem is proved. □

**Corollary 3.** *The Rademacher chaos $\{r_{i_1,i_2,\dots i_d}\}_{1\le i_1<i_2<\dots<i_d<\infty}$ is a system of strict convergence in the wide sense, but not in the narrow sense.*

**Proof.** First, in view of the analog of Carleson's theorem for the Walsh system (see, for example, [1, Theorem 9.2.1]), Rademacher chaos of any order is a system of convergence. Applying Theorem 3, we find that, moreover, it is a system of strict convergence in the wide sense. At the same time, consider the following series constructed with respect to the Rademacher chaos of order $d$:

$$\sum_{k=2d}^{\infty} c_k f_{d-1}(t) r_k(t), \qquad \text{where} \quad f_{d-1}(t) := \prod_{i=1}^{d-1} (r_{2i-1}(t) + r_{2i}(t)).$$

Here $\{c_k\}_{k=1}^{\infty}$ is an arbitrary sequence such that

$$\lim_{k\to\infty} c_k = 0 \qquad \text{and} \qquad \sum_{k=1}^{\infty} c_k^2 = \infty.$$

Since the Rademacher functions are independent, we have $f_{d-1}(t) = 0$ for all $t$ from some set $E$ of measure $1 - 2^{-d+1}$. Thus, this series converges to zero on $E$ as a series formed by some Walsh subsystem (and all the more in the sense of convergence of rectangular sums). Thus, Rademacher chaos is not a system of strict convergence in the narrow sense. □

**Remark 3.** In the case $d = 2$, the first of the assertions of Corollary 3 also follows from [4, Corollary 6.7.2].

## 4. ADDITIONAL REMARKS

The results obtained here show that Rademacher chaos indeed possesses "lacunary" properties. At the same time, its "lacunarity" is not embraced by the classical definitions: as was already stated (see the Introduction), the Rademacher chaos of fixed order $\geq 2$ does not belong to the class of lacunary sequences of Walsh functions even in the case where these functions are understood in a sufficiently loose way. In this connection, an approach to lacunarity based on the notion of combinatorial dimension, which was introduced and studied in the monograph [13], may be of some interest. Let us present some definitions and notation contained in this monograph as well as in [14].

For arbitrary $d \in \mathbb{N}$ and $N \in \mathbb{N}$, we denote

$$\mathbb{N}^d := \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{d}, \qquad [N] := \{1, 2, \dots, N\}.$$

**Definition 4.** Suppose that $F \in \mathbb{N}^d$ and $F_N := F \cap [N]^d$, $(N \in \mathbb{N})$. We say that the set $F$ has *combinatorial dimension* $\alpha$ (written $\dim F = \alpha$) if there exists an $N_0 \in \mathbb{N}$ and positive constants $C_1$ and $C_2$ such that, for all $N \geq N_0$, the following inequality holds:

$$\operatorname{card}(F_N \cap (A_1 \times \cdots \times A_d)) \leq C_1 \Big(\max_{1\leq j\leq d} \operatorname{card} A_j\Big)^{\alpha},$$

where the sets $A_j \subset [N]$ $(j = 1, 2, \dots, d)$ are arbitrary and

$$\operatorname{card} F_N \geq C_2 N^{\alpha}.$$

Further, we restrict ourselves to the consideration of the sets

$$\Delta^d = \{(i_1, \dots, i_d) : 1 \leq i_1 < \cdots < i_d < \infty\}, \qquad d \in \mathbb{N},$$

which are the index sets for the chaoses. As proved in [13, Chap. XIII], for arbitrary $d \in \mathbb{N}$ and $\alpha \in [1, d]$, there exists a set $F \subset \Delta^d$ whose combinatorial dimension is $\alpha$.

Let us also define a set $\Delta^{\infty} \subset \mathbb{N}^{\infty}$ consisting of all sequences $(i_1, i_2, \dots, i_n, \dots)$ such that, for some $d \in \mathbb{N}$ (where $d$ depends on the sequence), we have $1 \leq i_1 < \cdots < i_d$ and $i_k = 0$ if $k > d$. Then an arbitrary set $E$ such that

$$E \subset \bigcup_{j=1}^{m} \Delta^{d_j}$$

can be regarded, in a natural way, as a subset of $\Delta^{\infty}$. In addition, it is easy to verify that

$$\dim\Big\{\bigcup_{k=1}^{d} \Delta^d\Big\} = \dim \Delta^d = d.$$

This gives rise to the following, apparently, natural conjecture: if the combinatorial dimension of a set $F \subset \Delta^{\infty}$ is $\alpha$, then the corresponding subsequence of Walsh functions $\{r_{i_1,\dots,i_d}\}_{(i_1,\dots,i_d)\in F}$ is a system of $2^{-\alpha}$-uniqueness. Nevertheless, the following simple example shows that, at least, in such a strong form, the above statement is not valid.

**Example 3.** Suppose that $F = \mathbb{N} \cup \{(n, n+1) : n \in \mathbb{N}\}$. It is easy to verify that $\dim F = 1$. At the same time, the product

$$f(t) := (1 - r_2(t))(r_1(t) + r_3(t))$$

is, on the one hand, a linear combination of the Walsh functions corresponding to the set $F$, but, on the other hand, $f(t) = 0$ for all $t \in E$ with measure $m(E) = 3/4$.

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