Extrapolation description of rearrangement invariant spaces and related problems

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Abstract. Problems of the extrapolation theory of operators in the scale of L_p–spaces are considered. Mainly we focus on the problem of an extrapolation description of rearrangement invariant (r.i.) spaces on $[0,1]$. Generalizing Jawerth–Milman's approach we introduce and study a new class $\mathcal{SE}$ of extrapolation r.i. spaces with respect to the scale of L_p–spaces. In spite of the fact that the class $\mathcal{SE}$ contains almost all known by now extrapolation spaces "close" to L_∞ it has a rather simple description. We completely characterize Marcinkiewicz and Lorentz spaces belonging to this class and present some examples of not extrapolation r.i. spaces X such that $X \subset L_p$ for every $p < \infty$. Under mild assumptions it is proved that Peetre's $\mathcal{K}$–functional of a couple (X, L_∞) has an extrapolation description if and only if $X \in \mathcal{SE}$. Moreover, it is investigated the problem when an Orlicz space is representable as an intersection of L_p–spaces with scalar weights. As an application, a sharp Yano type theorem for operators bounded in L_p is established.

1 Introduction and preliminaries

The starting point of the extrapolation theory of operators is the following remarkable theorem proved by Japanese mathematician S.Yano in 1951.

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Theorem 1.1 ([45])**.** *(a) Suppose that a linear operator T is bounded in L_p for every $p \in (1, p_0)$, $p_0 > 1$, and*

$$\|T\|_{L_p \to L_p} \leqslant C(p-1)^{-\beta}, \ \ p \in (1, p_0), \tag{1}$$

for some $\beta > 0$, with a constant $C > 0$ independent of p. Then,

$$T: \ L(\log L)^{\beta} \to L_1,$$

where the Zygmund space $L(\log L)^{\beta}$ consists of all measurable on [0,1] functions $x(t)$ such that

$$\|x\|_{L(\log L)^{\beta}} = \int_0^1 \log^{\beta}(e/t) x^*(t)\, dt < \infty.$$

Here and next, $x^*(t)$ stands for the non-increasing rearrangement of $|x(t)|$.

(b) If T is a linear operator bounded in L_p for all $p > p_0$ and

$$\|T\|_{L_p \to L_p} \leqslant C p^{1/\beta}, \ \ p \in (p_0, \infty), \tag{2}$$

for some $\beta > 0$, with a constant $C > 0$ independent of p, then

$$T: \ L_\infty \to \operatorname{Exp} L^{\beta}.$$

The Zygmund space $\operatorname{Exp} L^{\beta}$ consists of all measurable functions $x(t)$ such that

$$\|x\|_{\operatorname{Exp} L^{\beta}} = \sup_{0 < t \leqslant 1} \log^{-1/\beta}(e/t) x^*(t) < \infty.$$

In fact, the paper [45] contains only the part (a) of Theorem 1.1 but in the classical Zygmund's book "Trigonometric series" both parts were published (see [47, Theorem 12.4.41]). Yano's theorem at once has found valuable applications to studying classical operators of analysis such that the maximal Hardy-Littlewood operator and the operator of trigonometric conjugate function [45, 47]. Their norms in L_p–spaces satisfy estimates of the form (1) or (2) and therefore, by Theorem 1.1, we immediately obtain the appropriate estimates in Zygmund spaces. Thus Yano's theorem can be treated as a converse statement with respect to interpolation theorems: estimates of operator norms in L_p-spaces imply these in appropriate limiting spaces of this scale. So, Yano's result has been called an *extrapolation theorem.*

Soon after that some more papers devoted to extrapolation problems were appeared (see, for instance, [46, 43, 19]). In particular, in [46] and [43] Yano's result was really rediscovered.

Later, in nineties, B. Jawerth and M. Milman laid down the foundations of the general extrapolation theory, which studies natural limiting spaces associated with various interpolation scales and provides estimates for norms of appropriate operators [16, 17]. Mainly, they consider scales generated by the real method of interpolation.

Recall that for any Banach couple (A_0, A_1) (i.e., A_0 and A_1 are Banach spaces linearly and continuously embedded in a common Hausdorff topological vector space) and $t > 0$, the Peetre $\mathcal{K}$–functional is defined as follows:

$$\mathcal{K}(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_i \in A_i, i = 0, 1\}.$$

Then, for every $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$, the space $\vec{A}_{\theta,q}$ consists of all $a \in A_0 + A_1$ such that

$$\|a\|_{\vec{A}_{\theta,q}} := \left(\int_0^\infty \left(t^{-\theta}\mathcal{K}(t, a; A_0, A_1)\right)^q \frac{dt}{t}\right)^{1/q} < \infty \ \text{ if } \ q < \infty$$

and

$$\|a\|_{\vec{A}_{\theta,\infty}} := \sup_{0<t<\infty} t^{-\theta}\mathcal{K}(t, a; A_0, A_1) < \infty.$$

If we replace in the last formulas the function $t^{-\theta}$ with an arbitrary positive function $W(t)$ we get the space $\vec{A}_{W,q}$.

As extrapolation methods (functors) B.Jawerth and M.Milman introduce the sum and the intersection of families of Banach spaces. Let $\{A_\theta\}_{0<\theta<1}$ be a family of Banach spaces such that there exist two Banach spaces U and V for which the continuous inclusions

$$U \subset A_\theta \subset V \ \ (0<\theta<1)$$

hold with uniformly bounded norms. Then the sum

$$\Sigma(A_\theta) = \Big\{ a = \sum_\theta a_\theta \text{ in } V: \ a_\theta \in A_\theta \text{ for some } \theta \text{ and } \ \sum_\theta \|a_\theta\|_{A_\theta} < \infty \Big\},$$

with the norm $\|a\|_{\Sigma(A_\theta)} = \inf \sum\limits_\theta \|a_\theta\|_{A_\theta}$, where infimum is taken over all admissible representations of $a \in \Sigma(A_\theta)$. Analogously, the intersection $\Delta(A_\theta)$ consists of all $a \in \bigcap_{0<\theta<1} A_\theta$ such that

$$\|a\|_{\Delta(A_\theta)} = \sup_{0<\theta<1} \|a\|_{A_\theta} < \infty.$$

In what follows, the expression $f \asymp g$ means that $cf \leqslant g \leqslant Cf$ for some constants $c>0$ and $C>0$ independent of all or of a part of arguments of f and g. Finally, if X is a Banach space and $b>0$, then bX consists of the same elements as X and $\|x\|_{bX} = b\|x\|_X$.

Jawerth–Milman's approach to calculation of norms of extrapolation spaces is based on two key facts. Firstly, Σ– and Δ–functors commute with appropriate interpolation functors. For example,

$$\Delta_{0<\theta<1}\left(M(\theta)\vec{A}_{\theta,\infty}\right) = \vec{A}_{W,\infty}, \tag{3}$$

where $M(\theta)$ is an arbitrary positive function and $W(t) = \sup_\theta t^{-\theta} M(\theta)$. The second fact is valid for any *tempered* function $M(\theta)$ which satisfies the conditions

$$M(\theta) \asymp M(\theta/2) \ \text{ as } \theta \to 0 \ \text{ and } \ M(\theta) \asymp M\left(\frac{1+\theta}{2}\right) \ \text{ as } \theta \to 1.$$

By this assumption, we have the following stability: for all $1 \leqslant q, r \leqslant \infty$

$$\Delta_{0<\theta<1}\left(M(\theta)\langle \vec{A}_{\theta,q}\rangle\right) = \Delta_{0<\theta<1}\left(M(\theta)\langle \vec{A}_{\theta,r}\rangle\right),$$

where $\langle \vec{A}_{\theta,q}\rangle := (\theta(1-\theta)q)^{1/q}\vec{A}_{\theta,q}$. In particular, for the Banach couple $\vec{A} = (L_1, L_\infty)$ of the spaces on the segment $[0,1]$ we have

$$\langle \vec{A}_{\theta,1}\rangle \subset L_p = \langle \vec{A}_{\theta,p}\rangle \subset \langle \vec{A}_{\theta,\infty}\rangle \text{ for } \theta = 1 - \frac{1}{p}$$

uniformly with respect to $1 \leqslant p \leqslant \infty$. Therefore, assuming that $\omega(p) \asymp \omega(2p)$ as $p \to \infty$ we obtain

$$M(\theta) := \omega\left(\frac{1}{1-\theta}\right) = \omega(p) \asymp \omega(2p) = M\left(\frac{1+\theta}{2}\right) \quad \text{as} \quad \theta \to 1,$$

whence

$$\Delta_{1\leqslant p<\infty}\left(\omega(p)L_p\right) = \Delta_{0<\theta<1}\omega\left(\frac{1}{1-\theta}\right)\vec{A}_{\theta,\infty}.$$

Hence, using (3), we are able to find an extrapolation description of Zygmund spaces from Yano's theorem (see, for instance, [34, p. 22-23]). For every $\beta > 0$ and $p_0 > 1$ we have

$$\Sigma_{1<p<p_0}\Big((p-1)^{-\beta}L_p\Big) = L(\log L)^\beta \tag{4}$$

and

$$\Delta_{p\geqslant p_0}\Big(p^{-1/\beta}L_p\Big) = \operatorname{Exp} L^\beta. \tag{5}$$

These relations recover the spaces $L(\log L)^\beta$ and $\operatorname{Exp} L^\beta$ as limiting spaces of the scale of L_p-spaces and, in addition, immediately imply both parts of Yano's theorem. In fact, for example, assuming $\|T\|_{L_p\to L_p} \leqslant Cp^{1/\beta}$, by (5), we obtain

$$\|Tx\|_{\operatorname{Exp} L^\beta} \asymp \sup_{p\geqslant p_0} p^{-1/\beta}\|Tx\|_p \leqslant C\sup_{p\geqslant p_0}\|x\|_p = C\|x\|_\infty.$$

Next, we will concentrate on studying the extrapolation Δ–functor. By (5),

$$\|x\|_{\operatorname{Exp} L^\beta} \asymp \sup_{p\geqslant p_0} p^{-1/\beta}\|x\|_p = \Big\|\|x\|_p\cdot p^{-1/\beta}\Big\|_{L_\infty(p_0,\infty)}. \tag{6}$$

Therefore, the Zygmund space $\operatorname{Exp} L^\beta$ is the result of applying a weighted L_∞-norm to the function $f_x(p) := \|x\|_p$. In [18], L_∞ is replaced with the more general space L_r $(1 \leqslant r \leqslant \infty)$ which allows to obtain new extrapolation spaces. In [6] and [9], we focused on the following in a sense opposite problem: which rearrangement invariant spaces such that $X \subset L_p$ for every $p < \infty$ have an extrapolation description with respect to the scale of L_p-spaces or, in other words, when the norm of an r.i. space X can be expressed by L_p–norms?

First of all, recall basic definitions of the theory of r.i. spaces on $[0,1]$ (its detailed exposition can be found in [21, 22]). A Banach function space X is called a Banach lattice if the conditions $y = y(t) \in X$ and $|x(t)| \leqslant |y(t)|$ imply that $x = x(t) \in X$ and $\|x\| \leqslant \|y\|$. A Banach lattice X is said to be an rearrangement invariant (r.i.) space if from $y = y(t) \in X$ and $x^*(t) \leqslant y^*(t)$ it follows that $x = x(t) \in X$ and $\|x\|_X \leqslant \|y\|_X$.

The simplest and most important example of an r.i. space is given by L_p–space $(1 \leqslant p \leqslant \infty)$, with the usual norm

$$\|x\|_p = \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} \ (1 \leqslant p < \infty) \ \text{ and } \ \|x\|_\infty = \operatorname{ess} \sup_{0 \leqslant t \leqslant 1} |x(t)|.$$

Note that L_∞ is the smallest space among all r.i. spaces on $[0,1]$ and L_1 is the largest one [21, Theorem 2.4.1]. Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then, the space $L_{p,q}$ is defined as the set of all functions for which the following quasi-norm is finite:

$$\|x\|_{p,q} := \left(\frac{q}{p} \int_0^1 \left(t^{1/p} x^*(t)\right)^q \frac{dt}{t}\right)^{1/q} \quad (1 \leqslant q < \infty)$$

and

$$\|x\|_{p,\infty} = \sup_{0 < t \leqslant 1} t^{1/p} x^*(t).$$

Replacing in the previous formulas $x^*(t)$ by $x^{**}(t) := \frac{1}{t}\int_0^t x^*(s)\, ds$, we get in $L_{p,q}$ an equivalent r.i. norm for every $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. In particular, $L_p = L_{p,p}$.

An another natural generalization of L_p–spaces are Orlicz spaces [20]. Let $M(u)$ be an Orlicz function, that is, an increasing convex function on $[0,\infty)$ such that $M(0)=0$. The Orlicz space L_M consists of all measurable functions $x(t)$ on $[0,1]$ such that the function $M\left(\frac{|x(t)|}{u}\right)\in L_1$ for some $u>0$, with the norm

$$\|x\|_{L_M}=\inf\left\{u>0:\int\limits_0^1 M\left(\frac{|x(t)|}{u}\right)\,dt\leqslant 1\right\}.$$

In the case of the power function $M(u)=u^p$ ($1\leqslant p<\infty$) we obtain L_p–space; if $M(u)$ is equivalent to the function e^{u^β} ($\beta>0$) for all u large enough, L_M coincides with the Zygmund space $\operatorname{Exp} L^\beta$ appeared in Yano's theorem. Moreover, the exponential Orlicz space $\operatorname{Exp} L^N$ is generated by the function $e^{N(t)}-1$ ($N(t)$ is an Orlicz function).

Other examples of r.i. spaces are Lorentz and Marcinkiewicz spaces. Let $\varphi(t)$ be an increasing concave function on $[0,1]$, with $\varphi(0)=0$. The Marcinkiewicz space $\mathcal{M}(\varphi)$ consists of all measurable functions $x(t)$ such that

$$\|x\|_{\mathcal{M}(\varphi)}=\sup_{0<s\leqslant 1}\frac{\varphi(s)}{s}\cdot\int\limits_0^s x^*(t)dt<\infty.$$

The Lorentz space $\Lambda_r(\varphi)$ ($1\leqslant r<\infty$) is defined by the norm

$$\|x\|_{\Lambda_r(\varphi)}=\left(\int\limits_0^1 x^*(t)^r\,d\varphi(t)\right)^{1/r}.$$

An important characteristic of an r.i. space X is its fundamental function $\phi_X(t)=\|\chi_{(0,t)}\|_X$, where $\chi_{(0,t)}$ is the characteristic function of the interval $(0,t)$. In particular,

$$\phi_{L_p}(t)=t^{1/p},\quad \phi_{\mathcal{M}(\varphi)}(t)=\phi_{\Lambda_1(\varphi)}(t)=\varphi(t),\quad \phi_{L_M}(t)=\frac{1}{M^{-1}(1/t)}.$$

Moreover, $\Lambda(\varphi):=\Lambda_1(\varphi)$ is the smallest space and $M(\varphi)$ is the largest space among all r.i. spaces with the fundamental function $\varphi(t)$ [21, Theorems 4.5.5 and 4.5.7]. Every fundamental function $\varphi(t)$ is *quasi-concave* (this means that $\varphi(0)=0$, $\varphi(t)>0$, $\varphi(t)$ increases, and $\varphi(t)/t$

decreases). Every such function is equivalent to its least concave majorant $\tilde{\varphi}(t)$; more exactly, $1/2\tilde{\varphi}(t) \leqslant \varphi(t) \leqslant \tilde{\varphi}(t)$ $(0 \leqslant t \leqslant 1)$ [21, Theorem 4.1.1]. An r.i. space X is said to have the *Fatou property* if the conditions $0 \leqslant x_n \uparrow x$ a.e. on $[0,1]$ and $\sup_{n\in\mathbf{N}} \|x_n\| < \infty$ imply $x \in X$ and $\|x_n\| \uparrow \|x\|$.

An Orlicz space L_M "close" to L_∞ coincides with the Marcinkiewicz space with the same fundamental function φ. More precisely [23, 42], $L_M = \mathcal{M}(\varphi)$ if and only if $1/\varphi \in L_M$. In view of the expression for the fundamental function of L_M, the last condition is equivalent to the following: there exists a $\lambda > 0$ such that

$$\int_0^1 M\left(\frac{1}{\lambda}M^{-1}\left(\frac{1}{t}\right)\right)dt < \infty. \tag{7}$$

In particular, a Zygmund space $\operatorname{Exp} L^\beta$ and an exponential Orlicz space $\operatorname{Exp} L^N$ coincide with the Marcinkiewicz space $\mathcal{M}(\varphi)$ generated by the function $\varphi(t) \asymp \log^{-1/\beta}(e/t)$ and $\varphi(t) = 1/N^{-1}(\log(e/t))$, respectively.

Let us recall the definition from [6], where instead of weighted L_r-parameters arbitrary Banach lattices are considered. Let F be a Banach lattice of measurable functions on $[1,\infty)$ (it will be called an *extrapolation parameter*). The set $\mathcal{L}_F$ consists of all functions $x(t)$ for which the function $f_x(p) := \|x\|_p \in F$, with the norm

$$\|x\|_{\mathcal{L}_F} := \|f_x\|_F.$$

It is not hard to check that $\mathcal{L}_F$ is an r.i. space such that $\mathcal{L}_F \subset L_p$ for every $p < \infty$.

Definition 1.2 ([6])**.** We will say that an r.i. space X on $[0,1]$ is an *extrapolation space with respect to the scale of L_p–spaces* ($X \in \mathcal{E}$) if there is a Banach lattice F on $[1,\infty)$ such that $X = \mathcal{L}_F$ (with equivalence of norms).

The simplest example of such a space is the space L_∞. Since

$$\|f\|_{L_\infty} = \lim_{p\to+\infty} \|f\|_p,$$

then $L_\infty = \mathcal{L}_{L_\infty[1,\infty)}$. Other examples are Zygmund spaces $\operatorname{Exp} L^\beta$ ($\beta > 0$) (see (6)) and exponential Orlicz spaces $\operatorname{Exp} L^N$ since, by [4],

$$\|x\|_{\operatorname{Exp} L^N} \asymp \sup_{p \geqslant p_0} \frac{\|x\|_p}{N^{-1}(p)}. \tag{8}$$

Note that in both cases we can take for a parameter F an appropriate weighted L_∞–space.

Some basic properties of Banach spaces $\mathcal{L}_F$ were studied in [27]. Moreover, in [6] necessary and sufficient conditions under which important in applications Marcinkiewicz and Lorentz spaces belong to the class $\mathcal{E}$ have been found. Next, in the paper [9] the following interesting subclass of strong or nice extrapolation spaces was singled out. If X is an r.i. space on $[0, 1]$, then by $\tilde{X}$ we denote the Banach lattice of all measurable on $[1, \infty)$ functions f such that

$$\|f\|_{\tilde{X}} := \|f\big(\log_2(2/t)\big)\|_X < \infty.$$

Definition 1.3 ([9])**.** An r.i. space X is called a *strong extrapolation space* ($X \in \mathcal{SE}$) if $X = \mathcal{L}_{\tilde{X}}$ (with equivalence of norms).

In particular, relations (6) and (8) demonstrate that all Zygmund spaces $\operatorname{Exp} L^\beta$ ($\beta > 0$) and exponential Orlicz spaces $\operatorname{Exp} L^N$ are strong extrapolation spaces.

Thus, for a strong extrapolation space X the corresponding extrapolation parameter $\tilde{X}$ is completely and explicitly determined by X. We observe the following surprised phenomenon: the norm of any function in every strong extrapolation r.i. space X is equivalent to the norm in X of its L_p–norm. More precisely, $\|x\|_X \asymp \big\| \, \|x\|_{\log_2(2/t)} \, \big\|_X$.

In [9], a rather simple characterization of strong extrapolation spaces has been found. Namely, an r.i. space $X \in \mathcal{SE}$ if and only if the operator $Sx(t) = x(t^2)$ is bounded in X. Moreover, in a partial case including Marcinkiewicz and Lorentz spaces a simpler criterion holds. In Section 2, we review these results. Moreover, we introduce the concept of a *tempered* extrapolation parameter generalizing the Jawerth-Milman's idea of a tempered weight. Then, we prove that an r.i. space $X \in \mathcal{SE}$ if and only if $X = \mathcal{L}_F$ with a tempered extrapolation parameter

F (see Theorem 2.2). This implies, in particular, that every limiting space of the L_p–scale "near" to L_∞ whose norm can be calculated by Jawerth-Milman's approach is a strong extrapolation space. At the same time, in this section some examples of extrapolation but not strong extrapolation Marcinkiewicz spaces are presented. By Theorem 2.2, every such space is generated by a not tempered parameter and therefore cannot be obtained by Jawerth-Milman's approach. Section 3 contains a review of results from [9] about an extrapolation description of the $\mathcal{K}$-functional $\mathcal{K}(t, x; X, L_\infty)$ provided that X is a strong extrapolation space (Theorem 3.1). In Section 4, it is investigated the problem when an Orlicz space L_M is representable in the form:

$$L_M = \Delta_{1\leqslant p<\infty}\left(\omega(p)L_p\right), \tag{9}$$

with some bounded measurable function $\omega(p)$. We find some sufficient conditions (see Theorems 4.6 and 4.7) and prove extrapolation relations for corresponding dual spaces which are both Orlicz and Lorentz spaces. In concluding part of this section we provide some examples of equalities of the form (9) both with tempered and not tempered functions $\omega(p)$. At last, Section 5 contains some concluding remarks and applications. Firstly, we consider a discrete version of the extrapolation Δ–functor and show that in contrast to the case of the continuous scale any extrapolation Δ–functor always gives on the scale $\{L_k\}_{k=1}^\infty$ an Orlicz space. Secondly, we prove a sharp extrapolation Yano type theorem. Finally, using a description of strong extrapolation spaces derived in Theorem 2.2, we find sufficient conditions for the rearrangement invariance of Rademacher multiplicator spaces.

The *dilation function* of a positive function $\varphi(t)$ $(0 \leqslant t \leqslant 1)$ is defined as

$$\mathcal{M}_\varphi(s) = \sup_{0<t\leqslant\min(1,1/s)} \frac{\varphi(st)}{\varphi(t)},\ 0 < s < \infty.$$

There are the numbers

$$\gamma_\varphi = \lim_{s\to0+} \frac{\ln \mathcal{M}_\varphi(s)}{\ln s} \quad \text{and} \quad \delta_\varphi = \lim_{s\to+\infty} \frac{\ln \mathcal{M}_\varphi(s)}{\ln s}$$

which are called the *lower and upper dilation indices* of φ. For a quasi-concave φ $0 \leqslant \gamma_\varphi \leqslant \delta_\varphi \leqslant 1$ [21, p. 76]. Finally, given Banach couple

(X_0, X_1), a Banach space X is an *interpolation space* with respect to (X_0, X_1) if $X_0 \bigcap X_1 \subset X \subset X_0 + X_1$ and, for every linear operator T bounded in X_0 and in X_1, T is bounded in X.

2 Strong extrapolation spaces

The next definition is a generalization of the Jawerth-Milman idea of a tempered weight.

Definition 2.1. An extrapolation parameter F will be called *tempered* if the operator $Df(p) := f(2p)$ is bounded in F.

Recall that the concepts of extrapolation and strong extrapolation r.i. spaces (with respect to the scale of L_p–spaces) were been introduced in [6] and [9], respectively (see Definitions 1.2 and 1.3 in the Introduction). One of the main results of this article is the following characterization of strong extrapolation spaces.

Theorem 2.2. *For any r.i. space X the following conditions are equivalent:*

(a) $X \in \mathcal{SE}$;

(b) the operator $Sf(t) = f(t^2)$ is bounded in X.

(c) $X = \mathcal{L}_F$ with a tempered extrapolation parameter F;

To prove Theorem 2.2 and some other statements we need the following lemma.

Lemma 2.3. *The operator $Sx(t) = x(t^2)$ is bounded in an r.i. space X if and only if $\|Sx^*\|_X \leqslant C\|x\|_X$ for some $C > 0$ and all $x \in X$.*

Proof. Since necessity is obvious, we should prove only sufficiency. Let $x \in E$ and let $\tau > 0$. If

$$A = \{t \in [0,1] : |\, x(t) \,| > \tau\} \text{ and } B = \{t \in [0,1] : |\, x(t^2) \,| > \tau\},$$

then $B = \kappa(A)$ with $\kappa(u) = \sqrt{u}$. Hence (see, for instance, [36, Lemma 9.5.1]),

$$\mu\{t:\ |Sx(t)| > \tau\} = \mu(B) = \int\limits_A \frac{du}{2\sqrt{u}} \leqslant$$

$$\leqslant \int\limits_0^{\mu\{t:\, x^*(t)>\tau\}} \frac{du}{2\sqrt{u}} = \mu\{t:\ Sx^*(t) > \tau\}.$$

Thereby $(Sx)^*(t) \leqslant Sx^*(t)$ and, by assumption, we obtain

$$Sx \in X \quad \text{and} \quad \|Sx\|_X \leqslant \|Sx^*\|_X \leqslant C\|x\|_X.$$

□

Proof of Theorem 2.2. Firstly, show that the conditions (a) and (b) are equivalent.

Let $x = x(t)$ be a measurable function on $[0,1]$. Consider the function $\tilde{x}(t) := \|x\|_{\log_2 2/t}$ $(0 < t \leqslant 1)$. Since

$$\|x\|_p \geqslant \left\|x^*\chi_{[0,2^{-p+1}]}\right\|_p \geqslant \frac{1}{2}x^*(2^{-p+1}) \ \ (p \geqslant 1),$$

then $\|\tilde{x}\|_X \geqslant \frac{1}{2}\|x\|_X$. Therefore, by the definition of a strong extrapolation space, condition (a) of the theorem is equivalent to the inequality

$$\|\tilde{x}\|_X \leqslant C\|x\|_X, \tag{10}$$

which holds for all $x \in X$ and some $C > 0$.

First, we assume S to be bounded in X. Since $\|S^n x\|_X \leqslant \|S\|^n\|x\|_X$

$(n = 1, 2, \dots)$, we infer

$$\begin{aligned}\tilde{x}(2^{-p+1}) = \|x\|_p \leqslant 2\big\|x^* \cdot \chi_{[0,2^{-p})}\big\|_p &\leqslant 2\sum_{n=0}^{\infty}\Big\|x^* \cdot \chi_{[2^{-2^{n+1}p},2^{-2^n p})}\Big\|_p \\ &\leqslant 2\sum_{n=0}^{\infty} 2^{-2^n} x^*\left(2^{-2^{n+1}p}\right),\end{aligned}$$

for all $p \geqslant 1$, or equivalently

$$\tilde{x}(2t) \leqslant 2\sum_{n=0}^{\infty} 2^{-2^n} S^{n+1}x^*(t) \ \ (0 < t \leqslant 1/2).$$

Hence,

$$\|\tilde{x}(2t)\|_X \leqslant 2\sum_{n=0}^{\infty} 2^{-2^n}\|S\|^{n+1}\|x\|_X \leqslant C_1\|x\|_X.$$

Since the dilation operator $\sigma_2 x(t) = x(t/2)$ is bounded in every r.i. space and $\|\sigma_2\|_{X\to X} \leq 2$ [21, Theorem 2.4.4], the last inequality implies (10) with $C = 2C_1$.

Let now conversely $X = \mathcal{L}_{\tilde{X}}$, i.e., $\tilde{x} \in X$ whenever $x \in X$ and inequality (10) holds. If $t = 2^{-p+1}$ $(p \geqslant 1)$, then

$$x^*(t^2) \leqslant x^*(2^{-2p}) \leqslant 4\big\|x^* \cdot \chi_{[0,2^{-2p}]}\big\|_p \leqslant 4\|x^*\|_p = 4\tilde{x}(t).$$

Therefore, $\|Sx^*\|_X \leqslant 4\|\tilde{x}\|_X \leqslant 4C\|x\|_X$. Applying Lemma 2.3, we conclude that S is bounded in X.

Next, prove that conditions (a) and (b) imply (c). In fact, with (a) at hand we may take $F = \tilde{X}$. Then, by condition (b) and again by the boundedness of the dilation operator $\sigma_2 x(t) = x(t/2)$ in X, for an arbitrary $f = f(p) \in F$ we have

$$\|f(2p)\|_F = \|f(2\log_2(2/t))\|_X = \|f(\log_2(4/t^2))\|_X \leqslant$$

$$\leqslant C\|f(\log_2(4/t))\|_X \leqslant 2C\|f(\log_2(2/t))\|_X = 2C\|f\|_F.$$

At last, we check that (c) implies (b). Suppose that $X = \mathcal{L}_F$, where F is a tempered extrapolation parameter. Applying the Hölder–Rogers inequality, we obtain

$$\int_0^1 |Sx(t)|^p\, dt = \int_0^1 \left|x(t^2)\right|^p\, dt = \int_0^1 |x(s)|^p\, \frac{ds}{2\sqrt{s}} \leqslant$$

$$\leqslant \left(\int_0^1 |x(s)|^{4p}\, ds\right)^{\frac{1}{4}} \left(\int_0^1 \frac{1}{(2\sqrt{s})^{\frac{4}{3}}}\, ds\right)^{\frac{3}{4}},$$

whence $\|Sx\|_p \leqslant C_1\|x\|_{4p}$. Since F is a tempered parameter, then

$$\|Sx\|_X \leqslant C_2\Big\|\|Sx\|_p\Big\|_F \leqslant C_1C_2\Big\|\|x\|_{4p}\Big\|_F \leqslant C_3\Big\|\|x\|_p\Big\|_F \leqslant C_4\|x\|_X,$$

which implies that the operator S is bounded in X. □

Corollary 2.4. *An r.i. space $X \in \mathcal{SE}$ if and only if there exists a constant $C > 0$ such that*

$$\Big\|\sum_{n=1}^{\infty} \|x^*\chi_{[2^{-2n},2^{-2n+1})}\|_n \chi_{[2^{-n},2^{-n+1})}\Big\|_X \leqslant C\|x\|_X$$

for all $x \in X$.

Proof. First of all, using the boundedness of the dilation operator σ_2 in X [21, Theorem 2.4.4] once more, we obtain

$$\frac{1}{2}\Big\|\sum_{k=1}^{\infty} x^*(2^{-k})\chi_{[2^{-k},2^{-k+1})}\Big\|_X \leqslant \|x\|_X \leqslant 2\Big\|\sum_{k=1}^{\infty} x^*(2^{-k+1})\chi_{[2^{-k},2^{-k+1})}\Big\|_X$$

This together with the following elementary inequality

$$x^*(2^{-k+1}) \leqslant 2\Big\|x^*\chi_{[2^{-k},2^{-k+1})}\Big\|_k \leqslant x^*(2^{-k}) \quad (k=1,2,\dots)$$

imply that

$$\Big\|\sum_{k=1}^{\infty}\Big\|x^*\chi_{[2^{-k},2^{-k+1})}\Big\|_k\chi_{[2^{-k},2^{-k+1})}\Big\|_X \leqslant \|x\|_X \leqslant$$

$$\leqslant 4\Big\|\sum_{k=1}^{\infty}\Big\|x^*\chi_{[2^{-k},2^{-k+1})}\Big\|_k\chi_{[2^{-k},2^{-k+1})}\Big\|_X.$$

Applying Theorem 2.2, we complete the proof. □

Theorem 2.2 implies that every r.i. space being an interpolation space with respect to a couple of strong extrapolation spaces is a strong extrapolation space as well. In particular, we obtain

Corollary 2.5. *Let N_1 and N_2 be two Orlicz functions and X be an interpolation r.i. space with respect to the couple* $(\mathrm{Exp}\, L^{N_1}, \mathrm{Exp}\, L^{N_2})$. *Then X is a strong extrapolation space.*

Remark 2.6. As it is easily seen, instead of the norm in $\tilde{X}$ we can use in Theorem 2.2 the seminorm

$$\|f\|_{\tilde{X},p_0} = \|f\chi_{[p_0,\infty)}\|_{\tilde{X}},$$

with some fixed $p_0 > 1$.

Remark 2.7. Recall that using Jawerth-Milman's theory [18] we may identify an extrapolation space (with respect to the L_p-scale) of the form $\Delta^{(r)}(\omega(p)L_p)$ with the norm

$$\|x\|_{\Delta^{(r)}} \asymp \left(\int_{p_0}^{\infty}\Big(\omega(p)\|x\|_p\Big)^r\, dp\right)^{\frac{1}{r}} \text{ if } r<\infty$$

and

$$\|x\|_{\Delta^{(\infty)}} \asymp \sup_{p \geqslant p_0} \Big(\omega(p)\|x\|_p\Big)$$

whenever

$$\omega(p) \asymp \omega(2p) \ \text{ as } \ p \to \infty. \tag{11}$$

Without loss of generality, we may assume that $\omega(p/2) \leqslant C\omega(p)$ if $p \geqslant 2p_0$. Thus,

$$\begin{aligned}\left(\int_{p_0}^{\infty} (\omega(p)f(2p))^r \, dp\right)^{1/r} &= \left(\int_{2p_0}^{\infty} (\omega(p/2)f(p))^r \, \frac{dp}{2}\right)^{1/r} \\ &\leqslant C\left(\int_{p_0}^{\infty} (\omega(p)f(p))^r \, dp\right)^{1/r},\end{aligned}$$

which implies that the extrapolation parameter $F = L_r(\omega)$ is tempered. Therefore, all r.i. spaces of the form $\Delta^{(r)}(\omega(p)L_p)$, with $\omega(p)$ satisfying condition (11), possess property (c) from Theorem 2.2 and, according to this theorem, are strong extrapolation spaces.

Remark 2.8. The construction resulting in strong extrapolation spaces as well as the $\Delta^{(r)}$-functor with a tempered weight is in a sense stable. Consider instead of the scale of L_p–spaces the scale of $L_{p,q}$-spaces ($1 < p < \infty$), where a $q \in [1, \infty]$ is fixed, and a given Banach lattice F on $[1, \infty)$ define the space $\mathcal{L}_{F,q}$ consisting of all functions $x = x(t)$ measurable on $[0, 1]$ such that $f_{x,q}(p) := \|x\|_{p,q} \in F$. Arguing in the same way as in the proof of Theorem 2.2, it can be showed that either of its conditions (a), (b), and (c) is equivalent to either of the following ones:

(d) for every $q \in [1, \infty]$ $X = \mathcal{L}_{\tilde{X},q}$, i.e., $\|x\|_X \asymp \big\| \|x\|_{\log_2(2/t),q} \big\|_X$ $(x \in X)$;

(e) there is a $q \in [1, \infty]$ such that $X = \mathcal{L}_{\tilde{X},q}$.

In some important cases condition (b) under which an r.i. space belongs to the class $\mathcal{SE}$ is equivalent to a much simpler condition [6].

Definition 2.9. We say that a nonnegative function $\varphi(t)$ on $[0, 1]$ satisfies *the* Δ^2*–condition* ($\varphi \in \Delta^2$) whenever there exists a $C > 0$ such that

$$\varphi(t) \leqslant C\varphi(t^2),\ 0 \leqslant t \leqslant 1. \tag{12}$$

Proposition 2.10. *Given an r.i. space X on $[0, 1]$, consider the following conditions:*

(i) the operator $Sx(t) = x(t^2)$ is bounded in X;

(ii) the fundamental function $\phi_X(t)$ of X satisfies the Δ^2–condition.

Then (i) implies (ii). Conversely, if X is an interpolation space with respect to a couple $(\Lambda(\varphi), M(\varphi))$ of Lorentz and Marcinkiewicz spaces with the same fundamental function $\varphi(t)$ (in particular, if X coincides with $\Lambda(\varphi)$ or with $M(\varphi)$), then (ii) implies (i).

Proof. Since the implication $(i) \Rightarrow (ii)$ is obvious it suffices to prove the opposite one.

Let $X = \Lambda(\varphi)$. It is well known that the boundedness in a Lorentz space of a linear operator that is continuous in the space of all finite a.e. measurable functions with respect to convergence in measure is equivalent to its boundedness on the set of characteristic functions [21, Lemma 2.5.2]. Since for the operator S the last is equivalent to (12), then in this case we done.

Examine now the case of a Marcinkiewicz space. Firstly, if $\varphi \in \Delta^2$, then for every $x \in M(\varphi)$

$$\varphi(t)Sx^*(t) \leqslant C\varphi(t^2)x^*(t^2) \leqslant C \sup_{0<s\leqslant 1} \varphi(s)x^*(s) \leqslant C\|x\|_{M(\varphi)}.$$

It is immediate that the condition $\varphi \in \Delta^2$ implies $\delta_\varphi = 0$. Hence, using the last inequality and [21, Theorem 2.5.3], we infer

$$\|Sx^*\|_{M(\varphi)} \leqslant C' \sup_{0<t\leqslant 1} Sx^*(t)\varphi(t) \leqslant C'C\|x\|_{M(\varphi)}.$$

Thereby it remains to refer to Lemma 2.3.

At last, let X be an interpolation space with respect to a couple $(\Lambda(\varphi), M(\varphi))$. Then, obviously, $\varphi \asymp \phi_X$ and therefore $\varphi \in \Delta^2$. Since by the previous arguments S is bounded in $\Lambda(\varphi)$ and in $M(\varphi)$, then it is bounded also in X. $\square$

Theorem 2.2 and the last proposition imply the following result (a weaker version was proved earlier in [26]).

Theorem 2.11. *Let φ be an increasing concave function on $[0,1]$ and let X be an interpolation space with respect to the couple $(\Lambda(\varphi), M(\varphi))$ (in particular, X may be $\Lambda(\varphi)$ or $M(\varphi)$). Then X is a strong extrapolation space if and only if $\varphi \in \Delta^2$.*

Corollary 2.12. *Suppose $\varphi \in \Delta^2$ and $1 \leqslant r < \infty$. Then*

$$\|x\|_{\mathcal{M}(\varphi)} \asymp \sup_{1 \leqslant p < \infty} \varphi(2^{-p}) \|x\|_p \tag{13}$$

and

$$\|x\|_{\Lambda_r(\varphi)} \asymp \left(\int_1^\infty \|x\|_p^r 2^{-p} \varphi'(2^{-p}) dp \right)^{\frac{1}{r}}. \tag{14}$$

Relations (13) and (14) were established for the first time in [6]. Earlier, in [25], Lukomskii has proved the second of them by a stronger assumption: $\varphi'(t) \leqslant Ct\varphi'(t^2)$ $(0 < t \leqslant 1)$. In the same paper, he has applied (14) by consideration of the problem of convergence of trigonometric series in r.i. spaces.

Relations (13) and (14) can be obtained also from Jawerth-Milman theory using the functors $\Delta^{(r)}$. Let us show that for Marcinkiewicz spaces. Suppose $\varphi \in \Delta^2$. Then, by [34, formula (4.6)],

$$\|x\|_{\Delta(\varphi(2^{-p})L_p)} \asymp \sup_{s>0} \frac{\mathcal{K}(s, x;\, L_1, L_\infty)}{\tau(s)}, \tag{15}$$

where

$$\tau(t) = \frac{t}{\sup_{p \geqslant 1} \varphi(2^{-p}) t^{1/p}}.$$

Firstly, $\sup_{p \geqslant 1} \varphi(2^{-p}) t^{\frac{1}{p}} \geqslant \varphi(t)/2$, for every $t < \frac{1}{2}$. Conversely, let $t = 2^{-p_1}$ and C be a constant from inequality (12). In the case, when $p \geqslant p_1$, we have $\varphi(2^{-p}) t^{1/p} \leqslant \varphi(t)$. If $p \leqslant p_1$, then $p_1 \in [2^{n-1}p, 2^n p]$, for some positive integer n. Thereby

$$\varphi(2^{-p}) t^{1/p} \leqslant C^n \varphi(2^{-2^n p}) 2^{-p_1/p} \leqslant C^n 2^{-2^{n-1}} \varphi(t) \leqslant C_1 \varphi(t),$$

where $C_1 = \sup_{n \in \mathbb{N}} C^n 2^{-2^{n-1}} < \infty$. Thus,

$$\sup_{p \geqslant 1} \varphi(2^{-p}) t^{1/p} \asymp \varphi(t),$$

which implies that $\tau(t) \asymp t/\varphi(t)$. Therefore, by (15),

$$\sup_{1\leqslant p<\infty} \varphi(2^{-p})\|x\|_p = \|x\|_{\Delta(\varphi(2^{-p})L_p)} \asymp \sup_{s>0} \frac{\varphi(s)}{s} \int_0^s x^*(t)\,dt = \|x\|_{\mathcal{M}(\varphi)},$$

and we come to (13).

In general, the condition $\varphi \in \Delta^2$ is not sufficient for an r.i. space X with the fundamental function φ to be strong extrapolation.

Theorem 2.13. *There is an r.i. space X which is a closed subspace of a Marcinkiewicz space $M(\varphi)$, with $\varphi \in \Delta^2$, such that $X \notin \mathcal{SE}$.*

Proof. Let $\varphi = \varphi(t)$ be an increasing concave function on $[0,1]$ such that $\varphi(+0) = \varphi(0) = 0$ and $\varphi \in \Delta^2$. Define a sequence $\{t_n\}_{n=0}^{\infty} \in [0,1]$ by induction as follows: $t_0 = 1$ and t_n are chosen arbitrarily so that

$$\frac{\varphi(t_{n-1}^2)}{\varphi(t_n)} > n.$$

Clearly, $t_n \to 0$. We put

$$\bar{x}(t) = \sum_{n=1}^{\infty} \frac{1}{\varphi(t_{n-1}^2)} \cdot \chi_{(t_n^2, t_{n-1}^2]}(t)$$

and

$$X_0 := \{y \in M(\varphi) : y^*(t) \leqslant B\bar{x}(\beta t) \text{ for some } B, \beta > 0 \,\text{and all}\, t \in (0,1]\}.$$

It is easy to see that X_0 is a linear subset of the Marcinkiewicz space $M(\varphi)$. Define X as the closure of X_0 in $M(\varphi)$. Then X is an r.i. space endowed with the norm of $M(\varphi)$. In particular, the fundamental function of this space $\phi_X(t)$ coincides with $\varphi(t)$ and therefore satisfies the Δ^2–condition.

Show now that the function $x_1(t) := \bar{x}(t^2) \notin X$. It suffices to check that

$$\inf_{y \in X_0} \|x_1 - y\|_{M(\varphi)} > 0. \tag{16}$$

Let $y \in X_0$ be arbitrary. Applying the well-known inequality [21, Theorem 2.3.1]

$$\|a - b\|_{M(\varphi)} \geqslant \|a^* - b^*\|_{M(\varphi)},$$

we obtain

$$\|x_1 - y\|_{M(\varphi)} \geqslant \|x_1 - y^*\|_{M(\varphi)} \geqslant \|(x_1 - y^*)\chi_{(t_n^2, t_n]}\|_{M(\varphi)}. \tag{17}$$

Choose a sufficiently large $n \in \mathbb{N}$ so that

$$1)\ \frac{\varphi(t_{n-1}^2)}{\varphi(t_n)} > 2B \ \text{ and } \ \left(1 + \frac{1}{\beta}\right) t_n^2 \leqslant t_n,$$

where $B > 0$ and $\beta \in (0,1)$ are such that $y^*(t) \leqslant B\bar{x}(\beta t)$. In this case

$$\begin{aligned}
\|(x_1 - y^*)\chi_{(t_n^2, t_n]}\|_{M(\varphi)} &\geqslant \|(x_1(t_n) - B\bar{x}(\beta t))\chi_{(t_n^2/\beta, t_n]}\|_{M(\varphi)} = \\
= \left(x_1(t_n) - B\bar{x}(t_{n-1}^2)\right)\varphi(t_n - t_n^2/\beta) &\geqslant \left(\bar{x}(t_n^2) - B\bar{x}(t_{n-1}^2)\right)\varphi(t_n^2) = \\
= \left(\frac{1}{\varphi(t_n^2)} - \frac{B}{\varphi(t_{n-1}^2)}\right)\varphi(t_n^2) &\geqslant \left(\frac{1}{\varphi(t_n)} - \frac{B}{\varphi(t_{n-1}^2)}\right)\varphi(t_n^2) = \\
= \frac{1}{\varphi(t_n)}\left(1 - \frac{B\varphi(t_n)}{\varphi(t_{n-1}^2)}\right)\varphi(t_n^2) &> \frac{\varphi(t_n^2)}{\varphi(t_n)} \cdot \frac{1}{2} \geqslant \frac{1}{2C}.
\end{aligned}$$

Hence, (17) implies that for every $y \in X_0$

$$\|x_1 - y\|_{M(\varphi)} \geqslant \frac{1}{2C},$$

and thus (16) is proved. Finally, Theorem 2.2 guarantees that the r.i. space X is not a strong extrapolation space. □

By [6, Theorem 1], a Marcinkiewicz space $M(\varphi)$ is an extrapolation space if and only if

$$\varphi(t) \leqslant C \cdot \sup_{p\geqslant 1} \frac{t^{1/p}}{\|1/\varphi\|_p}, \quad 0 < t \leqslant 1. \tag{18}$$

Moreover, the last condition implies that

$$\|x\|_{M(\varphi)} \asymp \sup_{p\geqslant 1} \frac{\|x\|_p}{\|1/\varphi\|_p}. \tag{19}$$

Now, assume that $\varphi \in \Delta^2$. Then, by [6, Lemma 1],

$$\|1/\varphi\|_p \asymp \frac{C}{\varphi(2^{-p})} \ \ (p \geqslant 1),$$

and therefore we have (18). Moreover, it is obvious that in this case (19) implies (13). In this connection, a natural question arises: Is there an extrapolation Marcinkiewicz space $M(\varphi)$ generated by a function $\varphi \notin \Delta^2$? In other words, is there an extrapolation space which cannot be obtained by Jawerth-Milman's theory ? We give an example showing that the answer is affirmative (other examples of this sort will be presented in Section 4).

Example 2.14. Consider an increasing concave function $\varphi(t)$ such that $\varphi(t) \asymp \psi(t) = e^{1-\ln^\theta e/t}$ for some $\theta \in (0,1)$. Such a function exists since $\psi(t) > 0, \ \psi'(t) > 0, \ \psi''(t) < 0$ for $t \in (0,t_0)$ (where t_0 is sufficiently small) and $\lim_{t\to 0+} \psi(t) = 0$.

Show that $M(\varphi) \in \mathcal{E}$. It suffices to prove (18) with ψ instead of φ. Firstly,

$$\|1/\psi\|_p^p = \int_0^1 e^{p\ln^\theta (e/t) - p} dt = e^{-p+1} \cdot \int_1^{+\infty} e^{ps^\theta - s}\, ds.$$

Let us represent the integrand in the form: $e^{ps^\theta - s} = e^{ps^\theta - \beta s} \cdot e^{-(1-\beta)s}$, where $\beta \in (0,1)$ is a parameter, and consider the function $f_\beta(s) =$

$ps^\theta - \beta s$ $(s \geqslant 1)$. It is not hard to check that $\max_{s\geqslant 1} f_\beta(s) = (1-\theta)\cdot p^{\frac{1}{1-\theta}} \cdot (\theta/\beta)^{\frac{\theta}{1-\theta}}$. Consequently,

$$\begin{aligned}\|1/\psi\|_p^p &\leqslant e^{-p+1}\cdot\exp\left((1-\theta)p^{\frac{1}{1-\theta}}\cdot\left(\frac{\theta}{\beta}\right)^{\frac{\theta}{1-\theta}}\right)\cdot\int\limits_0^\infty e^{-(1-\beta)s}ds\\ &= e^{-p+1}\cdot\exp\left((1-\theta)p^{\frac{1}{1-\theta}}\cdot\left(\frac{\theta}{\beta}\right)^{\frac{\theta}{1-\theta}}\right)\cdot\frac{1}{1-\beta},\end{aligned}$$

whence

$$\|1/\psi\|_p \leqslant \exp\left((1-\theta)\cdot\left(\frac{\theta p}{\beta}\right)^{\frac{\theta}{1-\theta}}\right)\cdot\left(\frac{1}{1-\beta}\right)^{1/p}.$$

Put $\beta = p^\gamma/(1+p^\gamma)$, where $\gamma = \theta/(1-\theta)$. Then

$$(1-\beta)^{-1/p} = (1+p^\gamma)^{1/p} \leqslant 2p^{\gamma/p} < C$$

for some $C > 0$ and all $p \geqslant 1$. Moreover, since $(1+x)^\gamma \leqslant 1 + C_\gamma x$ for all $x \in [0,1]$, we have

$$(1-\theta)\cdot\left(\frac{\theta p}{\beta}\right)^{\frac{\theta}{1-\theta}} = p^\gamma\cdot\left(1+\frac{1}{p^\gamma}\right)^\gamma\cdot(1-\theta)\theta^{\frac{\theta}{1-\theta}} \leqslant (1-\theta)\theta^{\frac{\theta}{1-\theta}}(p^\gamma + C_\gamma),$$

and thus

$$\|1/\psi\|_p \leqslant C_1\cdot\exp((1-\theta)\cdot\theta^{\frac{\theta}{1-\theta}}\cdot p^{\frac{\theta}{1-\theta}}). \tag{20}$$

To prove (18), verify that the inequality

$$\psi(t)\,\|1/\psi\|_p \leqslant C_2 t^{1/p} \tag{21}$$

holds for $p = \theta^{-1}\ln^{1-\theta} e/t$ with the constant $C_2 = C_1\cdot e$. In fact, on the one hand, by (20),

$$\psi(t)\,\|1/\psi\|_p \leqslant C_1\exp((1-\ln^\theta e/t)+(1-\theta)\ln^\theta e/t) = C_2\exp(-\theta\ln^\theta e/t).$$

On the other hand,

$$t^{1/p} = \exp((\ln t)/p) = \exp(\theta(\ln^{\theta-1} e/t)(1 - \ln e/t)) \geqslant \exp(-\theta \ln^{\theta} e/t).$$

Thus, (21) is proved and $M(\varphi) \in \mathcal{E}$. At the same time, it is not hard to check that $\varphi \notin \Delta^2$.

3 Extrapolation description of Peetre's $\mathcal{K}$–functional

The extrapolation theory developed in [16, 17, 18, 34] allows us to reverse in a sense the interpolation process. In particular, in the case of scales generated by the real method of interpolation it is possible to recover the $\mathcal{K}$–functional corresponding to the initial Banach couple. More precisely, for any Banach couple $\vec{A} = (A_0, A_1)$ we have

$$\|a\|_{\Sigma_{0<\theta<1}\left(t^{\theta}\langle \vec{A}_{\theta,q}\rangle\right)} \asymp \mathcal{K}(t, a; A_0, A_1),$$

with a constant independent of $a \in A_0 + A_1$ and $t > 0$ [35, formula (B.26)]. Here, as above, $\langle \vec{A}_{\theta,q}\rangle := (\theta(1-\theta)q)^{\frac{1}{q}} \vec{A}_{\theta,q}$.

Let us consider the problem when a $\mathcal{K}$–functional $\mathcal{K}(t, f; X, L_\infty)$, where X is an r.i. space on $[0, 1]$, may be described by the extrapolation procedure from the previous section. We show that under certain assumptions it is possible if and only if X is a strong extrapolation space.

Theorem 3.1. *Let X be an r.i. space on* $[0, 1]$.

(i) If $X \in \mathcal{SE}$, then

$$\mathcal{K}(t, x; X, L_\infty) \asymp \mathcal{K}(t, \|x\|_{\log_2(2/u)}; X, L_\infty), \tag{22}$$

with constants independent of $x \in X$ and $t > 0$;

(ii) If there is a Banach lattice F on $[1,\infty)$, $F \supset L_\infty$, such that

$$\mathcal{K}(t,x;X,L_\infty) \asymp \mathcal{K}(t,\|x\|_p;F,L_\infty[1,\infty)), \tag{23}$$

with constants independent of $x \in X$ and $t > 0$, then the fundamental function φ of X satisfies the Δ^2–condition.

To prove this theorem, we need several auxiliary assertions. The first of them is proved in [28, Example 3]. Since the proof of the second one is similar, it is omitted as well.

Let X be an r.i. space on $[0,1]$, $X \neq L_\infty$. Then its fundamental function $\varphi(t)$ satisfies the condition $\lim_{t\to 0+}\varphi(t) = 0$. Without loss of generality, we may assume that $\varphi(t)$ is strictly increasing on $[0,1]$ and $\varphi(1) = 1$.

Lemma 3.2. *For every $x \in X$ and $0 < t \leqslant 1$ we have*

$$\|x^*\chi_{[0,\varphi^{-1}(t)]}\|_X \leqslant \mathcal{K}(t,x;X,L_\infty) \leqslant 2\cdot\|x^*\chi_{[0,\varphi^{-1}(t)]}\|_X.$$

Let now $X \in \mathcal{E}$, i.e. $X = \mathcal{L}_F$, with a Banach lattice F on $[1,\infty)$. If $X \neq L_\infty$, then there exists $x \in X$ such that $\lim_{p\to+\infty}\|x\|_p = \infty$. Hence, the function

$$\psi(u) := \|\chi_{[u,\infty)}\|_F \quad (u \geqslant 1) \tag{24}$$

satisfies the condition $\lim_{u\to+\infty}\psi(u) = 0$. Without loss of generality, it may be assumed that ψ is strictly decreasing on $[1,\infty)$ and $\psi(1) = 1$.

Lemma 3.3. *Let $a \in F$ be an increasing nonnegative function on $[1,\infty)$ and $0 < t \leqslant 1$. Then*

$$\|a\chi_{[\psi^{-1}(t),\infty)}\|_F \leqslant \mathcal{K}(t,a;F,L_\infty[1,\infty)) \leqslant 2\cdot\|a\chi_{[\psi^{-1}(t),\infty)}\|_F.$$

In what follows, a connection between the fundamental function φ of an extrapolation space $X = \mathcal{L}_F$ and the function ψ defined by (24) plays an important role. In particular, if $X \in \mathcal{SE}$, then

$$\psi(u) = \|\chi_{[u,\infty)}\|_{\tilde{X}} = \|\chi_{[u,\infty)}(\log_2(2/s))\|_X = \varphi(2^{1-u}) \quad (u \geqslant 1),$$

whence

$$\varphi(t) = \psi(\log_2(2/t)) \ \ (0 < t \leqslant 1). \tag{25}$$

In the general case, we have the following: for every $h \in (0,1)$ there is $A = A(h) > 0$ such that

$$\psi\left(h \log_2 \frac{2}{t}\right) \leqslant A\varphi(t) \ \text{ if } 0 < t \leqslant 2^{1-1/h}. \tag{26}$$

In fact, if $p \geqslant h\log_2(2/t)$, then we have $t^{1/p} \geqslant t^{\frac{1}{h\log_2 2/t}} \geqslant 2^{-1/h}$, and the definitions of φ and ψ imply that $\varphi(t) \geqslant C^{-1}2^{-1/h}\psi\left(h\log_2 \frac{2}{t}\right)$, where C is the constant of equivalence of norms in X and $\mathcal{L}_F$. Thereby we obtain (26), with $A = 2^{1/h}C$.

In the proof of the second part of Theorem 3.1 we will use the next statement.

Lemma 3.4. *Let $X = \mathcal{L}_F$, where F is a Banach lattice on $[1,\infty)$, and let φ be the fundamental function of X. Then the following conditions are equivalent:*

(a) there exists $C_1 > 0$ such that

$$\varphi(t) \leqslant C_1 \left\| t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),\infty)}\right\|_F \tag{27}$$

for all $0 < t \leqslant 1$;

(b) there exist $C_2 > 0$ and $0 < h < 1$ such that

$$\varphi(t) \leqslant C_2\psi\left(h\log_2\frac{2}{t}\right) \tag{28}$$

for all $0 < t \leqslant 2^{1-1/h}$.

Proof. First, we assume that (b) holds. In the case when $0 < t \leqslant 2^{1-1/h}$ the definition of ψ yields

$$\begin{aligned} \varphi(t) \leqslant C_2 \left\|\chi_{[h\log_2 2/t,\infty)}\right\|_F &\leqslant 2^{1/h}C_2\left\|t^{\frac{1}{h\log_2 2/t}}\chi_{[h\log_2 2/t,\infty)}\right\|_F \\ &\leqslant 2^{1/h}C_2\left\|t^{1/p}\chi_{[h\log_2 2/t,\infty)}\right\|_F. \end{aligned}$$

Moreover, since the function ψ^{-1} decreases, then (26) implies that

$$\psi^{-1}(A\varphi(t)) \leqslant h\log_2(2/t) \ \text{ if } 0<t\leqslant 2^{1-1/h}.$$

Combining this with the previous inequality, we infer

$$\varphi(t) \leqslant 2^{1/h}C_2 \left\| t^{1/p}\chi_{[\psi^{-1}(A\varphi(t)),\infty)}\right\|_F.$$

At the same time, Lemma 3.3 and the concavity of the $\mathcal{K}$–functional yield

$$\left\| t^{1/p}\chi_{[\psi^{-1}(A\varphi(t)),\infty)}\right\|_F \leqslant \mathcal{K}(A\varphi(t),t^{1/p};F,L_\infty[1,\infty)) \leqslant$$
$$\leqslant A\mathcal{K}(\varphi(t),t^{1/p};F,L_\infty[1,\infty)) \leqslant 2A\left\| t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),\infty)}\right\|_F.$$

Thus, (27) is proved for $0<t\leqslant 2^{1-1/h}$ and hence for all $0<t\leqslant 1$.

Prove now the opposite implication $(a)\Rightarrow(b)$. Let h be for now an arbitrary number from the interval $(0,1/2)$. If $0<t\leqslant 2^{1-1/h}$ then $t<1/2$. Thereby $\frac{\log_2 t}{1-\log_2 t}\leqslant -\frac{1}{2}$, which implies $t^{h^{-1}\log_2^{-1}2/t}\leqslant 2^{-1/(2h)}$. Therefore, assuming that $h\log_2(2/t)>\psi^{-1}(\varphi(t))$, we obtain that

$$\left\| t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),h\log_2(2/t))}\right\|_F \leqslant t^{h^{-1}\log_2^{-1}2/t}\left\|\chi_{[\psi^{-1}(\varphi(t)),\infty)}\right\|_F$$
$$\leqslant 2^{-1/(2h)}\varphi(t).$$

Thus, choosing $h\in(0,1/2)$ so that $2C_1 2^{-1/(2h)}<1$, by (27), we have

$$\varphi(t)\leqslant C_1\left\| t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),\infty)}\right\|_F \leqslant$$
$$\leqslant C_1\left\| t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),h\log_2(2/t))}\right\|_F + C_1\left\| t^{1/p}\chi_{[h\log_2(2/t),\infty)}\right\|_F \leqslant$$
$$\leqslant C_1 2^{-1/(2h)}\varphi(t) + C_1\psi\left(h\log_2(2/t)\right) \leqslant (1/2)\varphi(t) + C_1\psi\left(h\log_2(2/t)\right),$$

whence

$$\varphi(t) \leqslant 2C_1\psi\left(h\log_2(2/t)\right) \quad (0 < t \leqslant 2^{1-1/h}).$$

Since in the case when $h\log_2 2/t \leqslant \psi^{-1}(\varphi(t))$ the last inequality is a straightforward consequence of (27), the proof is completed. □

Proof of Theorem 3.1. Since for $X = L_\infty$ both statements of the theorem are obvious, we confine exposition to the case when $X \neq L_\infty$. Therefore, we may assume that the fundamental function φ of X satisfies the conditions mentioned before Lemma 3.2. Since $\varphi(1) = 1$, we have that $\|x\|_X \leqslant \|x\|_\infty$ $(x \in L_\infty)$, whence

$$\mathcal{K}(t, x; X, L_\infty) = \|x\|_X \quad (t \geqslant 1). \tag{29}$$

Moreover, $\|a\|_{\tilde{X}} \leqslant \|a\|_{L_\infty[1,\infty)}$ $(a \in L_\infty[1,\infty))$ and, therefore, if $a \in \tilde{X}$ then

$$\mathcal{K}(t, a; \tilde{X}, L_\infty[1,\infty)) = \|a\|_{\tilde{X}} \quad (t \geqslant 1).$$

The last equality and (29) imply that under the condition $X = \mathcal{L}_{\tilde{X}}$ we have

$$\mathcal{K}(t, x; X, L_\infty) \asymp \mathcal{K}(t, \|x\|_p; \tilde{X}, L_\infty[1,\infty)) \quad (t \geqslant 1).$$

Thus, taking into account Lemmas 3.2 and 3.3 and equality (25), we see that (i) will be proved as soon as we will show that

$$\Big\|\Big(\int_0^{\varphi^{-1}(t)} x^*(s)^p\,ds\Big)^{1/p}\Big\|_{\tilde{X}} \asymp \|\|x\|_p\chi_{[V(t),\infty)}\|_{\tilde{X}} \quad (0 < t \leqslant 1), \tag{30}$$

where $V(t) := \log_2\left(2/\varphi^{-1}(t)\right)$.

First of all, in the case when $p \geqslant V(t)$

$$\Big(\int_0^{\varphi^{-1}(t)} x^*(s)^p\, ds\Big)^{1/p} \geqslant \varphi^{-1}(t)^{1/p} \cdot \|x\|_p \geqslant \varphi^{-1}(t)^{1/V(t)} \cdot \|x\|_p.$$

Since $\frac{\log_2 z}{1-\log_2 z} \geqslant -1$ $(0 < z \leqslant 1)$, the definition of $V(t)$ and the previous inequality yield

$$\Big(\int_0^{\varphi^{-1}(t)} x^*(s)^p\, ds\Big)^{1/p} \geqslant \frac{1}{2}\|x\|_p \ \ (p \geqslant V(t)),$$

whence

$$\Big\|\Big(\int_0^{\varphi^{-1}(t)} x^*(s)^p\, ds\Big)^{1/p}\Big\|_{\tilde{X}} \geqslant \frac{1}{2}\big\|\|x\|_p \chi_{[V(t),\infty)}\big\|_{\tilde{X}} \ \ (0 < t \leqslant 1). \tag{31}$$

To prove the opposite inequality, we may assume that

$$\{s \in [0,1] :\ x^*(s) > 0\} \subset [0, \varphi^{-1}(t)]. \tag{32}$$

If $2^{1-p} \leqslant \varphi^{-1}(t)$ (which is equivalent to the inequality $p \geqslant V(t)$), then

$$x^*(2^{1-p}) \leqslant 2 \cdot \Big(\int_0^{2^{1-p}} x^*(s)^p\, ds\Big)^{1/p} \leqslant 2 \cdot \|x\|_p \chi_{[V(t),\infty)}(p).$$

From this and (32) it follows

$$\|x\|_X = \|x^*(2^{1-p})\|_{\tilde{X}} \leqslant 2 \cdot \big\|\|x\|_p \chi_{[V(t),\infty)}\big\|_{\tilde{X}}.$$

On the other hand, since X is a strong extrapolation space, then

$$\Big\|\Big(\int_0^{\varphi^{-1}(t)} x^*(s)^p\, ds\Big)^{1/p}\Big\|_{\tilde{X}} \leqslant C\|x\|_X.$$

The last two relations yield

$$\Big\|\Big(\int_0^{\varphi^{-1}(t)} x^*(s)^p\, ds\Big)^{1/p}\Big\|_{\tilde{X}} \leqslant 2C\big\|\|x\|_p \chi_{[V(t),\infty)}\big\|_{\tilde{X}},$$

which combined with (31) implies (30). Thus, the proof of (i) is completed.

Proceed with the proof of (ii). Firstly, since $F \supset L_\infty[1,\infty)$, we have

$$\mathcal{K}(t,a;F,L_\infty[1,\infty)) \asymp \|a\|_F \ \ (t \geqslant 1)$$

for all $a \in F$. Combining this with (29) and (23) we obtain that $X = \mathcal{L}_F$. Hence, applying Lemmas 3.2 and 3.3 (we may assume that the function ψ defined by (24) satisfies the conditions stated before the last of them) and hypothesis (23), we infer

$$\Big\|\Big(\int_0^{\varphi^{-1}(t)} x^*(s)^p\,ds\Big)^{1/p}\Big\|_F \asymp \|\|x\|_p\chi_{[\psi^{-1}(t),\infty)}\|_F \ \ (0<t\leqslant 1).$$

Choosing $x = \chi_{[0,u)}$ and $t = \varphi(u)$, where $0 < u \leqslant 1$, we conclude that

$$\varphi(u) \asymp \|u^{1/p}\|_F \leqslant C\left\|u^{1/p}\chi_{[\psi^{-1}(\varphi(u)),\infty)}\right\|_F.$$

Thus, Lemma 3.4 yields (28). This fact and (26) imply that

$$\begin{aligned}\varphi(u) &\leqslant C_2\psi\left(h\log_2(2/u)\right) = C_2\psi\left((h/2)\log_2(2/u)^2\right)\\ &\leqslant C_2\psi\left((h/2)\log_2(2/u^2)\right) \leqslant C_2A(h/2)\varphi(u^2)\end{aligned}$$

for some $h \in (0,1)$ and all $0 < u \leqslant 2^{1/2-1/h}$, i.e., $\varphi \in \Delta^2$. The proof is completed. □

As it was showed there are Marcinkiewicz spaces $M(\varphi) \in \mathcal{E}$ such that $\varphi \notin \Delta^2$ (see Example 2.14). At the same time, from Theorems 2.11 and 3.1 it follows the following result.

Corollary 3.5. *If X is an interpolation space with respect to a Banach couple $(\Lambda(\varphi), M(\varphi))$ (in particular, if $X = \Lambda(\varphi)$ or $X = M(\varphi)$), then the following conditions are equivalent:*

(i) there exists a Banach lattice F on $[1,\infty)$ such that $F \supset L_\infty$ and equivalence (23) *is valid with constants independent of $x \in X$ and $t > 0$;*

(ii) X is a strong extrapolation space.

In both cases relation (23) *is fulfilled for $F = \tilde{X}$.*

Example 3.6. Consider an exponential Orlicz space $\mathrm{Exp}L^\Phi$ generated by the function $e^{\Phi(u)} - 1$, where Φ is an increasimg convex function on the half-line $(0,\infty)$ and $\Phi(0) = 0$. As it is well known [23, 42], we have $\mathrm{Exp}L^\Phi = M(\varphi)$ (with equivalence of norms), where $\varphi(u) = 1/\Phi^{-1}\left(\log_2(2/u)\right)$. Since Φ^{-1} is a concave function, then

$$\Phi^{-1}\left(\log_2(2/u^2)\right) \leqslant \Phi^{-1}\left(2\log_2(2/u)\right) \leqslant 2\Phi^{-1}\left(\log_2(2/u)\right).$$

Therefore, $\varphi(u) \leqslant 2\varphi(u^2)$, which implies that $\varphi \in \Delta^2$. Thus, $\mathrm{Exp}L^\Phi \in \mathcal{SE}$ and

$$\mathcal{K}(t, f; \mathrm{Exp}L^\Phi, L_\infty) \asymp \left\| \|f\|_p \chi_{[\psi^{-1}(t),\infty)} \right\|_F \quad (0 < t \leqslant 1).$$

In this case, $\psi^{-1}(t) = \Phi(1/t)$ $(0 < t \leqslant 1)$ and, by (13), $F = L_\infty(\varphi(2^{-p}))$. Thereby

$$\mathcal{K}(t, f; \mathrm{Exp}L^\Phi, L_\infty) \asymp \sup_{p \geqslant \Phi(1/t)} \frac{\|f\|_p}{\Phi^{-1}(p)} \quad (0 < t \leqslant 1),$$

whence

$$\mathcal{K}(t, f; L_\infty, \mathrm{Exp}L^\Phi) \asymp t\mathcal{K}(1/t, f; \mathrm{Exp}L^\Phi, L_\infty) \asymp t \sup_{p \geqslant \Phi(t)} \frac{\|f\|_p}{\Phi^{-1}(p)} \quad (t \geqslant 1).$$

The last relation coincides with the statement of Proposition 1 in [4] (see also Theorem 2 in [1]) to within change of variables.

In the power case, i.e., $\Phi(t) = t^\beta$, we arrive at the classical Zygmund space $\mathrm{Exp}L^\beta$ and the equivalence

$$\mathcal{K}(t, f; L_\infty, \mathrm{Exp}L^\beta) \asymp t \sup_{p \geqslant t^\beta} \frac{\|f\|_p}{p^{1/\beta}} = t \sup_{p \geqslant t} \frac{\|f\|_{p^\beta}}{p}$$

is valid for every $\beta > 0$ [2, Theorem 2].

4 Extrapolation description of Orlicz spaces.

In this section we look for conditions under which an Orlicz space is a limiting space of the L_p–scale or, in other words, when it is possible to describe the norm of an Orlicz space by L_p–norms. Analogous problems and various applications to differential equations, imbedding theorems, probability theory, orthogonal series have been considered in [1, 2, 4, 6, 12-14, 26, 29-33, 37-40]. In particular, Mamontov [29] — [33] has developed the techniques of special integral transformations and used them to prove some imbedding theorems for Orlicz spaces and intersections of L_p–spaces.

Note that our exposition is close to that by Mamontov and Ostrovsky [38] — [40]. The main idea is to look at L_p–spaces as Orlicz spaces and a key observation is the following: an intersection of Orlicz spaces generated by functions M_α ($\alpha \in I$) often coincides with the Orlicz space, which is generated by the envelope of the family $\{M_\alpha\}_{\alpha \in I}$, i.e., by the function $M(u) = \sup_\alpha M_\alpha(u)$.

Lemma 4.1. *Let $M_\alpha(u)$ ($\alpha \in I$) be arbitrary Orlicz functions and let $L_\Delta := \Delta_{\alpha \in I} L_{M_\alpha}$, where L_{M_α} are Orlicz spaces generated by the functions M_α. Then*

$$L_M \subset L_\Delta \quad \text{and} \quad \|x\|_{L_\Delta} \leqslant \|x\|_{L_M}, \tag{33}$$

where $M(u) := \sup_\alpha M_\alpha(u)$ ($u > 0$).

Moreover, $\phi_{L_M}(t) = \phi_{L_\Delta}(t)$ ($0 \leqslant t \leqslant 1$).

Proof. Since (33) is obvious, we should prove only the equality of fundamental functions. By the definition of the space L_Δ, we have $\phi_{L_{M_\alpha}}(t) \leqslant \phi_{L_\Delta}(t)$. Therefore,

$$\int_0^1 M_\alpha\left(\frac{\chi_{[0,t]}(s)}{\phi_{L_\Delta}(t)}\right)\, ds \leqslant 1$$

for all $t \in (0,1]$ and $\alpha \in I$. On the other hand, for any $\varepsilon > 0$ and $t \in (0,1]$ there is $\alpha = \alpha(\varepsilon, t)$ such that

$$M\left(\frac{1}{\phi_{L_\Delta}(t)}\right) \leqslant (1+\varepsilon) M_\alpha\left(\frac{1}{\phi_{L_\Delta}(t)}\right).$$

Thus,

$$\int_0^1 M\left(\frac{\chi_{[0,t]}(s)}{\phi_{L_\Delta}(t)}\right)\, ds \leqslant (1+\varepsilon)\int_0^1 M_\alpha\left(\frac{\chi_{[0,t]}(s)}{\phi_{L_\Delta}(t)}\right)\, ds \leqslant 1+\varepsilon,$$

which implies $\phi_{L_M}(t) \leqslant \phi_{L_\Delta}(t)$. The opposite inequality is a consequence of (33). □

Let $\Delta(\omega L_p) := \Delta_{1\leqslant p<\infty}(\omega(p)L_p)$, where $\omega(p)$ is a nonnegative bounded measurable function on $[1,\infty)$. It is not hard to check that $\Delta(\omega L_p)$ is an r.i. space on $[0,1]$ and $\Delta(\omega L_p) = \Delta_{p_0\leqslant p<\infty}(\omega(p)L_p)$ with equivalence of norms whenever $\omega(p_0) > 0$ $(p_0 > 1)$.

Setting $M_\alpha(u) = M_p(u) = \omega(p)^p u^p$, we see that the space $\Delta(\omega L_p)$ coincides with the space L_Δ from Lemma 4.1 with equality of norms. We will examine under which conditions $L_\Delta = L_M$, where M is the envelope of the family $\{M_p\}_{1\leqslant p<\infty}$, i.e., $M(u) = \sup_{1\leqslant p<\infty} \omega(p)^p u^p$ $(u > 0)$. Note that it can be assumed that $\omega(p) \to 0$ as $p \to \infty$ since, otherwise, $M(u) = \infty$ for $u > 1$, or, equivalently, $\Delta(\omega L_p) = L_\infty$. We begin with the following result.

Theorem 4.2. *Let $M_p(u) = \omega(p)^p u^p$ and let*

$$M(u) := \sup_{p\geqslant 1} M_p(u) \leqslant \int_1^\infty M_p(u) R(p)\, dp, \quad \text{for } u > u_0,$$

where a nonnegative function $R(p)$ satisfies the condition

$$\int_1^\infty R(p)\lambda^{-p}\,dp < \infty, \tag{34}$$

with some $\lambda > 0$. Then $\Delta(\omega L_p) = L_M$.

Proof. Suppose that $x \in \Delta(\omega L_p)$ and $\|x\|_{\Delta(\omega L_p)} < 1$. By the definition of the functions M_p, we have

$$\int_0^1 M_p\left(\frac{|x(t)|}{\lambda}\right) dt \leqslant \frac{1}{\lambda^p},$$

with λ from hypothesis (34). Therefore,

$$\int_0^1 M\left(\frac{|x(t)|}{\lambda}\right) dt \leqslant M(u_0) + \int_0^1 \int_1^\infty M_p\left(\frac{|x(t)|}{\lambda}\right) R(p)\,dp\,dt =$$

$$= M(u_0) + \int_1^\infty \left(\int_0^1 M_p\left(\frac{|x(t)|}{\lambda}\right) dt\right) R(p)\,dp \leqslant M(u_0) + \int_1^\infty \frac{R(p)}{\lambda^p}\,dp < \infty,$$

which implies that $x \in L_M$. The opposite imbedding follows from Lemma 4.1. □

Next, we want to clarify when an Orlicz function may be represented in the form

$$M(u) = \sup_{p\geqslant 1} \psi(p)u^p, \tag{35}$$

with some function $\psi(p)$.

Lemma 4.3. *A function $M(u)$ may be represented in the form (35) if and only if*

$$\sup_{p\geqslant 1}\left(\inf_{v>0}\frac{M(v)}{v^p}\right)u^p = M(u). \tag{36}$$

Proof. Firstly, if equality (36) holds then, setting $\psi(p) := \inf_{v>0} \frac{M(v)}{v^p}$, we obtain (35).

Conversely, assume that (35) is fulfilled with some $\psi(p)$. Then $\psi(p) \leqslant \frac{M(u)}{u^p}$, for all $u > 0$ and $p \geqslant 1$. Therefore, $\psi(p) \leqslant \inf_{v>0} \frac{M(v)}{v^p}$ and, finally,

$$M(u) = \sup_{p\geqslant 1} \psi(p)u^p \leqslant \sup_{p\geqslant 1} \left(\inf_{v>0} \frac{M(v)}{v^p} \right) u^p \leqslant \sup_{p\geqslant 1} \left(\frac{M(u)}{u^p} \right) u^p = M(u).$$

□

Suppose that a function $M(u)$ may be represented in the form (35). Putting $\tilde{\psi}(p) := \sup_{q\geqslant p} \psi(q)$, we see that

$$M(u) = \sup_{p\geqslant 1} \tilde{\psi}(p)u^p$$

for all $u \geqslant 1$. Since Orlicz spaces generated by equivalent functions have equivalent norms we may assume that $\psi(p)$ decreases and $\psi(1) = 1$. In this case $M(u) = u$ if $0 \leqslant u \leqslant 1$.

After change of variables $u = e^t$, $v = e^s$ and taking the logarithm of both sides of equality (36) we obtain that

$$\log M(e^t) = \sup_{p\geqslant 1} \left\{ pt - \sup_{s\in\mathbb{R}} \{sp - \log M(e^s)\} \right\}, \ t \in \mathbb{R}.$$

Since $\log M(e^t) = t$ if $t < 0$, we may put in the right hand side of the last equality the supremum over all $p \in \mathbb{R}$. By the well-known Fenchel–Moreau theorem [44, Remark 1.63, p. 26], the last means that the function $N(t) := \log M(e^t)$ is convex. Recall that the Legendre conjugate function N^* for a function N is defined by

$$N^*(p) = \sup_{t\in\mathbb{R}} \{pt - N(t)\}.$$

In our case $N^*(p)$ increases for $p > 1$, $N^*(1) = 0$, and $N^*(p) = +\infty$ if $p < 1$. Thus, Lemma 4.3 implies the following statement.

Theorem 4.4. *An Orlicz function $M(u)$ may be represented in the form (35) if and only if*

$$M(u) = e^{N(\log u)}, \tag{37}$$

with some convex function N such that $N(t) = t$ if $t < 0$. Moreover, $\psi(p) = e^{-N^*(p)}$.

Remark 4.5. In the case when $N(t) = qt + C$ ($t > 0$, $q \geqslant 1$) we have $M(u) = C_1 u^q$ ($u > 1$) and $\psi(p) = 0$ for $p > q$.

The following results follow from Theorems 4.2, 4.4 and Lemma 4.1.

Theorem 4.6. *Let a function M be of the form (37) with some convex function $N(t)$ such that $\lim_{t\to+\infty} N(t)/t = \infty$. If*

$$e^{N(t)} \leqslant \int_1^\infty e^{pt-N^*(p)+Cp}\,dp, \quad \text{for } t \geqslant t_0, \tag{38}$$

with some $C > 0$, then $L_M = \Delta(\omega L_p)$, where

$$\omega(p) = e^{-\frac{N^*(p)}{p}}. \tag{39}$$

Proof. According to Theorem 4.4

$$M(u) = \sup_{p\geqslant 1} M_p(u), \quad \text{where} \quad M_p(u) = \omega(p)^p u^p = e^{p\log u - N^*(p)}.$$

After change of variables $t = \log u$ in (38) we see that conditions of Theorem 4.2 hold for $R(p) = e^{Cp}$, $\lambda = e^{C+1}$, and $u_0 = e^{t_0}$. $\square$

Theorem 4.7. *Let a function M be of the form (37) with some convex function $N(t)$ and let the Orlicz space L_M coincide with a Marcinkiewicz space. Then $L_M = \Delta(\omega L_p)$, where ω is defined by (39).*

Proof. By Theorem 4.4 and Lemma 4.1, $L_M \subset \Delta(\omega L_p)$ and the fundamental functions of these spaces are equal. Since L_M is a Marcinkiewicz space being the largest space among all r.i. spaces with the same fundamental function [21, Theorem 2.5.7], we deduce the opposite imbedding. $\square$

Remark 4.8. In [38], a somewhat similar result is proved. Moreover, in the same article its applications can be found as well.

Remark 4.9. Note that the function $N^*(p)$ for p large enough is defined by values of $N(p)$ for "large" p. Therefore, by the standard extending argument, it is not hard to check that Theorems 4.6 and 4.7 will be still valid if equality (37) is assumed to be only for $u \geqslant u_0$ $(u_0 > 0)$, where $N(t)$ is a function convex for $t \geqslant t_0$.

Remark 4.10. Recall that an Orlicz space L_M coincides with a Marcinkiewicz space if and only if the function M satisfies condition (7). Suppose M can be represented in the form (37). Then after logarithmic change of variables we see that (7) is equivalent to the following:

$$\int_{s_0}^{\infty} e^{N(N^{-1}(s)-C)-s}\, ds < \infty, \tag{40}$$

with some constant $C > 0$. Moreover, the change $N^{-1}(s) = x$ allows to rewrite condition (40) as follows

$$\int_{x_0}^{\infty} e^{N(x-C)-N(x)} N'(x)\, dx < \infty. \tag{41}$$

Next, using the convexity of the function N together with (40) and (41), we obtain the following conditions guaranteeing the coincidence of the corresponding Orlicz and Marcinkiewicz spaces:

$$\int_{s_0}^{\infty} \exp\left(-\frac{Cs}{N^{-1}(s)}\right) ds < \infty \tag{42}$$

and

$$\int_{x_0}^{\infty} \exp\left(-\frac{CN(x)}{x}\right) N'(x)\, dx < \infty. \tag{43}$$

Analogously, by the inequality

$$-CN'(x) < N(x-C) - N(x),$$

we conclude that the Orlicz and Marcinkiewicz spaces do not coincide whenever for arbitrary $C > 0$ we have

$$\int_{x_0}^{\infty} e^{-CN'(x)} N'(x)\, dx = \infty. \tag{44}$$

Let us state a result dual with respect to Theorem 4.7.

Theorem 4.11. *Let φ be an increasing concave function on $[0,1]$ such that $\varphi(t) \asymp te^{L(\log(1/t))}$, with a concave positive function $L(u)$ on $[0,+\infty)$ satisfying the condition $\lim_{u\to+\infty} L(u)/u = 0$. If the Lorentz space $\Lambda(\varphi)$ coincides with an Orlicz space, then*

$$\Lambda(\varphi) = \Sigma_{1<p<p_0}\left(e^{\frac{N^*(p')}{p'}} L_p\right),$$

where $N^*(p)$ is a Legendre conjugate function to the Orlicz function $N(t) = L^{-1}(t)$ and $p' = p/(p-1)$.

Proof. Firstly, the Marcinkiewicz space $\mathcal{M}(\tilde{\varphi})$, where $\tilde{\varphi}(t) = t/\varphi(t)$, coincides with the Orlicz space L_M, $M(u) = e^{N(\log u)}$. According to Theorem 4.7

$$\|y\|_{\mathcal{M}(\tilde{\varphi})} \asymp \sup_{p_0' < p' < \infty} \exp\left(-\frac{N^*(p')}{p'}\right) \|y\|_{p'}. \tag{45}$$

By [21, Theorem 2.5.4],

$$\|x\|_{\Lambda(\varphi)} \leqslant \sup\left\{\int_0^1 x(t)y(t)\,dt \;:\; y \in L_\infty,\, \|y\|_{\mathcal{M}(\tilde{\varphi})} = 1\right\}.$$

Hence, for any representation $x = \sum_{i=1}^\infty x_i$ (convergence in L_1) we have

$$\begin{aligned}
\left|\int_0^1 x(t)y(t)\,dt\right| &\leqslant \sum_{i=1}^\infty \left|\int_0^1 x_i(t)y(t)\,dt\right| \leqslant \sum_{i=1}^\infty \|x_i\|_p \|y\|_{p'} \\
&= \sum_{i=1}^\infty e^{\frac{N^*(p')}{p'}} \|x_i\|_p e^{-\frac{N^*(p')}{p'}} \|y\|_{p'} \\
&\leqslant \sup_{p'}\left\{e^{-\frac{N^*(p')}{p'}} \|y\|_{p'}\right\} \cdot \left\{\sum_{i=1}^\infty e^{\frac{N^*(p')}{p'}} \|x_i\|_p\right\}.
\end{aligned}$$

Combining the last inequality with the previous one and with (45), we conclude that

$$\Lambda(\varphi) \supset \Sigma_{1<p<p_0}\left(e^{\frac{N^*(p')}{p'}} L_p\right).$$

To prove the opposite inequality, we estimate the fundamental function of the space $\Sigma := \Sigma_{1<p<p_0}\left(e^{\frac{N^*(p')}{p'}} L_p\right)$. Since by (45),

$$\frac{t}{\varphi(t)} = \|\chi_{(0,t)}\|_{\mathcal{M}(\tilde{\varphi})} \asymp \sup_{p_0'<p'<\infty} e^{-\frac{N^*(p')}{p'}} \|\chi_{(0,t)}\|_{p'} = \sup_{p_0'<p'<\infty} e^{-\frac{N^*(p')}{p'}} t^{\frac{1}{p'}},$$

then

$$\begin{aligned}\phi_\Sigma(t) = \|\chi_{(0,t)}\|_\Sigma &\leqslant \inf_{1<p<p_0} e^{\frac{N^*(p')}{p'}} \|\chi_{(0,t)}\|_p \\ &= \frac{t}{\sup_{p_0'<p'<\infty} e^{-\frac{N^*(p')}{p'}} t^{\frac{1}{p'}}} \asymp \varphi(t).\end{aligned}$$

Recall that a Lorentz space is the smallest one among all r.i. spaces with the same fundamental function [21, Theorem 2.5.5]. Therefore, the last inequality implies that

$$\Lambda(\varphi) \subset \Lambda(\phi_\Sigma) \subset \Sigma_{1<p<p_0}\left(e^{\frac{N^*(p')}{p'}} L_p\right),$$

which completes the proof. $\square$

It is instructive to compare Theorem 4.11 with the following result which may be proved using Jawerth-Milman's theory.

Consider the Banach couple $\vec{A} = (L_1, L_\infty)$ of the spaces on $[0,1]$ and the corresponding spaces $\vec{A}_{\theta,1,J}$ of the real J–method of interpolation (for its definition see, for instance, [10, § 3.2]). It is well known that in this case the spaces $\vec{A}_{\theta,1,J}$ coincide (up to equivalence of norms) with the Lorentz spaces $L_{p,1}$, for $\theta = 1 - 1/p$ [10, Theorem 5.2.1]. Therefore, by [34, Ch. 2, formula (2.16)],

$$\Sigma_{0<\theta<\theta_0}\left(M(\theta)\vec{A}_{\theta,1,J}\right) = \Lambda(\psi),$$

where

$$\psi(t) = t \cdot \tau\left(1/t\right), \quad \tau(t) := \inf_{0<\theta<\theta_0} M(\theta) t^\theta.$$

Moreover, if M is a tempered function as $\theta \to 0$, i.e. $M(\theta) \asymp M(2\theta)$, then from [34, Ch. 2, Theorem 5] it follows that

$$\Sigma_{0<\theta<\theta_0}\left(M(\theta)\vec{A}_{\theta,1,J}\right) = \Sigma_{0<\theta<\theta_0}\left(M(\theta)\langle\vec{A}_{\theta,r}\rangle\right)$$

(see Introduction as well). Let $\omega(q)$ be a nonnegative function on $(1,\infty)$ such that $\omega(q) \asymp \omega(2q)$ as $q \to \infty$. Setting $M(\theta) := 1/\omega(q)$, where

$\theta = 1/q$, we see that $M(\theta) \asymp M(2\theta)$ as $\theta \to 0$. Since

$$\langle \vec{A}_{\theta,1} \rangle \subset L_p = \langle \vec{A}_{\theta,p} \rangle \subset \langle \vec{A}_{\theta,\infty} \rangle, \quad \text{for} \ \ \theta = 1 - \frac{1}{p} = \frac{1}{p'}\,,$$

with uniformly bounded constants, then the previous relations imply that

$$\Sigma_{1<p<p_0}\left(\frac{1}{\omega(p')}L_p\right) = \Lambda(\psi), \ \text{where} \ \psi(t) = \frac{t}{\sup_{q_0<q<\infty}\omega(q)t^{1/q}},$$

provided if $\omega(q) \asymp \omega(2q)$ as $q \to \infty$.

We see that this result is completely consistent with Theorem 4.11, but the last one contains essentially weaker assumptions on a weight $\omega(q)$. We may say the same concerning to Theorem 4.7 as well.

In conclusion, we consider some examples of extrapolation Orlicz spaces. Emphasize that only the first of them (which appeared in the second part of Yano's theorem) may be derived using Jawerth–Milman's theory. In this connection, note that an Orlicz space L_M is a strong extrapolation space if and only if the function M satisfies the following condition: there is $C > 0$ such that

$$M^{-1}(u) \leqslant CM^{-1}(\sqrt{u}) \ \ (u \geqslant 1). \tag{46}$$

In fact, let ϕ be the fundamental function of L_M. Since $\phi(t) = \frac{1}{M^{-1}(1/t)}$, then (46) means that $\phi \in \Delta^2$. Therefore, by Theorem 2.2 and Proposition 2.10, the condition $L_M \in \mathcal{SE}$ implies (46). On the other hand, from (46) it follows that

$$\int_0^1 M\left(\frac{1}{C}M^{-1}\left(\frac{1}{t}\right)\right)dt \leqslant \int_0^1 M\left(M^{-1}\left(\frac{1}{\sqrt{t}}\right)\right)dt = 2.$$

Hence we obtain (7) and L_M coincides with the Marcinkiewicz space $M(\phi)$. Then, by Theorem 2.11, L_M is a strong extrapolation space.

In the following examples we will repeatedly make use of the property indicated in Remark 4.9.

Example 4.12. Let $\alpha > 0$ and $M(u) = e^{u^\alpha}$ if $u \geqslant u_0$, where $u_0 > 0$ is large enough. This function is representable in the form (37), with $N(t) = e^{\alpha t}$. It is not hard to verify that this function satisfies condition (40) and hence the corresponding Orlicz space $\operatorname{Exp} L^\alpha$ coincides with the Marcinkiewicz space $\mathcal{M}(\varphi)$, where $\varphi(t) = \log^{-1/\alpha}(e/t)$. Therefore, by Theorem 4.7, the corresponding Orlicz space $\operatorname{Exp} L^\alpha$ is an extrapolation one. Moreover, elementary calculations show that

$$N^*(p) = \frac{p}{\alpha}\log\left(\frac{p}{\alpha}\right) - \frac{p}{\alpha},$$

and we infer

$$\|x\|_{\operatorname{Exp} L^\alpha} \asymp \sup_{p\geqslant 1} \|x\|_p p^{-1/\alpha}.$$

Example 4.13. Let $\alpha > 1$ and $\gamma > 0$. Consider the Orlicz space $L_{M_{\alpha,\gamma}}$ generated by the function $M_{\alpha,\gamma}(u) = e^{\gamma \log^\alpha(u)}$ if $u \geqslant u_{\alpha,\gamma}$, where $u_{\alpha,\gamma} > 0$ is large enough. This function is of the form (37), with $N_{\alpha,\gamma}(t) = \gamma t^\alpha$. Moreover, since condition (42) holds, $L_{M_{\alpha,\gamma}}$ is a Marcinkiewicz space. Taking into account that

$$N^*_{\alpha,\gamma}(p) = \frac{p^\beta}{\beta(\gamma\alpha)^{\beta-1}} \ \text{ if } p \geqslant p_0, \ \text{where } \frac{1}{\beta} + \frac{1}{\alpha} = 1,$$

and using Theorem 4.7, we infer

$$L_{M_{\alpha,\gamma}} = \Delta\left(\exp\left(-\frac{p^{\beta-1}}{\beta(\gamma\alpha)^{\beta-1}}\right) L_p\right),$$

which means that

$$\|x\|_{L_{M_{\alpha,\gamma}}} \asymp \sup_{p\geqslant 1} \|x\|_p \cdot \exp\left(-\frac{p^{\beta-1}}{\beta(\gamma\alpha)^{\beta-1}}\right).$$

In the case when $\gamma = 1/\alpha$ these expressions are simplified and for the function $M_\alpha(u) = e^{\alpha^{-1}\log^\alpha(u)}$ $(u \geqslant u_\alpha)$ we have the following extrapolation formula

$$\|x\|_{L_{M_\alpha}} \asymp \sup_{p \geqslant 1} \|x\|_p \cdot \exp\left(-\frac{p^{\beta-1}}{\beta}\right).$$

Example 4.14. Let $\gamma \geqslant 1$. Consider the Orlicz space generated by the function of the form (37), with $N(t) = N_\gamma(t) = t\log^\gamma t$ if $t \geqslant t_\gamma$, where t_γ is large enough. Since $N_\gamma(t)$ grows at infinity essentially more slowly than the function $N_{\alpha,\gamma}(t)$ from the previous example, then the Orlicz space L_{M_γ}, where $M_\gamma(u) = e^{N_\gamma(\log u)}$, is wider than the Orlicz space $L_{M_{\alpha,\gamma}}$ from Example 4.13. However, L_{M_γ} is an extrapolation space as well. In fact, since

$$\int_{x_0}^{\infty} e^{-2\log^\gamma x} \log^\gamma x \, dx < \infty \;\text{ if }\; \gamma \geqslant 1,$$

then (43) holds. Therefore, L_{M_γ} coincides with the corresponding Marcinkiewicz space.

It is not hard to find the Legendre conjugate function to an Orlicz function which is equivalent to N_γ (see, for example, [20]). But it is insufficient because the corresponding Orlicz spaces may be different. At the same time, to find the Legendre conjugate function N_γ^* in the general case is not an easy thing. However, if $\gamma = 1$ and $\gamma = 2$ elementary calculations show that

$$N_1^*(p) = e^{p-1} \;\text{ and }\; N_2^*(p) = pe^{\sqrt{p+1}-1}/\sqrt{p+1},$$

respectively. Therefore, by Theorem 4.7, we have

$$\|x\|_{L_{M_1}} \asymp \sup_{p\geqslant 1} \|x\|_p \exp\left(-\frac{e^{p-1}}{p}\right)$$

and

$$\|x\|_{L_{M_2}} \asymp \sup_{p\geqslant 1} \|x\|_p \exp\left(-\frac{e^{\sqrt{p+1}-1}}{\sqrt{p+1}}\right),$$

where

$$M_1(u) = \exp\left(\log u \log\log u\right) \quad \text{and} \quad M_2(u) = \exp\left(\log u \log^2 \log u\right),$$

respectively.

Note that in the case $\gamma = 1$ we may make use of Theorem 4.6 as well, without exploiting the fact of the coincidence of the Orlicz and Marcinkiewicz spaces. We must check only condition (38), which is a consequence of the following inequality

$$e^{t\log t} \leqslant \int_0^\infty e^{pt-e^{p-1}+p}\, dp \quad \text{if} \quad t \geqslant t_0.$$

In fact, the integrand attains its maximum at the point $p_0 = \log(t+1)+1$. Therefore,

$$\int_0^\infty e^{pt-e^{p-1}+p}\, dp \geqslant \int_{p_0}^{p_0+\frac{1}{t}} e^{pt-e^{p-1}+p}\, dp \geqslant$$

$$\geqslant \frac{1}{t}\exp\left(\left(p_0+\frac{1}{t}\right)t - e^{p_0+\frac{1}{t}-1} + p_0 + \frac{1}{t}\right) =$$

$$= \exp\left((t+1)\log(t+1) + t + 2 + \frac{1}{t} - (t+1)e^{\frac{1}{t}} - \log t\right) \geqslant e^{t\log t}$$

if t is large enough, because of

$$\lim_{t\to+\infty}\left((t+1)\log(t+1) + t + 2 + \frac{1}{t} - (t+1)e^{\frac{1}{t}} - \log t - t\log t\right) = 1.$$

The last example shows that there are Orlicz spaces generated by functions of the form (37) that do not satisfy the condition of Theorem 4.7.

Example 4.15. Let N_γ be of the same form as in the previous example but now $\gamma \in (0,1)$. Since for arbitrary $C > 0$

$$\int_{x_0}^{\infty} e^{-C\log^\gamma x} \log^\gamma x \, dx = \infty,$$

then (44) holds, and the corresponding Orlicz space doesn't coincide with any Marcinkiewicz space. Analogously, Theorem 4.7 cannot be used for the function $N(t) = t \log\log t$ if $t \geqslant t_0$.

5 Concluding remarks and some applications.

5.1 Discrete extrapolation relations.

Here we confine to consideration of L_p–spaces with integer p. Firstly, show that in contrast to the case of the continuous scale any extrapolation functor of the type Δ always gives on the scale $\{L_k\}_{k=1}^\infty$ an Orlicz space. In other words,

$$\|x\|_{\Delta^d(\omega L_k)} := \sup_{k\in\mathbb{N}} \omega(k)\|x\|_k$$

is an Orlicz norm, for an arbitrary nonnegative sequence $\{\omega(k)\}_{k=1}^\infty$.

Taking into account Lemma 4.1, it is sufficient to check that

$$L_\Delta := \Delta_{k=1}^\infty L_{M_k} \subset L_M,$$

where

$$M_k(u) := \omega(k)^k u^k \quad (k \in \mathbb{N}), \;\; M(u) = \sup_{k=1,2,\dots} \omega(k)^k u^k.$$

Consider the function $G(u) := \sum_{k=1}^{\infty} M_k(u)$. If $x \in L_\Delta$ and $\|x\|_{L_\Delta} < 1$, then $\int_0^1 M_k(|x(t)|)\,dt \leqslant 1$ for all $k \in \mathbb{N}$. Therefore,

$$\int_0^1 G\left(\frac{|x(t)|}{2}\right) dt = \sum_{k=1}^{\infty} \int_0^1 M_k\left(\frac{|x(t)|}{2}\right) dt \leqslant \sum_{k=1}^{\infty} \frac{1}{2^k} = 1,$$

whence $\|x\|_{L_M} \leqslant \|x\|_{L_G} \leqslant 2$, i.e., $x \in L_M$.

Remark 5.1. Analogous arguments show that $\|x\| = \sup_{k>2} \omega(k)\|x\|_{\log k}$ is an Orlicz norm, for an arbitrary nonnegative sequence $\{\omega(k)\}_{k=1}^{\infty}$.

Now, compare a discrete extrapolation method with its corresponding continuous version. The following example shows that in the case when $\omega(p)$ tends to zero (as $p \to \infty$) not too fast these methods are equivalent, i.e., give the same extrapolation spaces.

Example 5.2. Let φ be an arbitrary quasi-concave function on $[0,1]$. Then, it follows easily that

$$\sup_{k\in\mathbb{N}} \varphi(2^{-k})\|x\|_k \asymp \sup_{p\geqslant 1} \varphi(2^{-p})\|x\|_p.$$

In particular, this implies that for the Orlicz spaces L_{M_α} from Example 4.13 if $\alpha \geqslant 2$ we obtain

$$L_{M_\alpha} = \Delta^d(\exp(-k^{\beta-1}/\beta)L_k), \quad \text{where } \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

i.e.,

$$\|x\|_{L_{M_\alpha}} \asymp \sup_{k\in\mathbb{N}} \|x\|_k e^{-k^{\beta-1}/\beta}.$$

Remark 5.3. It is not hard to prove that the analogous result holds for every extrapolation method generated by a tempered Banach lattice F (see Section 2). However, Example 5.2 shows that discrete and continuous methods may coincide by essentially weaker assumptions.

In general, a discrete extrapolation method and its corresponding continuous version give, of course, different spaces.

Example 5.4. Let $x(t) = e^{\log^{3/4} 1/t}$ $(0 < t \leqslant 1)$. Using asymptotic Laplace's method, it can be proved that

$$\|x\|_p \asymp e^{\gamma p^3}, \quad \text{where } \gamma = \frac{27}{256}. \tag{47}$$

In fact, after change of variables $\log 1/t = s$, we have

$$\|x\|_p^p = \int_1^\infty e^{ps^{3/4}-s}\, ds.$$

The integrand increases on the interval $[1, s_p)$ and decreases on (s_p, ∞), where $s_p := (3p/4)^4$. Therefore,

$$\|x\|_p^p \geqslant \int_{s_p-1}^{s_p} e^{ps^{3/4}-s}\, ds \geqslant \exp\left(p\left((3p/4)^4 - 1\right)^{3/4} - (3p/4)^4 + 1\right),$$

whence $\|x\|_p \geqslant ce^{\gamma p^3}$. On the other hand, if p is large enough, then

$$\|x\|_p^p \leqslant e^{p+\gamma p^4} + \int_{e^p}^\infty e^{-\frac{s}{2}}\, ds \leqslant 2e^{p+\gamma p^4}.$$

Combining this with the previous inequality, we obtain (47). Thus,

$$\lim_{p\to\infty} \frac{\|x\|_{p+1/2}}{\|x\|_p} = \infty. \tag{48}$$

Setting

$$\omega(p) := \sum_{k=1}^\infty \frac{1}{\|x\|_k} \chi_{[k,k+1)}(p)$$

and taking into account (48), we see that

$$\omega(p) \downarrow 0 \text{ as } p \to \infty, \quad x \in \Delta^d(\omega L_k), \quad \text{but } x \notin \Delta(\omega L_p).$$

5.2 A sharp extrapolation Yano type theorem.

As usual, let $x^{**}(t) = \frac{1}{t}\int\limits_0^t x^*(s)\,ds$. Given $\alpha > 0$ and an r.i. space X define the space $X(\log^{-\alpha})$ as follows: it consists of all functions $x(t)$ such that $x^{**}(t)\log_2^{-\alpha}(2/t) \in X$ and it is endowed with the norm $\|x\|_{X(\log^{-\alpha})} = \|x^{**}(t)\log_2^{-\alpha}(2/t)\|_X$. Such spaces arise in the theory of interpolation of weak type operators [21, § 2.6] as well as in studying geometry of r.i. spaces (for instance, see [5, Theorem 2.17]).

First, we show that the replacement of two "stars" with one "star" in the norm of $X(\log^{-\alpha})$ leads to an equivalent functional provided that X is a strong extrapolation space.

Proposition 5.5. *Let $X \in \mathcal{SE}$. Then $X(\log^{-\alpha})$ consists of all functions $x(t)$ such that $x^*(t)\log_2^{-\alpha}(2/t) \in X$ and*

$$\|x\|_{X(\log^{-\alpha})} \asymp \|x^*(t)\log_2^{-\alpha}(2/t)\|_X. \tag{49}$$

Proof. Since $x^{**}(t)$ decreases, we obtain the inequality

$$\begin{aligned}\|x^{**}(t)\log_2^{-\alpha}(2/t)\cdot\chi_{[1/2,1]}\|_X &\leqslant 2\|x^{**}(t)\cdot\chi_{[1/2,3/4]}\|_X \\ &\leqslant 2\cdot 3^\alpha\|x^{**}(t)\log_2^{-\alpha}(2/t)\cdot\chi_{[1/4,1/2]}\|_X.\end{aligned}$$

Hence,

$$\begin{aligned}\|x\|_{X(\log^{-\alpha})} &\leqslant \|x^{**}(t)\log_2^{-\alpha}(2/t)\cdot\chi_{[0,1/2]}\|_X \\ &+ \|x^{**}(t)\log_2^{-\alpha}(2/t)\cdot\chi_{[1/2,1]}\|_X \\ &\leqslant (1+2\cdot 3^\alpha)\|x^{**}(t)\log_2^{-\alpha}(2/t)\cdot\chi_{[0,1/2]}\|_X.\end{aligned}$$

Further, assume that $t \in [0,1/2]$. Then,

$$x^{**}(t) = \frac{1}{t}\sum_{n=0}^\infty \int\limits_{t^{2^{n+1}}}^{t^{2^n}} x^*(s)\,ds \leqslant \sum_{n=0}^\infty x^*\left(t^{2^{n+1}}\right) t^{2^n-1}.$$

Therefore,

$$x^{**}(t)\log_2^{-\alpha}(2/t) \leqslant \sum_{n=0}^{\infty} x^*\left(t^{2^{n+1}}\right)\log_2^{-\alpha}\left(2/t^{2^{n+1}}\right)\cdot\frac{\log_2^{\alpha}\left(2/t^{2^{n+1}}\right)}{\log_2^{\alpha}(2/t)}t^{2^n-1}$$

$$\leqslant \sum_{n=0}^{\infty} S^{n+1}\left(x^*(t)\log_2^{-\alpha}(2/t)\right)\cdot 2^{\alpha(n+1)}\cdot\left(\frac{1}{2}\right)^{2^n-1},$$

where, as above, $Sx(t) = x(t^2)$. By Theorem 2.2, the operator S is bounded in X and hence

$$\|x\|_{X(\log^{-\alpha})} = \|x^{**}(t)\log_2^{-\alpha}(2/t)\|_X$$

$$\leqslant (1+2\cdot 3^{\alpha})\cdot\Big(\sum_{n=0}^{\infty}\|S\|^{n+1}\cdot 2^{\alpha(n+1)+1-2^n}\Big)\left\|x^*(t)\log_2^{-\alpha}(2/t)\right\|_X$$

$$= C_\alpha\left\|x^*(t)\log_2^{-\alpha}(2/t)\right\|_X.$$

Since $x^*(t)\leqslant x^{**}(t)$, the opposite inequality is obvious. □

Corollary 5.6. *Under the conditions of the previous proposition, $X(\log^{-\alpha})$ is a strong extrapolation space. Moreover, we have that $X(\log^{-\alpha}) = \mathcal{L}_{\tilde{X}(p^{-\alpha})}$, where*

$$\|f\|_{\tilde{X}(p^{-\alpha})} := \|f(p)\cdot p^{-\alpha}\|_{\tilde{X}}.$$

Proof. Formula (49) indicates that the boundedness of S in X implies its boundedness in $X(\log^{-\alpha})$. Therefore, by Theorem 2.2, $X(\log^{-\alpha})$ is a strong extrapolation space as well, i.e., $X(\log^{-\alpha}) = \mathcal{L}_{\widetilde{X(\log^{-\alpha})}}$. The last assertion follows now from the isometric equality $\mathcal{L}_{\widetilde{X(\log^{-\alpha})}} = \mathcal{L}_{\tilde{X}(p^{-\alpha})}$, which can be checked directly. □

Theorem 5.7. *Let T be a bounded linear operator in L_p for all $p \geqslant p_0 \geqslant 1$ and for some $\alpha > 0$ and $C > 0$*

$$\|T\|_{L_p \to L_p} \leqslant C p^{\alpha} \ \ (p \geqslant p_0). \tag{50}$$

Then, for every strong extrapolation r.i. space X the operator T is bounded from X in $X(\log^{-\alpha})$.

Moreover, there exists a linear operator T_0 satisfying condition (50) *with the following property: if T_0 is bounded from $X \in \mathcal{SE}$ into an r.i. space Y, then*

$$X(\log^{-\alpha}) \subset Y. \tag{51}$$

Proof. Estimate (50), Corollary 5.6, and Remark 2.6 yield

$$\|Tx\|_{X(\log^{-\alpha})} \leqslant C_1 \big\| \, \|Tx\|_p \cdot p^{-\alpha} \, \big\|_{\tilde{X},p_0} \leqslant C_2 \| \, \|x\|_p \, \|_{\tilde{X},p_0} \leqslant C_3 \|x\|_X$$

and the first assertion of the theorem is proved.

Given $\alpha > 0$ we consider the operator

$$T_0 x(t) = \int_t^1 \frac{\log^{\alpha-1}(s/t)}{s} x(s)\, ds.$$

It is not hard to check that the associated operator to T_0 is defined as follows:

$$T_0' x(t) = \frac{1}{t} \int_0^t \log^{\alpha-1}(t/s) x(s)\, ds.$$

Let us estimate

$$\|T_0' x\|_q \leqslant \int_0^1 \log^{\alpha-1}(1/s) \|x(st)\|_{L_q(t)} ds \leqslant \int_0^1 s^{-1/q} \log^{\alpha-1}(1/s)\, ds \cdot \|x\|_q.$$

Therefore, by the equality

$$\int_0^1 s^{-1/q} \log^{\alpha-1}(1/s)\, ds = \int_0^\infty e^{t(1/q-1)} t^{\alpha-1}\, dt$$

$$= \left(\frac{q}{q-1}\right)^\alpha \int_0^\infty e^{-v} v^{\alpha-1}\, dv = \left(\frac{q}{q-1}\right)^\alpha \cdot \Gamma(\alpha),$$

where $\Gamma(x)$ is the Euler gamma-function, we obtain that

$$\|T_0\|_{L_p \to L_p} = \|T_0'\|_{L_{p'} \to L_{p'}} \leqslant \Gamma(\alpha) \cdot p^\alpha, \text{ where } \frac{1}{p} + \frac{1}{p'} = 1,$$

for every $1 < p < \infty$. Thereby T_0 satisfies condition (50). Suppose that $X \in \mathcal{SE}$ and that T_0 is bounded from X into an r.i. space Y.

Let $x \in X(\log^{-\alpha})$. Then, for some $y \in X$ we have

$$x^*(t) \leqslant y(t) \log_2^\alpha(2/t) \ \ (0 < t \leqslant 1)$$

and, by [21, p. 93], we obtain that

$$x^*(t) \leqslant (y(t)\log_2^\alpha(2/t))^* \leqslant y^*(t/2)\log_2^\alpha(4/t) \leqslant z^*(t)\log_2^\alpha(2/t), \tag{52}$$

with some $z \in X$. Put $z_1(t) = z^*(t^2)$. By Theorem 2.2 and by the assumption $X \in \mathcal{SE}$, we have that $z_1 \in X$ and

$$\begin{aligned} T_0 z_1(t) &\geqslant \int_t^{\sqrt{t}} \frac{\log^{\alpha-1}(s/t)}{s} z_1(s)\, ds \geqslant z_1(\sqrt{t}) \int_t^{\sqrt{t}} \frac{\log^{\alpha-1}(s/t)}{s}\, ds \\ &= \alpha^{-1} z^*(t) \log^\alpha(1/\sqrt{t}) = \alpha^{-1} 2^{-\alpha} z^*(t) \log^\alpha(1/t) \\ &\geqslant \alpha^{-1} 2^{-2\alpha} \log^\alpha 2 \cdot z^*(t) \log_2^\alpha(2/t) \cdot \chi_{[0,1/2]}(t). \end{aligned}$$

If T_0 is bounded from X into Y then $T_0 z_1 \in Y$ and from the last inequality and (52) it follows $x \in Y$, i.e., we have imbedding (51). $\square$

Remark 5.8. If $X = L_\infty$ then the space $X(\log^{-\alpha})$ coincides with the Zygmund space $\operatorname{Exp} L^{1/\alpha}$. In this case, from Theorem 5.7 it follows the assertion of the second part of classical Yano's theorem, i.e., if a linear operator T satisfies (50) then T is bounded from L_∞ into $\operatorname{Exp} L^{1/\alpha}$.

Using Theorem 5.7, we obtain the following sufficient conditions for the convergence of orthogonal series in r.i. spaces "close" to L_∞.

Corollary 5.9. *Let X be a strong extrapolation space and let $\{\psi_n\}_{n=1}^\infty$ be a complete in X orthonormal system of functions such that for every $x \in X$ and for all positive integers N we have*

$$\Big\| \sum_{n=1}^{N} c_n(x)\psi_n \Big\|_p \leqslant Cp^\alpha \|x\|_p \ \textit{if}\ p \geqslant p_0,$$

where $c_n(x)$ are Fourier coefficients of x with respect to $\{\psi_n\}_{n=1}^\infty$. Then, for every $x \in X$ the series $\sum\limits_{n=1}^{\infty} c_n(x)\psi_n$ converges to x in the space $X(\log^{-\alpha})$.

In particular, by [47, Theorems 4.3.16 and 7.6.4] and [15, Theorem 5.3.2], using the interpolation Marcinkiewicz theorem in the same way as in [26], we deduce that L_p–norms of trigonometric Fourier and Walsh–Fourier partial sums satisfy inequality (50) if $\alpha = 1$. Therefore, we obtain

Corollary 5.10. *Suppose X is a separable strong extrapolation r.i. space. Then the Fourier trigonometric and the Fourier–Walsh series of a function $x \in X$ converge to x in the space $X(\log^{-1})$.*

In the case of Lorentz spaces the last corollary was proved by Lukomskii [24, 25]. Moreover, he showed that this result is sharp. For Marcinkiewicz spaces the analogous result was proved in [26].

5.3 Rearrangement invariance of Rademacher multiplicator spaces.

The Rademacher functions are $r_n(t) := \operatorname{sign}\sin(2^n\pi t),\ t\in[0,1],\ n\in\mathbb{N}$. Let $\mathcal{R}$ denote the set of all functions of the form $\sum_{n=1}^{\infty} a_n r_n$, where the series converges a.e. For an r.i. space X on [0,1], let $\mathcal{R}(X)$ be the closed linear subspace of X given by $\mathcal{R}\cap X$. The *Rademacher multiplicator space* of X is the space $\Lambda(\mathcal{R},X)$ of all measurable functions $x\colon[0,1]\to\mathbb{R}$ such that $x\cdot\sum_{n=1}^{\infty} a_n r_n \in X$, for every $\sum_{n=1}^{\infty} a_n r_n\in\mathcal{R}(X)$. It is a Banach function lattice on [0,1] when endowed with the norm

$$\|x\|_{\Lambda(\mathcal{R},X)} = \sup\Big\{\Big\|x\cdot\sum_{n=1}^{\infty} a_n r_n\Big\|_X : \sum_{n=1}^{\infty} a_n r_n \in X, \Big\|\sum_{n=1}^{\infty} a_n r_n\Big\|_X \leqslant 1\Big\}.$$

The space $\Lambda(\mathcal{R},X)$ can be viewed as the space of operators from $\mathcal{R}(X)$ into the whole space X given by multiplication by a measurable function.

The Rademacher multiplicator space $\Lambda(\mathcal{R},X)$ was firstly considered in [11] where it was shown that for a broad class of classical r.i. spaces X the space $\Lambda(\mathcal{R},X)$ is not r.i. This result was extended in [5] to include all r.i. spaces such that the lower dilation index γ_{φ_X} of their fundamental function φ_X satisfies $\gamma_{\varphi_X}>0$. This result motivated the study of the symmetric kernel $\operatorname{Sym}(\mathcal{R},X)$ of the space $\Lambda(\mathcal{R},X)$, that is, the largest r.i. space embedded into $\Lambda(\mathcal{R},X)$. The space $\operatorname{Sym}(\mathcal{R},X)$ was studied in [5], where it was shown that, if X is an r.i. space satisfying the Fatou property and $X\supset \operatorname{Exp}L^2$, then $\operatorname{Sym}(\mathcal{R},X)$ is the r.i. space with the norm $\|x\| := \|x^*(t)\log^{1/2}(2/t)\|_X$. It was also shown that any space X that has the Fatou property and that is an interpolation space for the couple $(L\log^{1/2}L, L^\infty)$ can be realized as the symmetric kernel of a certain r.i. space. The opposite situation is when the Rademacher multiplicator space $\Lambda(\mathcal{R},X)$ is r.i. The simplest case of this situation is when $\Lambda(\mathcal{R},X)=L^\infty$. In [3], it was shown that $\Lambda(\mathcal{R},X)=L^\infty$ holds for all r.i. spaces X that are interpolation spaces for the couple $(L^\infty,\operatorname{Exp}L^2)$. More exactly, in [7], it was proved that $\Lambda(\mathcal{R},X)=L^\infty$ if and only if the function $\log^{1/2}(2/t)$ does not belong to the closure of L^∞ in X, which will be denoted by X_0.. In the paper [8], the case when $\Lambda(\mathcal{R},X)$ is an r.i. space different from L^∞ was investigated. In particular, the following sufficient condition for $\Lambda(\mathcal{R},X)$ to be r.i. has been obtained [8, Theorem 3.4].

Theorem 5.11. *If X is an r.i. space on $[0,1]$ such that the operator $Sx(s) = x(s^2)$ is bounded in X, then $\Lambda(\mathcal{R}, X)$ is an r.i. space.*

The proof in [8] is based on some distribution function estimates. An extrapolation proof presented here is essentially shorter showing thereby usefulness of the concept of a strong extrapolation space.

Proof. Note that if $\log^{1/2}(2/t) \notin X_0$ then, as it was proved in [7], $\Lambda(\mathcal{R}, X) = \mathrm{Sym}\,(\mathcal{R}, X) = L_\infty$. Thus, we may assume that $\log^{1/2}(2/t) \in X_0$. Let $\Delta_n^k := [\frac{k-1}{2^n}, \frac{k}{2^n})$, where $n \in \mathbb{N}$ and $k = 1, \cdots, 2^n$. In view of [8, Proposition 3.3], it is sufficient to prove the following: there exists $A > 0$ such that for all $n \in \mathbb{N}$ and every $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{2^n} \geqslant 0$ we have

(53)
$$\Big\|\sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\Big(\frac{2}{t}\Big)\Big\|_X \leqslant A\Big\|\sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\Big(\frac{2}{2^n t + 1 - k}\Big)\Big\|_X.$$

Firstly, let x be a measurable function such that $x \in L_p$ for all $p < \infty$. For every $0 < t \leqslant 1$, setting $p := \log(2/t)$, we have

$$\begin{aligned} \|x\|_p = \|x\|_{\log(2/t)} &\geqslant \Big(\int_0^t (x^*(s))^{\log(2/t)}\,ds\Big)^{1/\log(2/t)} \\ &\geqslant x^*(t)t^{1/\log(2/t)} \geqslant \frac{1}{e}x^*(t), \end{aligned}$$

since $t^{1/\log(2/t)} \geqslant 1/e$ for $0 < t \leqslant 1$. Therefore,

$$\Big\|x^*(t) \cdot \log^{1/2}\Big(\frac{2}{t}\Big)\Big\|_X \leqslant e\Big\|\,\|x\|_{\log(2/t)} \log^{1/2}\Big(\frac{2}{t}\Big)\Big\|_X. \tag{54}$$

If $1 \leqslant p < \infty$, then, by [11, Lemma 1], there exists a Rademacher sum $h_p = \sum_{k=n+1}^m b_k r_k$ such that $\|(b_k)\|_{l_2} = 1$ and

$$\|x\|_p \leqslant 3p^{-1/2}\|x \cdot h_p\|_p. \tag{55}$$

By [41] and the relation

$$\|x\|_{L_N} \asymp \sup_{0<t\leqslant 1} x^*(t) \log^{-1/2}(2/t),$$

for every $k = 1, \dots, 2^n$, we have

$$\begin{aligned}
\big(\chi_{\Delta_n^k} h_p\big)^*(s) &= \Big(\sum_{k=1}^{m-n} b_k r_k\Big)^*(2^n s) \\
&\leqslant C_1 \log^{1/2}\Big(\frac{2}{2^n s}\Big)\chi_{[0,2^{-n})}(s)\|(b_k)\|_{l_2} \\
&\leqslant C_1 \log^{1/2}\Big(\frac{2}{2^n s}\Big)\chi_{[0,2^{-n})}(s).
\end{aligned}$$

Let $x = \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k}$ with nonnegative and decreasing coefficients c_k. The last inequality implies that

$$\begin{aligned}
(xh_p)^*(s) &= \Big(\sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} h_p\Big)^*(s) \\
&\leqslant C_1 \Big(\sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \log^{1/2}\Big(\frac{2}{2^n s + 1 - k}\Big)\Big)^*(s).
\end{aligned}$$

Hence, by (55)

$$\|x\|_p \leqslant 3C_1 p^{-1/2} \Big\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \log^{1/2}\Big(\frac{2}{2^n s + 1 - k}\Big)\Big\|_p.$$

From this, setting again $p = \log(2/t)$, we obtain that

$$\Big\| \|x\|_{\log(2/t)} \log^{1/2}\Big(\frac{2}{t}\Big)\Big\|_X \leqslant$$

$$\leqslant 3C_1 \Big\| \Big\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \log^{1/2}\Big(\frac{2}{2^n s + 1 - k}\Big)\Big\|_{\log(2/t)}\Big\|_X.$$

By Theorem 2.2 and the hypotheses, the previous inequality implies that

$$\Big\| \|x\|_{\log(2/t)} \log^{1/2}\Big(\frac{2}{t}\Big)\Big\|_X \leqslant C_3 \Big\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \log^{1/2}\Big(\frac{2}{2^n s + 1 - k}\Big)\Big\|_X.$$

This inequality and (54) yield (53) and the proof is completed. $\square$

References

[1] S.V. Astashkin, *Some new extrapolation estimates for the scale of L_p–spaces*// Funct. Anal. Appl. 2003. V. 37, No. 3, 221-224.

[2] S.V. Astashkin, *Extrapolation properties of the scale of L_p–spaces*// Sbornik: Math. 2003. V. 194, No. 6, 813–832.

[3] S.V. Astashkin, *On the multiplier space generated by the Rademacher system*// 2004. Math. Notes. V. 75, No. 2, 158–165.

[4] S.V. Astashkin, *Extrapolation functors on a family of scales generated by the real interpolation method*// Siberian Math. J. 2005. V. 46, No. 2, 205-225.

[5] S.V. Astashkin and G.P. Curbera, *Symmetric kernel of Rademacher multiplicator spaces*// J. Funct. Anal. 2005. V. 226, 173–192.

[6] S.V. Astashkin and K.V. Lykov, *Extrapolatory description for the Lorentz and Marcinkiewicz spaces "close" to L_∞*// Siberian Math. J. 2006. V. 47, No. 5, 797–812.

[7] S.V. Astashkin and G.P. Curbera, *Rademacher multiplicator spaces equal to L^∞*// Proc. Amer. Math. Soc. 2008. V. 136, No. 10, 3493–3501.

[8] S.V. Astashkin and G.P. Curbera, *Rearrangement invariance of Rademacher multiplicator spaces*// J. Funct. Anal. 2009. V. 256, 4071–4094.

[9] S.V. Astashkin and K.V. Lykov, *Strong extrapolation spaces and interpolation*// Siberian Math. J. 2009. V. 50, No. 2, 199–213.

[10] J. Bergh and J. Löfström, *Interpolation Spaces, an Introduction.* Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[11] G.P. Curbera, *A note on function spaces generated by Rademacher series*// Proc. Edinburgh. Math. Soc. 1997. V. 40, 119–126.

[12] D.E. Edmunds and M. Krbec, *On decomposition in exponential Orlicz spaces*// Math. Nachr. 2000. V. 213, 77–88.

[13] D.E. Edmunds and M. Krbec, *Decomposition and Moser's lemma*// Revista Matematica Complutense. 2002. V. 15, No. 1, 57–74.

[14] A. Fiorenza and M. Krbec, *On an optimal decomposition in Zygmund spaces*// Georgian Math. J. 2002. V. 9, No. 2, 271–286.

[15] B. Golubov, A. Efimov, and V. Skvortsov, *Walsh Series And Transforms: Theory And Applications.* Kluwer Acad. Publishers, 1991.

[16] B. Jawerth and M. Milman, *Extrapolation spaces with applications*// Mem. Amer. Math. Soc. 1991. V. 89, No. 440. 82 pp.

[17] B. Jawerth and M. Milman, *New results and applications of extrapolation theory*// Interpolation Spaces and Related Topics, (M.Cwikel, M.Milman, and R.Rochberg, editors), Israel Math. Confer. Proc., 5. 1992, 81–105.

[18] G. Karadzhov and M. Milman, *Extrapolation theory: New results and applications*// J. Approx. Theory. 2005. V. 133, No. 1, 38–99.

[19] R.A. Kerman, *An integral extrapolation theorem with applications*// Stud. Math. 1983. V. 76, 183–195.

[20] M.A. Krasnoselskii and Ya.B. Rutickii, *Convex Functions and Orlicz Spaces.* Noordhoff Ltd., Groningen, 1961.

[21] S.G. Krein, Ju.I. Petunin, and E.M. Semenov, *Interpolation of Linear Operators.* Amer. Math. Soc., Providence R. I., 1982.

[22] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces 2, Function spaces.* Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[23] G.G. Lorentz, *Relations between function spaces*// Proc. Amer. Math. Soc. 1961. V. 12, 127–132.

[24] S.F. Lukomskii, *Convergence of Walsh series in near-L_∞ spaces*// Math. Notes. 2001. V. 70, No. 6, 804–811.

[25] S.F. Lukomskii, *Convergence of Fourier series in Lorentz spaces//* East J. Approx. 2003. V. 9, No. 2, 229-238.

[26] K.V. Lykov, *Extrapolation in the scale of L_p–spaces and convergence of orthogonal series in Marcinkiewicz spaces//* Vestnik Samarsk. Univ. 2006. No. 2(42), 28–43 (in Russian).

[27] K.V. Lykov, *A criterion of separability of an extrapolation space//* Vestnik Samarsk. Univ. 2006. No. 4 (44), 5–12 (in Russian).

[28] L. Maligranda, The K-functional for symmetric spaces, Lect. Notes in Math. 1984. V. 1070, 169–182. Proc. Conf. "Interpolation spaces and Allied Topics in Analysis", Lund, Aug. 29 – Sept. 1, 1983.

[29] A.E. Mamontov, *Extrapolation of linear operators from L_p into the Orlicz spaces generated by N–functions of slow or fast growth,* in: Current Problems of Modern Mathematics, Novosibirsk. Univ., Novosibirsk, 1996. V. 2, 95-103 (in Russian).

[30] A.E. Mamontov, *Scales of L_p–spaces and their relation to Orlicz spaces//* Vestnik Novosibirsk. Univ. Ser. "Mathematics, mechanics, computer science". 2006. V. 6, No. 2, 33-56 (in Russian).

[31] A.E. Mamontov, *Integral representations and transforms of N–functions. I//* Siberian Math. J. 2006. V. 47, No. 1, 97-116.

[32] A.E. Mamontov, *Integral representations and transforms of N–functions. II//* Siberian Math. J. 2006. V. 47, No. 4, 669–686.

[33] A.E. Mamontov, *Extrapolation from L_p into Orlicz spaces via integral transforms of Young functions//* J. Anal. Appl. 2006. V. 4, No 42, 77-118.

[34] M. Milman, *Extrapolation and Optimal Decompositions with Applications to Analysis.* Springer-Verlag, Berlin, Heidelberg, New York, 1994. Lect. Notes in Math. V. 1580. 162 pp.

[35] M. Milman, *A note on extrapolation theory//* J. Math. Anal. Appl. 2003. V. 282, 26–47.

[36] I.P. Natanson, *The Theory of Functions of Real Variable.* Nauka, Moscow, 1974 (in Russian).

[37] J. Neves, *On decompositions in generalized Lorentz–Zygmund spaces*// Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8). 2001. V. 4, No. 1, 239–267.

[38] E. Ostrovsky, *Exponential Orlicz Spaces: new Norms and Applications* // Electronic Publ., arXiv/FA/0406534, v.1, 25.06.2004.

[39] E. Ostrovsky, *A remark on the inequalities of Bernstein-Markov type in exponential Orlicz and Lorentz spaces*// Electronic Publ., arXiv/FA/0411617, v.1, 27.11.2004.

[40] E. Ostrovsky and L. Sirota, *Some new moment rearrangement invariant spaces; theory and applications*// Electronic Publ., arXiv/FA/0605732, v.1, 29.05.2006.

[41] V.A. Rodin and E.M. Semenov, *Rademacher series in symmetric spaces*// Anal. Math. 1975. V. 1, 207–222.

[42] Ya.B. Rutickii, *On some classes of measurable functions*// Uspekhi Matematicheskikh Nauk. 1965. V. 20, No. 4, 205–208 (in Russian).

[43] I.B. Simonenko, *Interpolation and extrapolation of linear operators in Orlicz spaces*// Matematicheskii Sbornik. 1964. V. 63, No 4, 536-553 (in Russian).

[44] S. Simons, *Minimax and Monotonicity.* Lect. Notes in Math. - Berlin: Springer-Verlag, 1998.

[45] S. Yano, *An extrapolation theorem*// J. Math. Soc. Japan. 1951. V. 3, No. 2, 296-305.

[46] V.I. Yudovicz, *On some estimates connected with integral operators and solutions of elliptic equations*// Doklady Akademii Nauk SSSR. 1961. V. 138, No. 4, 805–808 (in Russian).

[47] A. Zygmund, *Trigonometric Series. V.2.* Cambridge Univ. Press, 1959.

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