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On Probability Analogs of Rosenthal's Inequality

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Abstract—We obtain probability combinatorial inequalities for independent random variables, strengthening the well-known Rosenthal inequality. As a corollary, we prove that the generalized Rosenthal inequality for identically distributed independent functions remains valid in the case of quasinormed symmetric spaces.

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1. INTRODUCTION

For any $n \in \mathbb{N}$ and an arbitrary sequence of functions $\{f_k\}_{k=1}^n$, measurable on $[0,1]$, we consider the corresponding "disjunctive sum"

$$F(t) := \sum_{k=1}^{n} f_k(t-k+1)\chi_{(k-1,k]}(t), \qquad t > 0. \tag{1}$$

As we can easily see,

$$m\{t > 0 : |F(t)| > \tau\} = \sum_{k=1}^{n} m\{t \in [0,1] : |f_k(t)| > \tau\}, \qquad \tau > 0, \tag{2}$$

where m is the Lebesgue measure. Let $F^*(t)$ be a nonincreasing left-continuous rearrangement of the function $F(t)$ (for the definition, see below), and let $\chi_{[0,1]}$ be, as is customary, the characteristic function of the interval $[0,1]$. Then if X is an arbitrary quasinormed symmetric space on $[0,1]$, then, for any $n \in \mathbb{N}$ and each sequence of nonnegative independent functions $\{f_k\}_{k=1}^n \subset X$, the following inequality holds [1, Theorem 1]:

$$\|F^*\chi_{[0,1]}\|_X + \sum_{k=1}^{n} F^*(k) \le C_X \left\| \sum_{k=1}^{n} f_k \right\|_X. \tag{3}$$

In the classical case $X = L_p$, $(1 \le p < \infty)$, inequality (3) was proved in the remarkable paper [2] of Rosenthal and, therefore, it is often referred to as *Rosenthal's inequality*. At this point, let us make two remarks. First, the reverse inequality of (3) is not always valid; it holds only for symmetric spaces sufficiently "remote" from the space L_∞ (for details, see [1], [3], and [4]). Second, a similar relation for the sequence of symmetrically distributed independent functions in L_p was obtained in [2] (see also [5, Theorem 2] and [6]). Generalizations of this version of Rosenthal's inequality to arbitrary symmetric space are contained in [1] and [7].

The proof of relation (3) is reduced to the proof of two inequalities:

$$\|F^*\chi_{[0,1]}\|_X \le A_X \left\| \sum_{k=1}^{n} f_k \right\|_X,$$

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$$\sum_{k=1}^{n} F^*(k) \le B_X \Big\| \sum_{k=1}^{n} f_k \Big\|_X. \tag{4}$$

The first of them is a direct consequence of the well-known probability inequality (see, for example, [8, Proposition 2.1])

$$\begin{aligned}\frac{1}{2} m\{t \in [0,1] : F^*(t)\chi_{[0,1]}(t) > \tau\} &\le m\Big\{t \in [0,1] : \max_{k=1,2,\dots,n} f_k(t) > \tau\Big\} \\ &\le m\{t \in [0,1] : F^*(t)\chi_{[0,1]}(t) > \tau\}, \qquad \tau > 0,\end{aligned}$$

as well as of the definition of symmetric space. As to inequality (4), its proof in [1] was based on the use of the properties of quasinormed symmetric spaces and, in particular, on the application of Nikishin's factorization theorem [9]. The more general inequality

$$\Big\| \sum_{k=1}^{n} F^*(k) e_k \Big\|_E \le D_{X,E} \Big\| \Big\| \sum_{k=1}^{n} f_k e_k \Big\|_E \Big\|_X, \tag{5}$$

where E is an arbitrary symmetric space of sequences and $\{e_k\}_{k=1}^{\infty}$ is the standard basis in E, was proved in [10] (see also [11]) for any $n \in \mathbb{N}$ and an arbitrary sequence of independent functions $\{f_k\}_{k=1}^{n} \subset X$ for the case in which X and E are Banach symmetric spaces. In both these papers, the key element of the proof of inequality (5) is the use of duality, which is absent in the case of quasinormed spaces and, therefore, this proof, cannot, in principle, be generalized to such a case.

The main aim of this paper is to prove that, in fact, inequalities of types (4) and (5) are a consequence of some probabilistic lower bounds and, therefore, they are of purely probabilistic combinatorial nature. In particular, this approach allows us to prove (5) in the case of identically distributed functions for arbitrary *quasinormed* symmetric spaces E and X.

A great number of probability-theoretic papers are devoted to lower bounds for distributions (see, for example, [5], [6], [12]–[17]). Note, however, that the relations given here are of specific nature. First, they compare sums of *nonnegative* random variables with the corresponding "disjunctive sums" and act in the domain of normal deviations (in contrast, e.g., to estimates from [6], [13], and [16] acting in the domain of large and moderate deviations). Second, these estimates are oriented almost exclusively toward the functional-theoretic applications under consideration here.

Suppose that $x(t)$ is Lebesgue measurable on the semiaxis $(0,\infty)$. Then the function $|x(t)|$ is *equimeasurable* with its nonincreasing left-continuous *rearrangement*

$$x^*(t) = \inf\big\{\tau \ge 0 : m\{s \in [0,1] : |x(s)| > \tau\} < t\big\}, \qquad t > 0,$$

i.e.,

$$m\{t > 0 : |x(t)| > \tau\} = m\{t > 0 : x^*(t) > \tau\} \qquad \text{for all} \quad \tau > 0.$$

The Banach space X whose elements are Lebesgue measurable functions on $[0,1]$ is said to be *symmetric* if

1) from $|y(t)| \le |x(t)|$ a.e. and $x \in X$, it follows that $y \in X$ and $\|y\|_X \le \|x\|_X$;

2) from the fact that the functions $|x(t)|$ and $|y(t)|$ are equimeasurable on $[0,1]$ and the fact that $x \in X$, it follows that $y \in X$ and $\|y\|_X = \|x\|_X$.

If, instead of the triangle inequality, the following inequality holds in X:

$$\|x + y\|_X \le L(\|x\|_X + \|y\|_X)$$

with some $L > 1$, then the space X is called a *quasi-Banach* symmetric space. Banach and quasi-Banach symmetric spaces of sequences are defined similarly. More information about symmetric spaces can be obtained from the monographs [18] and [19].

By $[a]$ we further denote the integer part of a real number a.

2. LOWER BOUNDS FOR DISCRETE DISTRIBUTIONS

Let us begin with estimates for the sums of arbitrarily distributed nonnegative independent discrete random variables taking the same values.

Let $n \in \mathbb{N}$, and let $(a_{i,j})_{i,j=1}^n$ be an arbitrary bistochastic matrix, i.e.,

$$0 \le a_{i,j} \le 1, \qquad \sum_{i=1}^n a_{i,j} = 1, \quad j = 1, 2, \dots, n, \qquad \sum_{j=1}^n a_{i,j} = 1, \quad i = 1, 2, \dots, n.$$

For an arbitrary vector

$$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \qquad x_1 \ge x_2 \ge \dots \ge x_n \ge 0,$$

consider the collection of independent random variables ξ_i such that

$$\mathsf{P}\{\xi_i = x_j\} = a_{i,j}, \qquad i, j = 1, 2, \dots, n.$$

Proposition 1. *If $\xi := \sum_{i=1}^n \xi_i$, then*

$$\mathsf{P}\left\{\xi \ge \frac{1}{2}\sum_{i=1}^n x_i\right\} \ge \frac{1}{8}. \tag{6}$$

Proof. If $\mathsf{E}\xi$ is the expectation of ξ, then

$$\mathsf{E}\xi = \sum_{i=1}^n \sum_{j=1}^n x_j a_{i,j} = \sum_{j=1}^n x_j \sum_{i=1}^n a_{i,j} = \sum_{j=1}^n x_j,$$

and hence inequality (6) is equivalent to the following one:

$$\mathsf{P}\left\{\xi \ge \frac{1}{2}\mathsf{E}\xi\right\} \ge \frac{1}{8}. \tag{7}$$

In view of the Paley–Zygmund inequality [20, p. 19 (Russian transl.)],

$$\mathsf{P}\left\{\xi \ge \frac{1}{2}\mathsf{E}\xi\right\} \ge \left(1 - \frac{1}{2}\right)^2 \frac{(\mathsf{E}\xi)^2}{\mathsf{E}(\xi^2)} = \frac{1}{4}\frac{(\mathsf{E}\xi)^2}{\mathsf{E}(\xi^2)}. \tag{8}$$

Since the quantities ξ_i, $i = 1, 2, \dots, n$, are independent, we have

$$\begin{aligned}
\mathsf{E}(\xi^2) &= \sum_{i=1}^n \mathsf{E}(\xi_i^2) + \sum_{i \ne j} \mathsf{E}\xi_i \mathsf{E}\xi_j \le \sum_{i=1}^n \mathsf{E}(\xi_i^2) + \left(\sum_{i=1}^n \mathsf{E}\xi_i\right)^2 \\
&= (\mathsf{E}\xi)^2 + \sum_{i=1}^n \sum_{j=1}^n x_j^2 a_{i,j} = (\mathsf{E}\xi)^2 + \sum_{j=1}^n x_j^2 \le 2(\mathsf{E}\xi)^2.
\end{aligned}$$

As a result, relation (7) and hence also (6) follow from (8). ☐

More elaborate arguments show that, in the case of identically distributed quantities, a significantly stronger assertion holds.

Now suppose that

$$\mathsf{P}\{\xi_i = x_j\} = \frac{1}{n}, \qquad i, j = 1, 2, \dots, n,$$

where $n \in \mathbb{N}$ and $x_1 > x_2 > \dots > x_n \ge 0$. By $(\xi_i^\sharp(\omega))_{i=1}^n$ we denote the nonincreasing rearrangement of the vector $(\xi_j(\omega))_{j=1}^n$ for each ω from the probability space Ω.

Proposition 2. *There exists a constant $\alpha > 0$ independent of $n \in \mathbb{N}$ and a vector $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ such that, under the condition that the ξ_i are independent,*

$$\mathsf{P}\left\{\omega \in \Omega : \xi_i^\sharp(\omega) \geq x_{2i-1},\, i = 1, 2, \dots, \left[\frac{n}{2}\right]\right\} \geq \alpha. \tag{9}$$

Let us present a combinatorial statement for the last assertion. Suppose that, just as above,

$$n \in \mathbb{N}, \qquad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \quad x_1 > x_2 > \dots > x_n \geq 0,$$

as well as

$$S := \sum_{i=1}^{n} x_i.$$

Denote by Σ the set of all vectors $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ "composed" of the coordinates of the vector x, i.e.,

$$y_i \in \{x_1, x_2, \dots, x_n\}, \quad i = 1, 2, \dots, n \quad (x_k \text{ may repeat}).$$

Obviously, $\operatorname{card} \Sigma = n^n$. In addition, suppose that

$$\Sigma_1 = \left\{y = (y_1, y_2, \dots, y_n) \in \Sigma : y_i^\sharp \geq x_{2i-1},\, i = 1, 2, \dots, \left[\frac{n}{2}\right]\right\}.$$

Then, as we can easily see, inequality (9) is equivalent to the following one:

$$\operatorname{card} \Sigma_1 \geq \alpha \cdot n^n, \tag{10}$$

and, therefore, we obtain

Corollary 1. *There exists a universal constant $\alpha > 0$ such that, for each $n \in \mathbb{N}$, inequality* (10) *holds.*

3. LOWER BOUNDS FOR DISTRIBUTIONS OF INDEPENDENT FUNCTIONS

Using the results obtained in the previous part, we shall establish lower bounds for the distributions of the sums of independent functions measurable on $[0, 1]$. Let us recall that the "disjunctive sum" $F(t)$ corresponding to the sequence $\{f_k\}_{k=1}^n$ of functions measurable on $[0, 1]$ is defined by relation (1).

Theorem 1. *Suppose that $n \in \mathbb{N}$ and $\{f_k\}_{k=1}^n$ is an arbitrary sequence of nonnegative independent functions on* $[0, 1]$. *Then*

$$m\left\{t \in [0, 1] : \sum_{k=1}^{n} f_k(t) \geq \frac{1}{2} \sum_{k=1}^{n} F^*(k)\right\} \geq \frac{1}{8}, \tag{11}$$

where $F^*(t)$ is the nonincreasing rearrangement of the function $F(t)$.

Proof. First, suppose that

$$m\{t \in [0, 1] : f_i(t) = \tau\} = 0 \qquad \text{for all} \quad i = 1, 2, \dots, n, \quad \tau \in \mathbb{R}.$$

Setting $\tau_0 := \infty$ and arguing by induction, choose the numbers $\tau_1 > \tau_2 > \dots > \tau_n \geq 0$ as follows: if $k = 1, 2, \dots, n$ and τ_{k-1} have already been defined, then

$$\tau_k := \max\left\{\tau \in [0, \tau_{k-1}) : \sum_{i=1}^{n} m\{t \in [0, 1] : \tau \leq f_i(t) < \tau_{k-1}\} = 1\right\}. \tag{12}$$

If

$$a_{i,j} := m\{t \in [0, 1] : \tau_j \leq f_i(t) < \tau_{j-1}\}, \qquad i, j = 1, 2, \dots, n,$$

then $a_{i,j} \geq 0$ and, in view of (12),

$$\sum_{i=1}^{n} a_{i,j} = 1, \qquad j = 1, 2, \dots, n.$$

In addition, because $f_i \geq 0$, we have

$$\sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} m\{t \in [0,1] : \tau_j \leq f_i(t) < \tau_{j-1}\} = 1, \qquad i = 1, 2, \dots, n.$$

Thus, $(a_{i,j})_{i,j=1}^{n}$ is a bistochastic matrix.

Further, for each $\tau \in (\tau_k, \tau_{k-1})$ (see (2) and (12)),

$$\begin{aligned} m\{t > 0 : |F(t)| > \tau\} = \sum_{j=1}^{k-1} \sum_{i=1}^{n} m\{t \in [0,1] : \tau_j \leq f_i(t) < \tau_{j-1}\} \\ + \sum_{i=1}^{n} m\{t \in [0,1] : \tau < f_i(t) < \tau_{k-1}\} < k, \end{aligned}$$

whence, by the definition of the rearrangement,

$$F^*(k) \leq \tau_k, \qquad k = 1, 2, \dots, n. \tag{13}$$

Let us introduce the sequence of step-functions

$$g_i(t) := \sum_{j=1}^{n} \tau_j \chi_{\{\tau_j \leq f_i(t) < \tau_{j-1}\}}(t), \qquad i = 1, 2, \dots, n.$$

The functions g_i are independent,

$$m\{t \in [0,1] : g_i(t) = \tau_j\} = a_{i,j}, \qquad i, j = 1, 2, \dots, n,$$

and, for all $t \in [0,1]$,

$$g_i(t) \leq f_i(t), \qquad i = 1, 2, \dots, n. \tag{14}$$

Applying Proposition 1 to the functions g_i, we find that

$$m\biggl\{t \in [0,1] : \sum_{i=1}^{n} g_i(t) \geq \frac{1}{2} \sum_{i=1}^{n} \tau_i\biggr\} \geq \frac{1}{8}.$$

Combining this inequality with relations (13) and (14), we obtain

$$m\biggl\{t \in [0,1] : \sum_{i=1}^{n} f_i(t) \geq \frac{1}{2} \sum_{i=1}^{n} F^*(i)\biggr\} \geq m\biggl\{t \in [0,1] : \sum_{i=1}^{n} g_i(t) \geq \frac{1}{2} \sum_{i=1}^{n} \tau_i\biggr\} \geq \frac{1}{8}$$

and (11) is proved.

Now suppose that, for some $\tau \in \mathbb{R}$,

$$m\{t \in [0,1] : f_i(t) = \tau\} > 0.$$

Let us show that inequality (11) is also valid in this case. Indeed, for any $\epsilon > 0$, there exist functions h_i, $1 \leq i \leq n$, such that

$$\begin{gathered} f_i(t) \leq h_i(t) \leq f_i(t) + \epsilon \qquad \text{for all} \quad t \in [0;1], \\ m\{t \in [0,1] : h_i(t) = \tau\} = 0 \qquad \text{for any} \quad 1 \leq i \leq n, \quad \tau \in \mathbb{R}. \end{gathered}$$

Let $(q_i)_{i=1}^n$ is be the independent copies of the functions h_i (i.e., q_i is equimeasurable with h_i for each $i = 1, 2, \ldots, n$, and $q_1, q_2, \ldots, q_n$ are independent). Then, using the arguments given above, we obtain the inequality

$$m\biggl\{t \in [0,1] : \sum_{k=1}^n q_k(t) \geq \frac{1}{2} \sum_{k=1}^n Q^*(k)\biggr\} \geq \frac{1}{8},$$

where Q is the "disjunctive sum" for the functions q_i. Since q_i is equimeasurable with h_i and $h_i \geq f_i$, $i = 1, 2, \ldots, n$, we have

$$Q^*(k) \geq F^*(k) \qquad \text{for all} \quad k = 1, 2, \ldots, n.$$

Further, since

$$(f_i + \epsilon)^*(t) \geq q_i^*(t) \qquad \text{for all} \quad 1 \leq i \leq n, \quad t \in [0; 1],$$

it follows that, in view of the independence of the collections $\{f_i\}$ and $\{q_i\}$,

$$m\biggl\{t \in [0,1] : \sum_{k=1}^n (f_k(t) + \epsilon) \geq x\biggr\} \geq m\biggl\{t \in [0,1] : \sum_{k=1}^n q_k(t) \geq x\biggr\}$$

for any real x. Therefore,

$$m\biggl\{t \in [0,1] : \sum_{k=1}^n f_k(t) \geq \frac{1}{2} \sum_{k=1}^n F^*(k) - n\epsilon\biggr\} \geq \frac{1}{8},$$

where ϵ can be arbitrarily small. Thus, using the continuity of the Lebesgue measure, we obtain (11). $\square$

Corollary 2. *Let X be an arbitrary quasinormed symmetric space on $[0,1]$. Then there exists a constant $B_X > 0$ such that, for each $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n$ of nonnegative independent functions on $[0,1]$, inequality (4) holds.*

Similarly, using Proposition 2, we can prove the following statement.

Theorem 2. *Let $n \in \mathbb{N}$, and let $\{f_k\}_{k=1}^n$ be an arbitrary sequence of equimeasurable nonnegative independent functions on $[0,1]$. Then*

$$m\biggl\{t \in [0,1] : f_j^\sharp(t) \geq F^*(2j-1),\ j = 1, 2, \ldots, \biggl[\frac{n}{2}\biggr]\biggr\} \geq \alpha, \tag{15}$$

where α is the constant from Proposition 2.

Corollary 3. *Suppose that X and E are arbitrary quasinormed symmetric spaces of functions on $[0,1]$ and of sequences, respectively, and let $\{e_k\}_{k=1}^\infty$ be the standard basis in E. Then there exists a constant $D_{X,E}$ such that, for any $n \in \mathbb{N}$ and any sequence of equimeasurable nonnegative independent functions $\{f_k\}_{k=1}^n \subset X$, inequality (5) holds.*

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