Functional Analysis and Its Applications, Vol. 45, No. 1, pp. 64–68, 2011
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 45, No. 1, pp. 79–83, 2011

BRIEF COMMUNICATIONS

Geometry of Cesàro Function Spaces*

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Received March 6, 2009

Abstract. Geometric properties of Cesàro function spaces $\mathrm{Ces}_p(I)$, where $I = [0, \infty)$ or $I = [0, 1]$, are investigated. In both cases, a description of their dual spaces for $1 < p < \infty$ is given. We find the type and the cotype of Cesàro spaces and present a complete characterization of the spaces l^q that have isomorphic copies in $\mathrm{Ces}_p[0, 1]$ $(1 \leqslant p < \infty)$.

Key words: Cesàro space, Köthe dual space, dual space, q-concave Banach space, type and cotype of a Banach space, Dunford–Pettis property.

1. Introduction. The *Cesàro sequence spaces* ces_p $(1 \leqslant p \leqslant \infty)$ have became widely known in the late 1960s in connection with the problem of describing their duals, which was posed by the Dutch Mathematical Society (see [1]). Recall that the space ces_p consists of all sequences of real numbers $x = (x_k)_{k=1}^{\infty}$ such that

$$\|x\|_{\mathrm{ces}_p} = \left[\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right]^{1/p} < \infty$$

(with a natural modification in the case $p = \infty$).

The Cesàro function space $\mathrm{Ces}_p = \mathrm{Ces}_p(I)$ consists of all Lebesgue measurable functions f defined on $I = [0, 1]$ or $I = [0, \infty)$ such that

$$\|f\|_{\mathrm{Ces}_p} = \left[\int_I\left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right]^{1/p} < \infty \quad \text{if } 1 \leqslant p < \infty$$

and

$$\|f\|_{\mathrm{Ces}_\infty} = \sup_{x\in I,\, x>0} \frac{1}{x}\int_0^x |f(t)|\,dt < \infty \quad \text{if } p = \infty.$$

The space $\mathrm{Ces}_\infty[0, 1]$, which is known as the Korenblyum–Krein–Levin space K, first appeared as early as in 1948 in connection with studying problems of the theory of singular integrals (see [2] and [3]). Later, the spaces $\mathrm{Ces}_p[0, \infty)$ $(1 \leqslant p \leqslant \infty)$ were considered in [4]–[6].

This note is devoted to studying geometric properties of Cesàro function spaces. In particular, we describe the (Banach) dual space of $\mathrm{Ces}_p(I)$ $(1 < p < \infty)$ both for $I = [0, \infty)$ and for $I = [0, 1]$. We should mention that, in the first case, another (implicit) description of this space was obtained in [6]. Moreover, we determine the type and the cotype of Cesàro function spaces and present a complete characterization of the spaces l^q that have isomorphic copies in $\mathrm{Ces}_p[0, 1]$ $(1 \leqslant p < \infty)$.

Below we recall some definitions and notation, which we will need later on. Let $L^0 = L^0(I)$ be the set of all real-valued Lebesgue measurable functions (more precisely, the set of their equivalence classes) defined on I. By a *normed ideal space* $X = (X, \|\cdot\|)$ we understand a linear normed space $X \subset L^0(I)$ with the following property: if $|f| \leqslant |g|$ almost everywhere on I and $g \in X$, then $f \in X$ and $\|f\| \leqslant \|g\|$. If, in addition, X is a complete space, then it is called a *Banach ideal space*.

*The first author acknowledges the support of the Ministry of Education and Science of the Russian Federation (grant no. 3341). The second author acknowledges the support of the Swedish Research Council (VR) (grant no. 621-2008-5058).

 0016–2663/11/4501–0064

For a more detailed information on properties of normed ideal spaces we refer the reader to the monographs [7]–[9].

If X and Y are Banach ideal spaces on I, then $X \hookrightarrow Y$ means that $X \subset Y$ and the embedding is continuous; if, in addition, the inequality $\|x\|_Y \leqslant C\|x\|_X$ holds for all $x \in X$, then we write $X \overset{C}{\hookrightarrow} Y$. By $\operatorname{supp} f = \{t \in I : f(t) \neq 0\}$ we denote the support of a measurable function f. We begin with collecting the most elementary properties of Cesàro function spaces.

Theorem 1. (a) *If $1 < p \leqslant \infty$, then $\mathrm{Ces}_p(I)$ is a Banach ideal space, $\mathrm{Ces}_1[0,1] = L^1_w$ with the weight $w(t) = \ln(1/t)$ $(t \in (0,1])$, and $\mathrm{Ces}_1[0,\infty) = \{0\}$.*

(b) *The space $\mathrm{Ces}_p(I)$ is separable if $1 \leqslant p < \infty$ and nonseparable if $p = \infty$.*

(c) *If $1 < p \leqslant \infty$, then $L^p(I) \overset{p'}{\hookrightarrow} \mathrm{Ces}_p(I)$, where $p' = p/(p-1)$, and the embedding is strict.*

(d) *If $1 < p < \infty$, then $\{f \in \mathrm{Ces}_p[0,1] : \operatorname{supp} f \subset [0,a]\} \hookrightarrow L^1[0,a]$ for every $a \in (0,1)$ and $\{f \in \mathrm{Ces}_p[0,\infty) : \operatorname{supp} f \subset [0,a]\} \hookrightarrow L^1[0,a]$ for all $0 < a < \infty$. Moreover, $\mathrm{Ces}_\infty[0,1] \overset{1}{\hookrightarrow} L^1[0,1]$. At the same time, $\mathrm{Ces}_p[0,1] \not\subset L^1[0,1]$ and $\mathrm{Ces}_p[0,\infty) \not\subset L^1[0,\infty)$ $(1 < p < \infty)$.*

(e) *If $1 < p < q \leqslant \infty$, then $\mathrm{Ces}_q[0,1] \overset{1}{\hookrightarrow} \mathrm{Ces}_p[0,1]$ and the embedding is strict.*

(f) *The space $\mathrm{Ces}_p(I)$ is nonreflexive but strictly convex if $1 < p < \infty$. The latter means that $\|f\|_{\mathrm{Ces}_p} = \|g\|_{\mathrm{Ces}_p} = 1$ and $f \neq g$ imply $\|(f+g)/2\|_{\mathrm{Ces}_p} < 1$.*

2. Description of dual spaces. Let X be a normed ideal space on I. Recall that its *Köthe dual* (or *associated space*) X' is the set of all functions $f \in L^0(I)$ such that

$$\|f\|_{X'} := \sup_{g \in X,\, \|g\|_X \leqslant 1} \int_I |f(x)g(x)|\, dx < \infty.$$

This is a Banach ideal space on I. Let X^* denote the (Banach) dual of X; then $X' \subset X^*$. Moreover, $X' = X^*$ if and only if X is separable.

The following two theorems give a description of the dual (and Köthe dual) space for the space $\mathrm{Ces}_p(I)$ $(1 < p < \infty)$ in the cases $I = [0,\infty)$ and $I = [0,1]$, respectively. The appearance of the weight $(1 - x^{p-1})^{-1}$ in the second case was quite unexpected.

Let $1 < q < \infty$. Then $D(q)$ is the space of functions defined on $[0,\infty)$ with the norm $\|f\|_{D(q)} = \|\tilde{f}\|_{L^q}$, where $\tilde{f}(x) = \operatorname{ess\,sup}_{t \in [x,\infty)} |f(t)|$.

Theorem 2. *If $I = [0,\infty)$ and $1 < p < \infty$, then*

$$(\mathrm{Ces}_p)^* = (\mathrm{Ces}_p)' = D(p'), \quad \textit{where } p' = p/(p-1);$$

the norms on these spaces are equivalent.

For every $1 < q < \infty$, we define the Banach ideal space $U(q)$ on $[0,1]$ as the set of all functions with finite norm

$$\|f\|_{U(q)} = \left[\int_0^1 \left(\frac{\tilde{f}(x)}{1 - x^{1/(q-1)}}\right)^q dx\right]^{1/q}, \quad \text{where } \tilde{f}(x) = \operatorname*{ess\,sup}_{t \in [x,1]} |f(t)|.$$

It is easy to see that an equivalent norm on $U(q)$ (with the constant of equivalence depending on q) can be defined as follows:

$$\|f\|^0_{U(q)} = \left[\int_0^1 \left(\frac{\tilde{f}(x)}{1 - x}\right)^q dx\right]^{1/q}.$$

Theorem 3. *If $I = [0,1]$ and $1 < p < \infty$, then*

$$(\mathrm{Ces}_p)^* = (\mathrm{Ces}_p)' = U(p'), \qquad p' = p/(p-1);$$

the norms on these spaces are equivalent.

The proofs of these theorems use, in particular, some factorization relations similar to those considered in the discrete situation in [10]. We confine ourselves to the case of the semi-axis. Let

$G(p)$ $(1 \leqslant p < \infty)$ be the p-convexification of the space $\mathrm{Ces}_\infty[0,\infty)$ endowed with the norm

$$\|f\|_{G(p)} = \||f|^p\|_{\mathrm{Ces}_\infty}^{1/p} = \sup_{x>0}\left(\frac{1}{x}\int_0^x |f(t)|^p\,dt\right)^{1/p}.$$

Proposition 1. *If $1 < p < \infty$, then $\mathrm{Ces}_p = L^p \cdot G(p')$, i.e., $f \in \mathrm{Ces}_p$ if and only if $f = gh$, where $g \in L^p$, $h \in G(p')$, and*

$$\|f\|_{\mathrm{Ces}_p} \approx \inf\{\|g\|_{L^p}\|h\|_{G(p')} : f = gh,\ g \in L^p,\ h \in G(p')\}.$$

Proposition 2. *If $1 < p < \infty$, then $D(p) \cdot G(p) = L^p$ and*

$$\|f\|_{L^p} = \inf\{\|g\|_{D(p)}\|h\|_{G(p)} : f = gh,\ g \in D(p),\ h \in G(p)\}.$$

Remark 1. Theorems 2 and 3 show, in particular, that, in both cases $I = [0,\infty)$ and $I = [0,1]$, the dual space $(\mathrm{Ces}_p)^*$ is nonseparable.

Remark 2. As already mentioned, the space $\mathrm{Ces}_\infty[0,1] := K$ was introduced by Korenblum, Krein, and Levin [2]. In [11], Luxemburg and Zaanen showed that $K' = \tilde{L}^1$; in particular, the norm on K' is equal to the norm on $\tilde{L}^1$, which is defined by $\|f\|_{\tilde{L}^1} = \|\tilde{f}\|_{L^1}$, where $\tilde{f}(x) = \operatorname{ess\,sup}_{t\in[x,1]}|f(t)|$. Somewhat earlier, in [12], Tandori described the dual space for K_0 being the separable part of the space K; namely, $(K_0)^*$ is the space $\tilde{L}^1$ (with the same norm).

3. Copies of l^q in Ces_p. In [13] it was shown that, for every $1 < p < \infty$, the Cesàro space $\mathrm{Ces}_p(I)$ contains an asymptotically isometric copy of the space l^1, i.e., there exists a sequence $\{\varepsilon_n\} \subset (0,1)$, $\varepsilon_n \to 0$ as $n \to \infty$, and a sequence of functions $\{f_n\} \subset \mathrm{Ces}_p(I)$ such that

$$\sum_{n=1}^\infty (1-\varepsilon_n)|\alpha_n| \leqslant \left\|\sum_{n=1}^\infty \alpha_n f_n\right\|_{\mathrm{Ces}_p} \leqslant \sum_{n=1}^\infty |\alpha_n|$$

for any $\{\alpha_n\} \in l^1$. This property shows the similarity of the spaces $\mathrm{Ces}_p(I)$ and $L^1(I)$; at the same time, the following result indicates a connection between the spaces $\mathrm{Ces}_p(I)$ and $L^p(I)$. Consider the characteristic functions $g_n = \chi_{[2^{-n-1},2^{-n}]}$ $(n = 1,2,\dots)$. It is not hard to show that $\|g_n\|_{\mathrm{Ces}_p} \approx \|g_n\|_{L^p} \approx 2^{-n/p}$ $(n = 1,2,\dots)$.

Theorem 4. *For every $1 < p < \infty$, the space $\mathrm{Ces}_p(I)$ contains an order isomorphic and complemented copy of l^p. Namely, if $\bar{g}_n = g_n/\|g_n\|_{\mathrm{Ces}_p}$ $(n = 1,2,\dots)$, then the sequence $\{\bar{g}_n\}_{n=1}^\infty$ is equivalent to the canonical basis in l^p and generates a complemented subspace in $\mathrm{Ces}_p(I)$.*

Recall that a Banach space X has the *Dunford–Pettis property* if, for any weakly null sequences $\{x_n\}$ in X and $\{f_n\}$ in the dual space X^*, we have $f_n(x_n) \to 0$. Classical examples of Banach spaces with the Dunford–Pettis property are the AL- and AM-spaces [9, Sec. 1b].

Corollary 1. *The spaces $\mathrm{Ces}_p(I)$ $(1 < p < \infty)$ do not have the Dunford–Pettis property.*

It is well known [14, Theorem 6.4.19] that if $1 \leqslant p \leqslant 2$ $(2 < p < \infty)$, then the space l^q is isomorphically embedded in $L^p[0,1]$ if and only if $p \leqslant q \leqslant 2$ ($q = p$ or $q = 2$, respectively). A similar description of isomorphic copies of l^q can be given also for the Cesàro spaces.

Theorem 5. (a) *If $1 \leqslant p \leqslant 2$, then the space l^q is isomorphically embedded in $\mathrm{Ces}_p[0,1]$ if and only if $q \in [1,2]$.*

(b) *If $2 < p < \infty$, then the space l^q is isomorphically embedded in $\mathrm{Ces}_p[0,1]$ if and only if $q \in [1,2]$ or $q = p$.*

The proof of necessity in the last theorem in the case $p > 2$ uses the well-known Kadec–Pełczyński procedure [15] for constructing a sequence of disjointly supported functions equivalent to a given basis sequence of normalized functions in the space $\mathrm{Ces}_p[0,1]$ converging to zero in Lebesgue measure. The difference between this theorem and the result for $L^p[0,1]$ mentioned above is explained mainly by the fact that the space $\mathrm{Ces}_p[0,1]$ contains a copy of $L^1[0,1]$; namely, the norms of these spaces are equivalent on the subspace $\{f \in \mathrm{Ces}_p[0,1] : \operatorname{supp} f \subset [h, 1-h]\}$ for each $h \in (0,1/2)$.

4. Rademaher type of Cesàro spaces. A Banach ideal space X is called *q-concave* ($1 \leqslant q < \infty$) with constant $L > 0$ if

$$\left(\sum_{k=1}^n \|x_k\|^q\right)^{1/q} \leqslant L \left\|\left(\sum_{k=1}^n |x_k|^q\right)^{1/q}\right\|$$

for any $x_1, \dots, x_n$ from X. In particular, the space $L^p(I)$ ($1 \leqslant p < \infty$) is p-concave with constant 1. It turns out that the space $\mathrm{Ces}_p(I)$, where $I = [0, \infty)$ or $I = [0, 1]$, possesses the same property.

Using the fact that the space $L^1[0, x]$ is p-concave with constant 1 for each $x > 0$, it is not hard to prove the following assertion.

Theorem 6. *For every p, $1 < p < \infty$, the Cesàro space $\mathrm{Ces}_p(I)$ is p-concave with constant* 1.

Let $r_n\colon [0, 1] \to \mathbb{R}$ ($n \in \mathbb{N}$) be the Rademacher functions, i.e., $r_n(t) = \mathrm{sign}(\sin 2^n \pi t)$. A Banach space X is said to have *type* p, $1 \leqslant p \leqslant 2$, if there is a constant $K > 0$ such that, for arbitrary vectors $x_1, \dots, x_n$ from X, the following inequality holds:

$$\int_0^1 \left\|\sum_{k=1}^n r_k(t) x_k\right\| dt \leqslant K \left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}.$$

A Banach space X has *cotype* $q \geqslant 2$ if there is a constant $K > 0$ such that, for arbitrary vectors $x_1, \dots, x_n$ from X,

$$\left(\sum_{k=1}^n \|x_k\|^q\right)^{1/q} \leqslant K \int_0^1 \left\|\sum_{k=1}^n r_k(t) x_k\right\| dt.$$

This definition makes sense also in the case $q = \infty$ provided that the left-hand side of the inequality is replaced by $\max_{1 \leqslant k \leqslant n} \|x_k\|$. We say that the space X has *trivial type* (*trivial cotype*) if it does not have any type larger than 1 (any finite cotype, respectively).

As already mentioned, the space $Ces_p(I)$ contains a copy of L^1. Therefore, Theorems 4 and 6 yield the following result.

Corollary 2. *If $1 < p < \infty$, then the space $\mathrm{Ces}_p(I)$ has trivial type and cotype* $\max(p, 2)$. *The space $\mathrm{Ces}_\infty(I)$ has trivial type and trivial cotype.*

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