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SYMMETRIC QUASI-NORMS OF SUMS OF INDEPENDENT RANDOM VARIABLES IN SYMMETRIC FUNCTION SPACES WITH THE KRUGLOV PROPERTY

BY

S. V. Astashkin*

Department of Mathematics and Mechanics, Samara State University
443011 Samara, Acad. Pavlov, 1, Russian Federation
e-mail: astashkn@ssu.samara.ru

AND

F. A. Sukochev*

School of Mathematics and Statistics, University of New South Wales
Kensington NSW 2052, Australia
e-mail: f.sukochev@unsw.edu.au

ABSTRACT

Let X be a symmetric Banach function space on $[0,1]$ and let E be a symmetric (quasi)-Banach sequence space. Let $\mathbf{f} = \{f_k\}_{k=1}^n$, $n \geq 1$ be an arbitrary sequence of independent random variables in X and let $\{e_k\}_{k=1}^\infty \subset E$ be the standard unit vector sequence in E. This paper presents a deterministic characterization of the quantity

$$\left\| \left\| \sum_{k=1}^n f_k e_k \right\|_E \right\|_X$$

in terms of the sum of disjoint copies of individual terms of $\mathbf{f}$. We acknowledge key contributions by previous authors in detail in the introduction, however our approach is based on the important recent advances in the study of the Kruglov property of symmetric spaces made earlier by the authors.

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1. Introduction

Let X be a symmetric Banach function space on $[0,1]$. For an arbitrary sequence $\mathbf{f} := \{f_k\}_{k=1}^n \subset X$ consider its disjointification, that is, the function

$$F(u) := \sum_{i=1}^{n} \bar{f}_i(u) \quad (u>0), \tag{1}$$

where the sequence $\{\bar{f}_k\}_{k=1}^n$ is a disjointly supported sequence of equimeasurable copies of the individual elements from the sequence $\mathbf{f}$. Denote by F^* the decreasing rearrangement of $|F|$ (see relevant definitions in the next section).

The purpose of this paper is to provide sharp conditions on X under which the following estimates hold:

$$C^{-1}\Big(\|F^*\chi_{[0,1]}\|_X + \Big\|\sum_{k=1}^n F^*(k)e_k\Big\|_E\Big) \le \Big\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_E\Big\|_X \tag{2}$$

$$\le C\Big(\|F^*\chi_{[0,1]}\|_X + \Big\|\sum_{k=1}^n F^*(k)e_k\Big\|_E\Big) \tag{3}$$

for every sequence $\mathbf{f} \subset X$ of independent random variables (i.r.v.'s). Various special cases of this problem have been considered by various authors. The first result in this direction is the remarkable Rosenthal's inequality [27] which can be interpreted as the validity of (2) and (3) in the case when $E = l_1$ and $X = L_p$ $(1 \le p < \infty)$. This was extended by Carothers and Dilworth [10] to the case when X is a Lorentz space $L_{p,q}$ $(1 < p < \infty, 1 \le q \le \infty)$, and further by Johnson and Schechtman [14] to the case when X is any symmetric space such that $X \supset L_p$ for $p < \infty$ and $E = l_1$. Subsequently, Montgomery-Smith [23, Theorem 1] showed that inequality (2) holds for an arbitrary symmetric function space X and symmetric Banach sequence space E and inequality (3) holds for any symmetric Banach sequence space E when $X \supset L_p$ for some $p < \infty$. In all these works, a crucial component of the proof is an application of a well-known inequality by Hoffman-Jørgensen [13] or its versions. On the other hand, in [4, 3, 5, 6] the current authors investigated (3) using the operator approach to the study of the Kruglov property introduced and extensively used earlier by Braverman [8]. It is important to point out that the latter property of a symmetric space X is less restrictive than the assumption $X \supset L_p$ for some $p < \infty$. In particular, there exist symmetric spaces with the Kruglov property which do not contain any reflexive L_p-spaces. A standard example is given by an exponential Orlicz space $Exp(L_p)$, $0 < p < \infty$, which has the Kruglov property if and only if $p \in (0, 1]$. In [4, Theorem 6.7], estimate (3) was obtained for any symmetric space X with the Kruglov property but only for a special case $E = l_p$, $1 \leq p \leq \infty$. Here, we extend this result to an arbitrary Banach symmetric sequence space. Moreover, using also some ideas from the recent work of Junge [16], we establish (3) for any quasi-Banach symmetric sequence spaces satisfying some rather mild restrictions. It is important to point out that our main result here (see Theorem 1) is sharp in the following sense. If estimate (3) holds for $E = l_1$, for every $n \geq 1$ and for an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s, where X is a symmetric Banach function space with the Fatou property, then X has the Kruglov property [4, Theorem 3.5]. As a byproduct we obtain a rather general version of Theorem 27 from [24] related to random permutations (see Corollary 5).

In the final section of this paper we provide two applications of our main result.

Firstly, we briefly discuss the problem of finding the deterministic expression for the functional $\| \| \{\alpha_i f_i\}_{i=1}^\infty \|_{l_N} \|_{L_M}$, where L_M is an Orlicz function space on $[0, 1]$ and l_N is an Orlicz sequence space. Such a problem was first studied by Gordon, Litvak, Schütt and Werner in [11] for the special case $L_M = L_1$ and later by Montgomery-Smith in [23] under the assumption that L_M contains an L_q-space for some $q < \infty$. Our main result (see Theorem 1 below) extends results from [11, 23] to an arbitrary Orlicz space L_M with the Kruglov property and, in Corollary 9, we present such an extension for the case $L_M = Exp(L_p)$, $0 < p \leq 1$.

Secondly, we consider the problem of describing the space $A_{is}(p, q)$ consisting of all sequences $\alpha = (\alpha_k)_{k=1}^\infty \subset \mathbb{R}$ such that the series

$$\sum_{k=1}^\infty \alpha_k f_k$$

converges with respect to the norm of the space $L_{p,q} = L_{p,q}[0, 1]$ for any sequence $(f_k)_{k=1}^\infty \subset L_{p,q}$ of independent identically distributed symmetric random variables. This problem has been previously studied by S.Ya. Novikov who established in [25, Theorems 3.5.6 and 4.3.8] that $A_{is}(2, \infty)$ coincides with the Orlicz sequence space l_{M_0}, where $M_0(t) := t^2 \cdot \ln(1 + 1/t)$ $(t > 0)$. We shall present an easy and straightforward proof of this result as a consequence of our disjointification formula from Theorem 1.

2. Preliminaries

2.1. Symmetric function and sequence spaces. In this subsection we present some definitions from the theory of symmetric spaces. For more details on the latter theory we refer to [17, 21, 7].

Let I denote either $[0,1]$ or $(0,\infty)$ with Lebesgue measure λ. If $f \in L_1(I)+L_\infty(I)$ we denote by f^* the decreasing rearrangement of $|f|$, i.e.,

$$f^*(t) = \inf_{\lambda(A)=t} \sup_{s\in I\setminus A} |f(s)|.$$

A Banach function space X on I is said to be symmetric if the conditions $f \in X$ and $g^* \le f^*$ imply that $g \in X$ and $\|g\|_X \le \|f\|_X$. We will assume the normalization that $\|\chi_{(0,1)}\|_X = 1$, where χ_A is the characteristic function of a set $A \subset I$. Let $\varphi_X(t) = \|\chi_{(0,t)}\|_X$ be the **fundamental function** of X. A symmetric space X is said to have the **Fatou property** if for every sequence $(f_n)_{n=1}^\infty \subset X$ of nonnegative functions such that $f_n \uparrow f$ a.e. and $\lim_{n\to\infty} \|f_n\|_X < \infty$ we have $f \in X$ and $\|f\|_X = \lim_{n\to\infty}\|f_n\|_X$.

Let us recall some classical examples of symmetric spaces on $[0,1]$.

Let $M(t)$ be an increasing convex function on $[0,\infty)$ such that $M(0)=0$. By L_M we denote the Orlicz space on $[0,1]$ (see, e.g., [17, 21]) endowed with the norm

$$\|x\|_{L_M} = \inf\left\{\lambda > 0 : \int_0^1 M(|x(t)|/\lambda)dt \le 1\right\}.$$

We recall that $x \in L_{p,q}(0,1)$, $1<p<\infty$, $1\le q\le\infty$, if

$$\|x\|_{L_{p,q}} = \begin{cases} \left(\frac{q}{p}\int_0^1 (x^*(t)t^{1/p})^q \frac{dt}{t}\right)^{1/q}, & q<\infty, \\ \sup_{0<t\le 1} x^*(t)t^{1/p}, & q=\infty, \end{cases}$$

is finite. The expression $\|\cdot\|_{L_{p,q}}$ is a norm for $q \le p$ and is equivalent to a norm for $q>p$ [21, p. 142].

We next discuss briefly symmetric sequence spaces. If $\xi = (\xi_n)_{n=1}^\infty$ is a bounded sequence, then $\xi^* = (\xi_n^*)_{n=1}^\infty$ denotes its decreasing rearrangement:

$$\xi_n^* = \inf_{|\mathbb{A}|=n-1} \sup_{k\in\mathbb{N}\setminus\mathbb{A}} |\xi_k|.$$

where $|A|$ is a cardinality of A.

A quasi-Banach sequence space E is called **symmetric** if the conditions $a \in E$ and $b^* \le a^*$ imply that $b \in E$ and $\|b\|_E \le \|a\|_E$. So, the functional $\|\cdot\|_E$ satisfies the generalized triangle inequality

$$\|a+b\|_E \le L(\|a\|_E + \|b\|_E), \quad a, b \in E, \tag{4}$$

where $L \ge 1$. If we may take in the last inequality $L = 1$, then E said to be a **(Banach) symmetric sequence space**.

We shall assume (without loss of generality) that $\|e_k\|_E = 1$, $k \ge 1$, where

$$e_k := (0, 0, \dots, 0, \underbrace{1}_{k^{\text{th}} \text{ place}}, 0, 0, \dots).$$

Moreover, we will need further to have placed some restrictions upon a quasi-Banach symmetric sequence space E. Let, for every $k \in \mathbb{N}$, the dilation operator

$$\sigma_k(\{a_j\}_{j\ge1}) := (\underbrace{a_1, \dots, a_1}_{k \text{ times}}, \underbrace{a_2, \dots, a_2}_{k \text{ times}}, \dots, \underbrace{a_j, \dots, a_j}_{k \text{ times}}, \dots)$$

act boundedly in E and let

$$\|\sigma_k\|_{E\to E} \le k, \quad k \in N. \tag{i}$$

We shall assume also sometimes that the quasi-Banach sequence space E satisfies the condition

$$l_1 \subseteq E. \tag{ii}$$

Observe that in the case when E is a Banach symmetric sequence space, the latter assumptions are satisfied automatically.

2.2. The Kruglov property in symmetric function spaces. Let f be a measurable function (a random variable) on $[0,1]$. By $\pi(f)$ we denote the random variable $\sum_{i=1}^N f_i$, where f_i's are independent copies of f and N is a Poisson random variable with parameter 1 independent of the sequence $\{f_i\}$.

Definition: A symmetric function space X is said to have the **Kruglov property** if and only if $f \in X \Longleftrightarrow \pi(f) \in X$.

This property has been studied by M. Sh. Braverman [8] who used some probabilistic constructions of V. M. Kruglov [18] and by the authors in [4, 3, 5, 6] via an operator approach. We refer to the latter papers for various equivalent characterizations of the Kruglov property. Note that only the implication $f \in X \Longrightarrow \pi(f) \in X$ is nontrivial, since the implication $\pi(f) \in X \Longrightarrow f \in X$

is always satisfied [8, p. 11]. Moreover, a symmetric space X has the Kruglov property if $X \supseteq L_p$ for some $p < \infty$ [8, Theorem 1.2] and [4, Corollaries 5.4, 5.6]. At the same time, some exponential Orlicz spaces which do not contain L_q for any $q < \infty$ also possess this property (see [8], [4] and the preceding section).

3. Main result

Our main result concerns the general version of the inequality $(6.1)'$ from [4] in which the space l_p is replaced by an arbitrary Banach sequence space or even by an arbitrary quasi-Banach symmetric sequence space E satisfying properties (i) and (ii) above.

Theorem 1: *If a symmetric Banach function space X has the Kruglov property and if E is an arbitrary symmetric Banach sequence space, then there exists a constant $C > 0$ which depends only on X such that for all $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s, inequalities* (2) *and* (3) *hold. Moreover, inequality* (3) *holds for an arbitrary quasi-Banach symmetric sequence space E satisfying assumptions (i) and (ii).*

Proof. In fact, inequality (2) holds in the case of arbitrary symmetric function space X and symmetric Banach sequence space E (see, e.g., [23, Theorem 1]).

Therefore, we concentrate on proving (3). Our approach is based on a key observation that a central role in question is played by the weak l_1-space, $l_{1,\infty}$ (see also [16, the proof of Theorem 0.2]). The latter space is defined as the space of all sequences $a = \{a_k\}_{k\geq 1}$ such that

$$\|a\|_{1,\infty} := \sup_{k\geq 1} k a_k^* < \infty.$$

It is obvious that $l_{1,\infty}$ is a quasi-Banach sequence space such that

$$\|a+b\|_{1,\infty} \leq 2(\|a\|_{1,\infty} + \|b\|_{1,\infty}) \tag{5}$$

for all $a.b \in l_{1,\infty}$.

We shall begin therefore with the proof of (3) in this special case. Recall that the notation F was introduced in (1).

Proposition 2: *If a symmetric function space X has the Kruglov property, then there exists a universal constant $C_0 > 0$ such that for all $n \in \mathbb{N}$ and any* *sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s, the following inequality holds:*

$$\Big\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_{l_{1,\infty}}\Big\|_X \le C_0\Big(\|F^*\chi_{[0,1]}\|_X + \Big\|\sum_{k=1}^n F^*(k)e_k\Big\|_{l_{1,\infty}}\Big). \tag{6}$$

Proof. It is obvious that we may (and shall) assume that the sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s consists of nonnegative functions. Set $t_0 := F^*(1)$ and observe that

$$\sum_{k=1}^n \lambda\{s \in [0,1]:\ f_k(s) > t_0\} = \lambda\{s > 0:\ F(s) > t_0\} \le 1.$$

Therefore, setting

$$g_k := f_k\chi_{\{f_k > t_0\}}, \quad h_k := f_k\chi_{\{f_k \le t_0\}}, \quad k = 1, 2, \ldots, n,$$

we have $f_k = g_k + h_k$ and $\sum_{k=1}^n \lambda(supp(g_k)) \le 1$. We shall estimate quantities $\|\|\sum_{k=1}^n g_k e_k\|_{l_{1,\infty}}\|_X$ and $\|\|\sum_{k=1}^n h_k e_k\|_{l_{1,\infty}}\|_X$ separately. Let G and H be functions defined for sequences $\{g_k\}_{k=1}^n$ and $\{h_k\}_{k=1}^n$ respectively analogously to the definition of the function F in (1). Then, from the definition we have the equality $G^* = G^*\chi_{[0,1]} = F^*\chi_{[0,1]}$. Since $\|a\|_{l_{1,\infty}} \le \|a\|_{l_1}$ $(a \in l_1)$, it follows from [8, Lemma 4] that there exists a (universal) constant α such that

$$\Big\|\Big\|\sum_{k=1}^n g_k e_k\Big\|_{l_{1,\infty}}\Big\|_X \le \Big\|\sum_{k=1}^n g_k\Big\|_X \le \alpha\|G^*\|_X = \alpha\|F^*\chi_{[0,1]}\|_X. \tag{7}$$

Now, we shall concentrate on obtaining the estimate analogous to (6) for the sequence $\{h_k\}_{k=1}^n$ of uniformly bounded i.r.v's. Firstly, we intend to prove (6) for this sequence when $X = L_p$, $p \ge 1$, with the constant equivalent in order to $p/\sqrt{\ln(p+1)}$ and then infer the required inequality for general X's with the Kruglov property. To this end, observe that

$$H(s) \le t_0 \quad \text{and} \quad H^*(s) \le F^*(s+1)\ (s > 0). \tag{8}$$

Therefore,

$$\begin{aligned}\sup_{t>0} t\lambda\{s>0 : H(s) > t\} &= \sup_{0<t\le t_0} t\lambda\{s > 0:\ H^*(s) > t\}\\ &\le \sup_{0<t\le t_0} t\lambda\{s > 0 : F^*(s+1) > t\}\\ &= \sup_{0<t\le t_0} t\lambda\{s > 1:\ F^*(s) > t\}\\ &= \sup_{s\ge 1} F^*(s)(s-1) \le \sup_{k\ge 1} kF^*(k).\end{aligned}$$

Setting

$$A := \sup_{k\geq 1} kF^*(k) = \Big\| \sum_{k=1}^n F^*(k)e_k \Big\|_{l_{1,\infty}}$$

we arrive at

$$\sup_{t>0} t\lambda\{s>0:\ H(s)>t\} \leq A. \tag{9}$$

Now, consider a sequence $\{h'_k\}_{k\geq 1}$ such that $(h'_k)^* = h^*_k$ for every $k \geq 1$ and such that the family $\{h_k, h'_k\}_{k\geq 1}$ consists of i.r.v.'s. In view of (9), we have

$$t\sum_{k=1}^n \lambda\{h'_k > t\} = t\sum_{k=1}^n \lambda\{h_k > t\} = t\lambda\{s>0:\ H(s)>t\} \leq A.$$

Using an elementary inequality $(a+b)^p \leq 2^p(a^p+b^p)$ (for $a,b>0$), we have for every $v \in [0,1]$ that

$$\begin{aligned}
&\int_0^1 \Big\| \sum_{k=1}^n h_k(u)e_k \Big\|^p_{l_{1,\infty}} du \\
&\quad = \int_0^1 \Big(\sup_{t>0} t\sum_{k=1}^n \chi_{\{h_k(u)>t\}} \Big)^p du \\
&\quad \leq \int_0^1 \Big(\Big| \sup_{t>0} \Big[t\sum_{k=1}^n (\chi_{\{h_k(u)>t\}} - \lambda\{h'_k(v)>t\}) \Big] \Big| \\
&\qquad\qquad + \sup_{t>0} t\sum_{k=1}^n \lambda\{h'_k(v)>t\} \Big)^p du \\
&\quad \leq 2^p \int_0^1 \Big| \sup_{t>0} \Big[t\sum_{k=1}^n (\chi_{\{h_k(u)>t\}} - \lambda\{h'_k(v)>t\}) \Big] \Big|^p du + 2^pA^p.
\end{aligned}$$

By an elementary convexity argument, we deduce

$$\begin{aligned}
&\int_0^1 \Big\| \sum_{k=1}^n h_k(u)e_k \Big\|^p_{l_{1,\infty}} du \\
&\qquad \leq 2^p \int_0^1\int_0^1 \Big| \sup_{t>0} \Big[t\sum_{k=1}^n (\chi_{\{h_k(u)>t\}} - \chi_{\{h'_k(v)>t\}}) \Big] \Big|^p du dv + 2^pA^p.
\end{aligned}$$

Since the sequence $\{\chi_{\{h_k(u)>t\}} - \chi_{\{h'_k(v)>t\}}\}_{k=1}^n$ consists of symmetrically distributed i.r.v.'s, we can further modify the estimate above as

$$\int_0^1 \Big\| \sum_{k=1}^n h_k(u) e_k \Big\|_{l_{1,\infty}}^p du \\ \le 2^p \int_0^1 \int_0^1 \Big| \sup_{t>0} \Big[t \sum_{k=1}^n r_k (\chi_{\{h_k(u)>t\}} - \chi_{\{h'_k(v)>t\}}) \Big] \Big|^p du dv + 2^p A^p,$$

where $\{r_k\}_{k\ge1}$ is a sequence of identically and symmetrically distributed Bernoulli variables taking on values ±1 and independent of the family $\{h_k, h'_k\}_{k\ge1}$. If functions $x, y \in L_p$ are equidistributed, then obviously

$$\int_0^1 \int_0^1 |x(u) + y(v)|^p du dv \le 2^p \int_0^1 |x(u)|^p du.$$

Combining this observation with the previous estimates we arrive at the inequality

$$(10) \quad \int_0^1 \Big\| \sum_{k=1}^n h_k(u) e_k \Big\|_{l_{1,\infty}}^p du \le 2^{2p} \int_0^1 \sup_{t>0} t^p \Big| \sum_{k=1}^n r_k \chi_{\{h_k(u)>t\}} \Big|^p du + 2^p A^p.$$

At this moment in the proof, we have to pause and establish one auxiliary result. The proof of the latter is almost identical to the proof of [22, Lemma 3.4] when $p = 2$. However, we need to estimate the order of the constant with respect to p.

Lemma 3: *Let $1 \le p < \infty$ and $n \in \mathbb{N}$. For every nonnegative scalar sequence $\{u_k\}_{k=1}^n$, we have*

$$\int_0^1 \sup_{t>0} t^p \Big| \sum_{k=1}^n r_k(s) \chi_{\{u_k>t\}} \Big|^p ds \le 2^p p^{p/2} \Big(\sum_{k=1}^n u_k^2 \Big)^{p/2}.$$

Proof. Let $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ be a permutation such that $u_{\pi(k)} = u_k^*$ $(1 \le k \le n)$. Then

$$(11) \qquad \sup_{t>0} t^p \Big| \sum_{k=1}^n r_k(s) \chi_{\{u_k>t\}} \Big|^p = \sup_{1\le i\le n} (u_i^*)^p \Big| \sum_{k=1}^i r_{\pi(k)}(s) \Big|^p.$$

Denote

$$S_i := u_i^* \sum_{k=1}^i r_{\pi(k)}\ ,\ T_i := \sum_{k=1}^i u_k^* r_{\pi(k)}\ (1 \le i \le n) \quad \text{and} \quad T_0 = 0.$$

Applying Abel's transform, we estimate

$$S_i = u_i^* \sum_{k=1}^{i} (T_k - T_{k-1})(u_k^*)^{-1} = T_i - u_i^* \sum_{k=1}^{i-1} T_k((u_{k+1}^*)^{-1} - (u_k^*)^{-1})$$
$$\leq |T_i| + u_i^* \sum_{k=1}^{i-1} \sup_{1\leq k\leq i} |T_k|((u_{k+1}^*)^{-1} - (u_k^*)^{-1}),$$

whence

$$\sup_{1\leq i\leq n} |S_i|^p \leq 2^p \sup_{1\leq i\leq n} |T_i|^p.$$

Since the sequence $\{T_i\}_{i=1}^n$ is distributed exactly as the sequence $\{\sum_{k=1}^i u_k^* r_k\}_{i=1}^n$, we obtain via the maximal Khintchine inequality (see, e.g., [26, §5])

$$\int_0^1 \sup_{1\leq i\leq n} |T_i|^p \, ds = \int_0^1 \sup_{1\leq i\leq n} \left| \sum_{k=1}^{i} u_k^* r_k(s) \right|^p ds$$
$$\leq p^{p/2} \left(\sum_{k=1}^{n} (u_k^*)^2 \right)^{p/2} = p^{p/2} \left(\sum_{k=1}^{n} u_k^2 \right)^{p/2}.$$

Combining the last two inequalities yields

$$\int_0^1 \sup_{1\leq i\leq n} |S_i|^p \, ds \leq 2^p p^{p/2} \left(\sum_{k=1}^{n} u_k^2 \right)^{p/2}.$$

In view of (11) this completes the proof of Lemma 3. ∎

Let us continue the proof of Proposition 2. Integrating inequality (10), we obtain

$$\int_0^1 \left\| \sum_{k=1}^{n} h_k(u) e_k \right\|_{l_{1,\infty}}^p du \leq 2^{2p} \int_0^1 \int_0^1 \sup_{t>0} t^p \left| \sum_{k=1}^{n} r_k(s) \chi_{\{h_k(u)>t\}} \right|^p duds + 2^p A^p$$
$$= 2^{2p} \int_0^1 \int_0^1 \sup_{t>0} t^p \left| \sum_{k=1}^{n} r_k(s) \chi_{\{h_k(u)>t\}} \right|^p dsdu + 2^p A^p$$
$$\leq 2^{3p} p^{p/2} \int_0^1 \left(\sum_{k=1}^{n} h_k^2(u) \right)^{p/2} du + 2^p A^p. \tag{12}$$

Now we are going to estimate the quantity

$$\int_0^1 \left(\sum_{k=1}^{n} h_k^2(u) \right)^{p/2} du = \left\| \left(\sum_{k=1}^{n} h_k^2 \right)^{1/2} \right\|_p^p.$$

Firstly, Theorem 13 from [6] and inequalities (8) show that

$$\int_0^1 \Big(\sum_{k=1}^n h_k^2(u)\Big)^{p/2} du \le \alpha^p \Big(\frac{p}{\ln(p+1)}\Big)^{p/2} \Big(F^*(1) + \Big(\sum_{k=1}^n F^*(k)^2\Big)^{1/2}\Big)^p$$

$$\le (2\alpha)^p \Big(\frac{p}{\ln(p+1)}\Big)^{p/2} \Big(\sum_{k=1}^n F^*(k)^2\Big)^{p/2}.$$

Since $l_{1,\infty}$ is continuously embedded into l_2, we have for some universal constant $\beta > 0$ that

$$\Big(\sum_{k=1}^n F^*(k)^2\Big)^{1/2} \le \beta \sup_{k\ge 1} kF^*(k) = \beta A.$$

Thus, it follows from (12) that for some universal constant $\gamma > 0$

$$\int_0^1 \Big\|\sum_{k=1}^n h_k(u)e_k\Big\|_{l_{1,\infty}}^p du \le 2^{4p} p^{p/2} \alpha^p \Big(\frac{p}{\ln(p+1)}\Big)^{p/2} \beta^p A^p + 2^p A^p$$

$$= \gamma^p A^p \Big(\frac{p}{\sqrt{\ln(p+1)}}\Big)^p,$$

or, equivalently,

$$\Big\|\Big\|\sum_{k=1}^n h_k(u)e_k\Big\|_{l_{1,\infty}}\Big\|_p \le \gamma A \frac{p}{\sqrt{\ln(p+1)}}, \quad p \ge 1,$$

implying further

$$\sup_{p\ge 1}\Big(\frac{\sqrt{\ln(p+1)}}{p}\Big\|\Big\|\sum_{k=1}^n h_k(u)e_k\Big\|_{l_{1,\infty}}\Big\|_p\Big) \le \gamma A.$$

The quantity standing on the left-hand side in this inequality is equivalent to the norm of the function $u \mapsto \|\sum_{k=1}^n h_k(u)e_k\|_{l_{1,\infty}}$ in the Orlicz space L_M, where $M(u) = e^{u\sqrt{\ln(u+1)}} - 1$ (see, e.g., [1, Corollary 1] or [15, Proposition 3.6]). Since X has the Kruglov property, it follows from [4, Theorem 7.2] that $L_M \subseteq X$ and so, for some universal constant $\gamma' > 0$, we have

$$\Big\|\Big\|\sum_{k=1}^n h_k(u)e_k\Big\|_{l_{1,\infty}}\Big\|_X \le \gamma' A.$$

Combining this estimate with (7) and using the fact that $f_k = h_k + g_k$, $k \ge 1$ along with (5), we infer (6). This completes the proof of Proposition 2. ∎

Before we proceed with the completion of the proof of Theorem 1, we need some additional notation. Fix a quasi-Banach symmetric space E. Given $n \in \mathbb{N}$, we first define an operator U_n^E acting on the set of all square $n \times n$ matrices into n^n-dimensional space of step-functions in $L_1(0,1)$ spanned by characteristic functions of the intervals

$$\Delta_i := \Big[\frac{i-1}{n^n}, \frac{i}{n^n}\Big), \quad i = 1, \dots, n^n.$$

To define this operator, we consider the set $\{1, 2, \dots, n\}^n$ and an arbitrary one-to-one mapping ϕ from this set onto the collection of intervals $\{\Delta_i\}_{i=1}^{n^n}$. Fix an $n \times n$ matrix $\alpha = (\alpha_{i,j}\}_{i,j=1}^n$. We define the image $U_n^E(\alpha)$ as the following function:

$$U_n^E \alpha(t) := \sum_{j_1, j_2, \dots, j_n = 1}^{n} \Big\| \sum_{k=1}^{n} \alpha_{k, j_k} e_k \Big\|_E \chi_{\Delta_{\phi(j_1, j_2, \dots, j_n)}}(t) \quad (0 \le t \le 1).$$

Of course, the definition of the operator U_n^E depends on the choice of the mapping ϕ. However, if X is a symmetric space on $[0,1]$, then the quantity $\|U_n^E(\alpha)\|_X$ remains the same for any choice of ϕ.

Next, we shall use bistochastic matrices $\mu = (\mu_{i,j}\}_{i,j=1}^n$, that is, matrices satisfying

$$\mu_{i,j} \ge 0, \quad \sum_{i=1}^{n} \mu_{i,j} = \sum_{j=1}^{n} \mu_{i,j} = 1, \quad \forall 1 \le i, j \le n.$$

For every such matrix, we define a family $\xi^\mu = \{\xi_k^\mu\}_{k=1}^n$ of i.r.v.'s on the probability space $[0,1]^n$ as follows. If $k \in [1, n]$ and $t = (t_1, t_2, \dots, t_n) \in [0,1]^n$, then

$$\xi_k^\mu(t) = \xi_k^\mu(t_k) = 1/i,$$

on a subset whose measure is equal to $\mu_{i,k}$. Here, i runs through the values $1, 2, \dots, n$. Observe that the assumption that μ is bistochastic guarantees

$$\sum_{k=1}^{n} \lambda\{\xi_k^\mu = 1/i\} = 1, \quad \forall i = 1, 2, \dots, n. \tag{13}$$

Observe also that if for $t \in [0,1]^n$ we have $\xi_k^\mu(t) = 1/i_k$ for some $1 \le i_k \le n$ and $1 \le k \le n$, then

$$\begin{aligned}\|\xi_k^\mu(t)\|_{l_{1,\infty}} &= \sup_{k=1,2,\dots,n} k \xi_k^\mu(t)^* = \sup_{r=1,2,\dots,n} \frac{1}{r} card\{j : \ \xi_j^\mu(t) \ge 1/r\} \\ &= \sup_{r=1,2,\dots,n} \frac{1}{r} card\{k : \ j_k \le r\}.\end{aligned}$$

In this notation, we now restate the assertion which is implicitly contained in the proof of [16, Proposition 2.1] as follows.

Proposition 4: *Let X be an arbitrary symmetric function space and let E be a quasi-Banach symmetric sequence space satisfying condition (i). Then for every matrix $\alpha = (\alpha_{i,j}\}_{i,j=1}^n$, there exists a bistochastic matrix $\mu = (\mu_{i,j}\}_{i,j=1}^n$ such that*

$$\|U_n^E \alpha\|_X \leq 2\| \max\{1, \|\xi^\mu\|_{1,\infty}\} \ \|_X \cdot \left\| \sum_{k=1}^n \alpha^*_{(k-1)n+1} e_k \right\|_E,$$

where $\{\alpha_j^*\}_{j=1}^{n^2}$ is the nonincreasing permutation of the sequence $\{|\alpha_{i,j}|\}_{i,j=1}^n$.

We shall combine results of Proposition 2 and Proposition 4 in the proof of Theorem 1 as follows. If a symmetric function space X has the Kruglov property, then by Proposition 2 we have

$$\|\|\xi^\mu\|_{1,\infty}\|_X \leq C_0 \left(\|F_\xi^* \chi_{[0,1]}\|_X + \left\| \sum_{k=1}^n F_\xi^*(k) e_k \right\|_{l_{1,\infty}} \right),$$

where F_ξ is a disjointification of the sequence $\xi^\mu = \{\xi_k^\mu\}_{k=1}^n$ (see (1)). It now immediately follows from (13) that

$$F_\xi^*(u) = \sum_{i=1}^n \frac{1}{i} \chi_{[i-1,i)}(u),$$

and therefore $\|F_\xi^* \chi_{[0,1]}\|_X \leq 1$ and $\| \sum_{k=1}^n F_\xi^*(k) e_k\|_{1,\infty} \leq 1$, which yields

$$\| \ \|\xi^\mu\|_{1,\infty} \ \|_X \leq 2C_0.$$

In particular, Proposition 4 now yields the following rather general version of Theorem 27 from [24] (see also [19]).

Corollary 5: *If a symmetric function space X has the Kruglov property, then for every $n \in \mathbb{N}$, for every quasi-Banach symmetric sequence space E satisfying condition (i) and for any matrix $\alpha = (\alpha_{i,j}\}_{i,j=1}^n$ we have*

$$\|U_n^E \alpha\|_X \leq K_0 \left\| \sum_{k=1}^n \alpha^*_{(k-1)n+1} e_k \right\|_E,$$

where K_0 depends only on X.

Now, we are ready to complete the proof of Theorem 1. Since f_k's ($k = 1, 2, \dots, n$) are i.r.v.'s and since E is symmetric, we may (and shall) assume that $f_1, f_2, \ldots, f_n$ are defined on $[0,1]^n$ and, for every $k = 1, 2, \ldots, n$, the function $f_k(s_1, s_2, \ldots, s_n) = f_k(s_k)$ is nonnegative and nonincreasing. Each of these functions we split into the sum of three disjointly supported summands as follows. If $t_0 = F^*(1)$, then

$$f_k^1 := f_k \chi_{\{f_k > t_0\}}, \quad f_k^2 := (f_k - f_k^1)\chi_{[0,1/n)}, \quad f_k^3 := f - f_k^1 - f_k^2.$$

Observe that for $i = 1, 2$ we have

$$\sum_{k=1}^n \lambda(\operatorname{supp} f_k^i) \le 1.$$

This observation and condition (ii) imposed on E allow us to appeal once more to [8, Lemma 4] and thus obtain

$$\Big\| \Big\| \sum_{k=1}^n f_k^i e_k \Big\|_E \Big\|_X \le \Big\| \sum_{k=1}^n f_k^i \Big\|_X \le \alpha \|F^* \chi_{[0,1]}\|_X \quad (i = 1, 2). \tag{14}$$

It remains to obtain a similar estimate for $\| \| \sum_{k=1}^n f_k^3 e_k \|_E \|_X$. To this end, we shall employ Corollary 5. Analogously to [16, proof of Theorem 0.2], we introduce the matrix $\alpha = (\alpha_{i,j}\}_{i,j=1}^n$ by

$$\alpha_{i,j} := \sup_{j/n < s_i \le (j+1)/n} f_i^3(s_i), \quad j = 1, 2, \ldots, n-1, \quad \alpha_{i,n} = 0.$$

Fix $(j_1, j_2, \ldots, j_n) \in \{1, 2, \ldots, n\}^n$. Then for an arbitrary $s := (s_1, s_2, \ldots, s_n)$ such that $j_i/n < s_i \le (j_i + 1)/n$ we have

$$\Big\| \sum_{i=1}^n f_i^3(s_i) e_i \Big\|_E \le \Big\| \sum_{i=1}^n \alpha_{i,j_i} e_i \Big\|_E$$

(observe that $f_i^3(s_i) = 0$ when $0 \le s_i \le 1/n$). Since for any $(j_1, j_2, \ldots, j_n) \in \{1, 2, \ldots, n\}^n$ we have

$$\lambda\{s \in [0,1]^n : j_i/n < s_i \le (j_i + 1)/n,\ i = 1, 2, \ldots, n\} = n^{-n},$$

we infer from Corollary 5 that

$$\Big\| \Big\| \sum_{k=1}^n f_k^3 e_k \Big\|_E \Big\|_X \le \|U_n^E \alpha\|_X \le K_0 \Big\| \sum_{k=1}^n \alpha^*_{(k-1)n+1} e_k \Big\|_E. \tag{15}$$

To estimate the right-hand side of (15) in the form required in Theorem 1, we introduce a family of i.r.v.'s $\{\tilde{f}_i\}_{i=1}^n$ on $[0,1]^n$ associated with a given matrix

$\alpha = (\alpha_{i,j}\}_{i,j=1}^n$ as follows:

$$\tilde{f}_i(t) = \tilde{f}_i(t_i) = \sum_{j=1}^n \alpha_{i,j}\chi_{[\frac{j-1}{n},\frac{j}{n}]}(t_i), \quad i = 1, 2, \dots, n.$$

Since $\tilde{f}_i(t) \leq f_i(t)$ $(i = 1, 2, \dots, n)$, then for the corresponding disjoint sums

$$F(t) := \sum_{i=1}^n f_i(t-i+1)\chi_{(i-1,i]}(t) \text{ and } \tilde{F}(t) := \sum_{i=1}^n \tilde{f}_i(t-i+1)\chi_{(i-1,i]}(t) \ \ (t > 0)$$

we have: $\tilde{F}(t) \leq F(t)$ $(t > 0)$. Moreover, $\tilde{F}^*(k) = \alpha^*_{kn+1}$ $(k = 0, 1, 2, \dots, n-1)$ and

$$\alpha_1^* = \sup_{i=1,2,\dots,n} \sup_{0<s_i\leq 1} f_i^3(s_i) \leq t_0 = F^*(1).$$

Hence, if L is the constant in the triangle inequality for the space E, then

$$\begin{aligned}\Big\|\sum_{k=1}^n \alpha^*_{(k-1)n+1}e_k\Big\|_E &\leq L\Big(\alpha_1^* + \Big\|\sum_{k=1}^{n-1}\alpha^*_{kn+1}e_k\Big\|_E\Big)\\ &\leq L\Big(F^*(1) + \Big\|\sum_{k=1}^{n-1}\tilde{F}^*(k)e_k\Big\|_E\Big)\\ &\leq L\Big(F^*(1) + \Big\|\sum_{k=1}^{n-1}F^*(k)e_k\Big\|_E\Big) \leq 2L\Big\|\sum_{k=1}^{n-1}F^*(k)e_k\Big\|_E.\end{aligned}$$

Hence, by (15), we have

$$\Big\|\Big\|\sum_{k=1}^n f_k^3 e_k\Big\|_E\Big\|_X \leq 2K_0L\Big\|\sum_{k=1}^{n-1}F^*(k)e_k\Big\|_E. \tag{16}$$

Now, the pointwise estimate

$$\Big\|\sum_{k=1}^n f_k(t)e_k\Big\|_E \leq L^2\Big(\Big\|\sum_{k=1}^n f_k^1(t)e_k\Big\|_E + \Big\|\sum_{k=1}^n f_k^2(t)e_k\Big\|_E + \Big\|\sum_{k=1}^n f_k^3(t)e_k\Big\|_E\Big)$$

together with (14) and (16) yield the assertion of Theorem 1. ∎

Corollary 6: *If a symmetric Banach function space X has the Kruglov property and if E is an arbitrary symmetric Banach sequence space, then there exists a constant $C > 0$ which depends only on X such that, for all $n \in \mathbb{N}$ and any* *sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s, the following inequality holds:*

$$C^{-1}\Big(\|F^*\chi_{[0,1]}\|_X + \Big\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_E\Big\|_{L_1}\Big)$$
$$\le \Big\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_E\Big\|_X$$
$$\le C\Big(\|F^*\chi_{[0,1]}\|_X + \Big\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_E\Big\|_{L_1}\Big). \tag{17}$$

Proof. From [12, Proposition 1] it follows that the estimate from below in (17) holds in an arbitrary symmetric Banach function space X. To obtain the estimate from the above in (17), we note that estimate (2) applied to the special case $X = L_1$ yields

$$\Big\|\sum_{k=1}^n F^*(k)e_k\Big\|_E \le C'\Big\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_E\Big\|_{L_1}.$$

This observation together with inequality (3) guarantee the right hand side inequality in (17). ∎

Remark 7: Theorem 1 and Corollary 6 strengthen and make precise a number of earlier results. For example, the estimates from the above in (3) and (17) were established earlier in [23, Theorem 1] and [12, Theorem 5], respectively, under the more restrictive assumption that $X \supset L_p$ for $p < \infty$. In both papers, the proof was based on an application of a well-known inequality by Hoffman-Jørgensen [13]. The methods employed in [23] and in [12] do not seem to extend to our present setting. In [4, Theorem 6.7] estimate (3) was obtained under the same assumption on the space X as in Theorem 1, but only for a special case $E = l_p$ $(1 \le p \le \infty)$. It is further important to point out that the statement of Theorem 1 is exact in the following sense. If the estimate from the above in (3) (as well as in (17)) holds for $E = l_1$, for every $n \in N$ and an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s, where X is a symmetric function space with the Fatou property, then X has the Kruglov property [4, Theorem 3.5].

Remark 8: It is easy to see that in the case when the spaces X and E have the Fatou property, inequalities (2), (3) and (17) hold also for all infinite sequences $\{f_k\}_{k=1}^\infty \subset X$ of i.r.v.'s.

4. Two applications

4.1. Orlicz norms of sequences of random variables. Let l_N be an Orlicz sequence space generated by an Orlicz function N and let $\{f_k\}_{k=1}^{\infty} \subset L_1$ be a sequence of identically distributed i.r.v.'s. It is shown in [11] that the mapping

$$(\alpha_i)_{i=1}^{\infty} \mapsto \mathbb{E}\left(\|\{\alpha_i f_i\}_{i=1}^{\infty}\|_{l_N}\right) := \int_0^1 \|(\alpha_i f_i(t))_{i=1}^{\infty}\|_{l_N} dt$$

defines a norm in some Orlicz space. The latter norm was explicitly computed in [11] in some interesting cases. This result was strengthened in [23] as follows. Suppose that L_M is an Orlicz space on $[0,1]$ containing an L_q-space for some $q < \infty$. Then for any sequence $\{f_k\}_{k=1}^{\infty} \subset L_M$ of independent identically distributed random variables, the functional $\| \, \|\{\alpha_i f_i\}_{i=1}^{\infty}\|_{l_N} \, \|_{L_M}$ is a norm in the Orlicz sequence space l_Λ, where

$$\Lambda(x) := \int_0^1 \theta(xf_1(u))du, \quad \theta(x) := N(x), \forall 0 < x \le 1, \text{ and } \theta(x) := M(x), \forall x > 1.$$

Using now the result of Theorem 1 (see also Remark 8) and arguing exactly as in [23, Lemma 7 and Theorem 8], the result of [23] can be extended to an arbitrary Orlicz space with Kruglov property. We state below only the special case when such a space is $Exp(L_p)$, $0 < p \le 1$ (see [8], [4]). Clearly the latter space does not contain an L_q-space for any $q < \infty$.

Corollary 9: *Suppose that N is an arbitrary increasing convex function on $[0,\infty)$ satisfying $N(0) = 0$ and suppose that $0 < p \le 1$. Then for any sequence $\{f_k\}_{k=1}^{\infty} \subset Exp(L_p)$ of identically distributed i.r.v's, the mapping*

$$(\alpha_i)_{i=1}^{\infty} \mapsto \|\|\{\alpha_i f_i\}_{i=1}^{\infty}\|_{l_N}\|_{Exp(L_p)}$$

is equivalent to the norm in the Orlicz space l_Λ, where

$$\Lambda(x) := \int_0^1 \theta(xf_1(u))du, \quad \theta(x) := N(x), \forall 0 < x \le 1, \text{ and } \theta(x) \asymp e^{x^p} - 1, \forall x > 1.$$

4.2. Classes of sequences generated by the convergent series of identically distributed symmetric i.r.v.'s. Let us recall the following notion (see [25] for more details and additional references). Fix $1 < p < \infty$, $1 \le q \le \infty$ and let $A_{is}(p,q)$ consist of all sequences $\alpha = (\alpha_k)_{k=1}^{\infty} \subset \mathbb{R}$ such that the series

$$\sum_{k=1}^{\infty} \alpha_k f_k$$

converges with respect to the norm of the space $L_{p,q} = L_{p,q}[0,1]$ for any sequence $(f_k)_{k=1}^{\infty} \subset L_{p,q}$ of identically distributed symmetric i.r.v.'s. In [25, Theorems 3.5.6 and 4.3.8], it is established that

$$A_{is}(2,\infty) = l_{M_0}, \tag{18}$$

where l_{M_0} is the Orlicz sequence space generated by $M_0(t) := t^2 \cdot \ln(1 + 1/t)$ $(t > 0)$ (see, e.g., [21]). We shall prove (18) with the help of our disjointification formula from Theorem 1.

Since $\|f\|_{L_{2,\infty}} \asymp \sup_{0<t\le 1} f^*(t) t^{1/2}$, then, by [20, Theorem 3.2.1], it is sufficient to verify that

$$\Big\| \sum_{k=1}^{\infty} \alpha_k g_k^{(2)} \Big\|_{2,\infty} \asymp \|\alpha\|_{l_{M_0}},$$

where $\{g_k^{(2)}\}_{k=1}^{\infty}$ is a sequence of identically distributed symmetric i.r.v.'s such that, for all $k \ge 1$,

$$\lambda\{t \in (0,1] : \ |g_k^{(2)}(t)| > \tau\} = \lambda\{t \in (0,1] : \ t^{-1/2} > \tau\} = \tau^{-2} \ \ (\tau > 1).$$

It is well known (see, for example, [9, Lemma 2.2] or in a more general setting [2, Theorem 1]) that

$$\Big\| \sum_{k=1}^{\infty} \alpha_k g_k^{(2)} \Big\|_{2,\infty} \asymp \| \|\{\alpha_k g_k^{(2)}\}_{k=1}^{\infty}\|_2 \|_{2,\infty}.$$

Therefore, by Theorem 1 (see also [23, Theorem 1]), we have

$$\Big\| \sum_{k=1}^{\infty} \alpha_k g_k^{(2)} \Big\|_{2,\infty} \asymp \|G^* \chi_{[0,1]}\|_{2,\infty} + \left(\int_1^{\infty} (G^*(t))^2 \, dt \right)^{1/2},$$

where G is the disjointification of the sequence $\alpha_k g_k^{(2)}$ $(k = 1, 2, \dots)$; see (1). Thus, it suffices to show that

$$\|G^* \chi_{[0,1]}\|_{2,\infty} + \left(\int_1^{\infty} (G^*(t))^2 \, dt \right)^{1/2} \asymp \|\alpha\|_{l_{M_0}}. \tag{19}$$

Without loss of generality, we may assume that $\alpha_k \downarrow 0$. Let us compute the distribution function of $|G|$. If $\tau > 0$, then

$$n_G(\tau) := \lambda\{t > 0 : \ |G(t)| > \tau\} = \sum_{k=1}^{\infty} \lambda\{0 < t \leq 1 : \ \alpha_k t^{-1/2} > \tau\}$$

$$= \sum_{k:\alpha_k > \tau} 1 + \sum_{k:\alpha_k \leq \tau} \frac{\alpha_k^2}{\tau^2} = \operatorname{card}\{k : \alpha_k > \tau\} + \frac{1}{\tau^2} \sum_{k:\alpha_k \leq \tau} \alpha_k^2. \tag{20}$$

First, we shall demonstrate that

$$\|G^* \chi_{[0,1]}\|_{2,\infty} = \|\alpha\|_2. \tag{21}$$

Indeed, letting $\tau_0 := G^*(1)$, we see that $\alpha_1 = (\alpha_1 g_1^{(2)})^*(1) \leq \tau_0$ and so $\alpha_k \leq \alpha_1 \leq \tau_0$ for all $k \geq 1$, and therefore it follows from (20) that $\frac{1}{\tau_0^2} \sum_{k \geq 1} \alpha_k^2 = 1$, that is, $\tau_0 = \|\alpha\|_2$. Thus, in view of (20), we obtain (21) as follows:

$$\|G^* \chi_{[0,1]}\|_{2,\infty} = \sup_{\tau \geq G^*(1) = \tau_0} (\tau n_G(\tau)^{1/2}) = \sup_{\tau \geq \|\alpha\|_2} \tau \cdot \frac{1}{\tau} \cdot \|\alpha\|_2 = \|\alpha\|_2.$$

To estimate the second summand on the left-hand side of (19), we first apply (20) to obtain

$$\begin{aligned} \int_1^{\infty} (G^*(t))^2 \, dt &= 2 \int_0^{\|\alpha\|_2} \tau n_G(\tau) d\tau \\ &= 2 \int_0^{\|\alpha\|_2} \tau \sum_{k:\alpha_k > \tau} 1 d\tau + 2 \int_0^{\|\alpha\|_2} \frac{1}{\tau} \sum_{k:\alpha_k \leq \tau} \alpha_k^2. \end{aligned} \tag{22}$$

For the first summand, we have

$$\begin{aligned} \int_0^{\|\alpha\|_2} \tau \sum_{k:\alpha_k > \tau} 1 d\tau &= \sum_{i=1}^{\infty} i \cdot \int_{\alpha_{i+1}}^{\alpha_i} \tau \, d\tau = \frac{1}{2} \sum_{i=1}^{\infty} i(\alpha_i^2 - \alpha_{i+1}^2) \\ &= \frac{1}{2} \Big(\sum_{i=1}^{\infty} i \cdot \alpha_i^2 - \sum_{i=2}^{\infty} i \cdot \alpha_i^2 + \sum_{i=2}^{\infty} \alpha_i^2 \Big) = \frac{1}{2} \|\alpha\|_2^2. \end{aligned} \tag{23}$$

Similarly, computing the second summand in (21), we have

$$\begin{aligned}\int_0^{\|\alpha\|_2} \frac{1}{\tau} \sum_{k:\alpha_k\le\tau} \alpha_k^2 &= \sum_{i=1}^{\infty} \int_{\alpha_{i+1}}^{\alpha_i} \frac{1}{\tau} \sum_{k=i+1}^{\infty} \alpha_k^2 d\tau + \int_{\alpha_1}^{\|\alpha\|_2} \frac{1}{\tau} \sum_{i=1}^{\infty} \alpha_k^2 d\tau \\ &= \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} \alpha_k^2 \ln\Big(\frac{\alpha_i}{\alpha_{i+1}}\Big) + \sum_{i=1}^{\infty} \alpha_i^2 \ln\Big(\frac{\|\alpha\|_2}{\alpha_1}\Big) \\ &= \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \alpha_k^2 \ln\Big(\frac{\alpha_i}{\alpha_{i+1}}\Big) + \sum_{k=1}^{\infty} \alpha_k^2 \ln\Big(\frac{\|\alpha\|_2}{\alpha_1}\Big) \\ &= \sum_{k=2}^{\infty} \alpha_k^2 \ln\Big(\prod_{i=1}^{k-1} \frac{\alpha_i}{\alpha_{i+1}}\Big) + \sum_{k=1}^{\infty} \alpha_k^2 \ln\Big(\frac{\|\alpha\|_2}{\alpha_1}\Big) \\ &= \sum_{k=2}^{\infty} \alpha_k^2 \ln\Big(\frac{\alpha_1}{\alpha_k}\Big) + \sum_{k=1}^{\infty} \alpha_k^2 \ln\Big(\frac{\|\alpha\|_2}{\alpha_1}\Big) \\ &= \sum_{k=1}^{\infty} \alpha_k^2 \ln\Big(\frac{\|\alpha\|_2}{\alpha_k}\Big).\end{aligned}$$

It follows from the last computation and from (21)–(23) that

$$\begin{aligned}\|G^*\chi_{[0,1]}\|_{2,\infty} &+ \Big(\int_1^{\infty} (G^*(t))^2\, dt\Big)^{1/2} \\ &\asymp \max\Big\{\|\alpha\|_2, \Big(\sum_{k=1}^{\infty} \alpha_k^2 \ln\Big(\frac{\|\alpha\|_2}{\alpha_k}\Big)\Big)^{1/2}\Big\}. \end{aligned} \tag{24}$$

Let us assume that $\|\alpha\|_{l_{M_0}} = 1$, that is, $\sum_{k=1}^{\infty} \alpha_k^2 \ln(1 + 1/\alpha_k) \le 1$. This assumption guarantees, in particular, that $\|\alpha\|_2 \le C_1'$ for some constant $C_1' > 0$. Thus

$$\ln\Big(\frac{\|\alpha\|_2}{\alpha_k}\Big) \le C_1'' \ln\Big(1 + \frac{1}{\alpha_k}\Big),$$

and we infer from (24) that

$$\|G^*\chi_{[0,1]}\|_{2,\infty} + \Big(\int_1^{\infty} (G^*(t))^2\, dt\Big)^{1/2} \le C_1$$

for some constant $C_1 > 0$. To establish the converse estimate, we may assume (in view of (24)) that $0 < \|\alpha\|_2 < 1$ and that $\sum_{k=1}^{\infty} \alpha_k^2 \ln(\|\alpha\|_2/\alpha_k) < 1$. Due to concavity of the logarithmic function, we have

$$\ln\Big(\frac{\|\alpha\|_2}{\alpha_k}\Big) \ge \|\alpha\|_2 \ln\Big(\frac{1}{\alpha_k}\Big) \ge c\|\alpha\|_2 \ln\Big(1 + \frac{1}{\alpha_k}\Big)$$

for some constant $c > 0$, which implies

$$\sum_{k=1}^{\infty} \alpha_k^2 \ln\left(1 + \frac{1}{\alpha_k}\right) < \infty.$$

The latter implies that $\alpha \in l_{M_0}$ and, invoking Banach's inverse mapping theorem, we conclude that

$$\|\alpha\|_{l_{M_0}} \leq C_2,$$

for some $C_2 > 0$. This completes the verification of (19) and (18).

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