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Independent functions and the geometry of Banach spaces

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Abstract. The main objective of this survey is to present the 'state of the art' of those parts of the theory of independent functions which are related to the geometry of function spaces. The 'size' of a sum of independent functions is estimated in terms of classical moments and also in terms of general symmetric function norms. The exposition is centred on the Rosenthal inequalities and their various generalizations and sharp conditions under which the latter hold. The crucial tool here is the recently developed construction of the Kruglov operator. The survey also provides a number of applications to the geometry of Banach spaces. In particular, variants of the classical Khintchine–Maurey inequalities, isomorphisms between symmetric spaces on a finite interval and on the semi-axis, and a description of the class of symmetric spaces with any sequence of symmetrically and identically distributed independent random variables spanning a Hilbert subspace are considered.

Bibliography: 87 titles.

Keywords: independent functions, Khintchine inequalities, Kruglov property, Rosenthal inequalities, Kruglov operator, symmetric space, Orlicz space, Marcinkiewicz space, Lorentz space, Boyd indices, K-functional, real method of interpolation, integral-uniform norm.

Contents

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1. Introduction

Sequences of independent functions are studied in probability theory as well as in functional analysis and the theory of functions. The main objective of this survey is to present the 'state of the art' of those parts of their theory which are related to the geometry of function (mostly, symmetric) spaces. The exposition is centred on the description of subspaces generated by sequences of independent functions in spaces of measurable functions. The pioneering work here is the paper [1] by Rosenthal containing remarkable inequalities making it possible to describe isomorphic types of such subspaces in L_p-spaces. Two thematically close survey articles published earlier in this journal ought to be mentioned. The first [2], written by Gaposhkin in 1966, focussed on the function-theoretic properties of sequences of independent as well as weakly dependent (lacunary) systems of functions (convergence and absolute convergence, integrability, limit theorems, the law of the iterated logarithm, and so on). The second survey [3], written much later by Peshkir and Shiryaev, is devoted to the simplest and yet very important system of independent functions, the Rademacher system, and to the question of whether its properties can be carried over to general martingale systems. We also remark that the behavior of this system in symmetric function spaces is also the subject of the recent publication [4].

The survey will mostly be connected with the following problem: suppose that $\{f_k\}_{k=1}^\infty$ is a sequence of independent random variables (r.v.'s) on some probability space. It is required to somehow estimate the 'size' of the sums $S_n=\sum_{k=1}^n f_k$ $(n=1,2,\dots)$. In the case when the sequence $\{f_k\}_{k=1}^\infty$ consists of identically distributed r.v.'s with mean zero and finite variance $\sigma^2>0$, this problem, of course, goes back to the foundations of probability theory. It has been known for quite some time that under these conditions the asymptotic value of the probabilities $\mathsf{P}\{S_n/(\sigma\sqrt{n}\,)>\tau\}$ as $n\to\infty$ is the standard Gaussian distribution. A vast number of articles and monographs (see any sufficiently detailed course in probability theory) have been devoted to generalizations and specializations of this fundamental fact. Our approach is different: we are interested in the approximate rather than the asymptotic solution of this problem. In this approach the 'measure' of $|S_n|$ is not the 'tail' distribution of the sum, that is, the quantity $\mathbb{P}\{|S_n|>\tau\}$ (estimates of which are contained in any number of publications), but the norm $\|S_n\|_X$ in this or that function space X.

First and foremost, of course, information on the classical moments $\|S_n\|_p = \bigl(\mathsf{E}|S_n|^p\bigr)^{1/p}$ $(p > 0)$ of the sums is especially important from this viewpoint. In this connection, the problem arises of finding a quantity A_n (as simple as possible) such that for some constant $C > 0$

$$C^{-1}A_n \leqslant \|S_n\|_p \leqslant CA_n \qquad (p > 0).$$

This question will be at the centre of our attention in the first part of the survey (§ 3).

The difference between the behavior of sums of independent functions in the spaces L_p with finite p and in L_∞ is well known. Moreover, the phenomena occurring cannot be properly explained if we stay 'inside' the L_p-scale, but they can be explained if we go outside that scale and consider general symmetric function spaces. This will be discussed in the second and main part of the survey (§§ 4–6). We shall be mainly interested in generalizations of the Rosenthal inequalities (see § 3.1), and in the boundaries of their extension to symmetric spaces. As we shall see below, the latter is directly related to the possibility of comparing sums of independent and disjointly supported functions. The centrepiece here is the construction of the so-called Kruglov operator, which was introduced by the authors and which has its roots in the (so-called) Kruglov property introduced for symmetric spaces by M. Sh. Braverman. This approach has made it possible to use the many advantages of the operator setting, for example, interpolation of operators. In the third and last part of the survey (§§ 7–10), applications to the geometry of symmetric spaces are considered. In particular, a characterization of symmetric spaces for which a version of the Khintchine inequality holds, together with the most general sufficient conditions to date under which symmetric spaces on $[0,1]$ and $(0,\infty)$ are isomorphic to each other are given in terms of properties of the Kruglov operator, and a number of other problems are resolved.

We mention two more important lines of inquiry, directly relevant to the present survey but not included for lack of space. The first direction is sufficiently well reflected in the paper [5], where a direct connection is established between the boundedness of the Kruglov operator and that of the operators generated by random permutations, which in a natural way describes the well-known inequalities due to Kwapień and Schütt [6], [7]. The second direction concerns non-commutative probability theory, where independent measurable functions are replaced by corresponding sequences of operators. We refer the reader to the paper [8], where non-commutative Rosenthal inequalities are obtained. In the authors' opinion the reference lists in [5] and [8] give a good account of the development of these directions.

Here we explain the most important classical results, which were the stimulus for subsequent investigations. In 1923 A. Ya. Khintchine proved in the paper "Über dyadische Brüche" [9] his famous inequalities, which we cite in the form common in the theory of functions, using the Rademacher functions $r_n(t) = \operatorname{sgn}\sin 2^n\pi t$, $0 \leqslant t \leqslant 1$ $(n \in \mathbb{N})$.

Theorem 1 (Khintchine inequalities). *For any $0 < p < \infty$ there are constants $A_p > 0$ and $B_p > 0$ such that for any $n \in \mathbb{N}$ and any $a = (a_k)_{k=1}^n \in \mathbb{R}^n$*

$$A_p\|a\|_2 \leqslant \left\| \sum_{k=1}^n a_k r_k \right\|_p \leqslant B_p\|a\|_2, \tag{1}$$

where as usual, $\|f\|_p$ is the norm in the space $L_p[0,1]$ and $\|a\|_2 = \left(\sum_k |a_k|^2\right)^{1/2}$. Moreover, $B_p \leqslant \sqrt{p}$ $(p \geqslant 1)$ and $B_p \sim \sqrt{p/e}$ as $p \to \infty$.

This shows that the L_p-norms of polynomials in the Rademacher system are equivalent to the ℓ_2-norms of their sequences of coefficients. From the viewpoint of the geometry of Banach space this means that the Rademacher system is equivalent in $L_p[0,1]$ $(0 < p < \infty)$ to the canonical basis in ℓ_2. The classical proof of Theorem 1 may be found in the monograph [10] (§ 4.5) or in the very beginning of the survey [3]. The Marcinkiewicz–Zygmund inequalities [11] are generalizations of the Khintchine inequalities. They were published in 1937 (it was proved later that under the assumptions of the following theorem a system of independent functions is even equivalent in distribution to the Rademacher system; [4], Theorem 8.4 and Corollary 8.3).

Theorem 2 (Marcinkiewicz–Zygmund inequalities). *Let $\{f_k\}_{k=1}^\infty$ be a system of independent functions on $[0,1]$ satisfying the conditions*

$$\|f_k\|_2 = 1, \quad \|f_k\|_\infty \leqslant M, \quad \int_0^1 f_k(t)\,dt = 0 \qquad (k \in \mathbb{N}). \tag{2}$$

Then for any $p \geqslant 1$ there exists a constant $C_{M,p} > 0$ depending only on M and p such that for any $n \in \mathbb{N}$ and any $a = (a_k)_{k=1}^n \in \mathbb{R}^n$

$$C_{M,p}^{-1}\|a\|_2 \leqslant \left\| \sum_{k=1}^n a_k f_k \right\|_p \leqslant C_{M,p}\|a\|_2. \tag{3}$$

The preceding theorem is a simple consequence of an exponential estimate for the distributions of polynomials in a system of independent functions satisfying the conditions (2). This estimate was proved in [11] (see also [12], Theorem 2.5).

Theorem 3. *If a system of independent functions $\{f_k\}_{k=1}^\infty$ satisfies the conditions (2), then for any $n \in \mathbb{N}$, $a_k \in \mathbb{R}$, and $\tau > 0$*

$$\lambda\left\{t \in [0,1] : \left| \sum_{k=1}^n a_k f_k(t) \right| > \tau\|a\|_2 \right\} \leqslant 2e^{-\tau^2/(4M^2)}. \tag{4}$$

We note that estimates for distributions of sums of independent functions similar to (4) were proved by Kolmogorov well before, in 1929 [13]. Moreover, in contrast to the previous results, the latter hold not just for systems of uniformly bounded functions.

Theorem 4 (Kolmogorov inequalities). *Let $\{f_k\}_{k=1}^{\infty}$ be a system of bounded independent r.v.'s such that* $\mathsf{E}f_k = 0 \ \ (k = 1, 2, \dots)$. *Let*

$$m_k = \|f_k\|_\infty, \quad M_n = \max_{k=1,\dots,n} m_k, \quad b_k = \mathsf{E}f_k^2, \quad B_n = \sum_{k=1}^{n} b_k.$$

Then for any $n \in \mathbb{N}$ and $\tau > 0$

$$\lambda\biggl\{t \in [0,1] : \sum_{k=1}^{n} f_k(t) > \tau\biggr\} \leqslant e^{-\tau/(4B_n)} \quad \text{if } \tau \geqslant \frac{B_n}{M_n}\,,$$

and

$$\lambda\biggl\{t \in [0,1] : \sum_{k=1}^{n} f_k(t) > \tau\biggr\} \leqslant e^{-\tau^2/(4B_n)} \quad \text{if } \tau \leqslant \frac{B_n}{M_n}\,.$$

Later, in 1959, Prokhorov [14] proved the following remarkable strengthening of these inequalities.

Theorem 5 (Prokhorov inequality). *Let the conditions of the preceding theorem hold. Then with the same notation,*

$$\lambda\biggl\{\sum_{k=1}^{n} f_k > t\biggr\} \leqslant \exp\biggl(-\frac{t}{2M_n}\operatorname{arcsinh}\frac{tM_n}{2B_n}\biggr).$$

2. Definitions, notation, and preliminaries

Let (T, Σ, μ) be a space with a σ-finite measure μ defined on a σ-algebra Σ of subsets of the set T. The set of all almost everywhere finite real-valued functions (equivalence classes) on T with the natural algebraic operations and the topology of convergence with respect to the measure μ on sets of finite measure is a linear topological space denoted by $S(T, \Sigma, \mu)$. As usual, functions that coincide almost everywhere are identified. The notation $x \leqslant y \ \ (x, y \in S(T, \Sigma, \mu))$ means that $x(t) \leqslant y(t)$ for almost all $t \in T$. Further, we will be mostly interested in the cases when T is either the closed interval $[0,1]$ with the Lebesgue measure λ, or the n-dimensional cube $[0,1]^n$ $(n \in \mathbb{N})$, or the infinite dimensional cube $[0,1]^\infty$ with the measures $\prod_{k=1}^{n} \lambda_k$ and $\prod_{k=1}^{\infty} \lambda_k$, respectively ($\lambda_k$ is the Lebesgue measure on $[0,1]$), or the semi-axis $(0,\infty)$ with the Lebesgue measure. In all cases except the last, we obtain probability spaces which are isomorphic to each other. So, for example, in the case of $[0,1]$ and $[0,1]^n$ this means that there exists a measure-preserving transformation δ which acts from $[0,1]$ onto $[0,1]^n$. In these cases the space $S(T, \Sigma, \mu)$ will be denoted by $S(0,1)$, $S([0,1]^n)$, $S([0,1]^\infty)$, and $S(0,\infty)$, respectively. Below we denote by $\chi_B(t)$ the characteristic function of a set B, that is, $\chi_B(t) = 1$ if $t \in B$, and $\chi_B(t) = 0$ if $t \notin B$.

This survey deals mainly with sequences of independent functions considered in symmetric (rearrangement invariant) spaces on $[0,1]$ and $(0,\infty)$. Only the most elementary facts from probability theory will be needed below.

Definition 1. A set $\{f_k\}_{k=1}^n$ of random variables (r.v.'s) defined on a probability space $(\Omega, \Sigma, \mathsf{P})$ is said to be *independent* if for any intervals I_k in $\mathbb{R}$

$$\mathsf{P}\{\omega \in \Omega : f_k(\omega) \in I_k,\ k = 1, \dots, n\} = \prod_{k=1}^{n} \mathsf{P}\{\omega \in \Omega : f_k(\omega) \in I_k\}.$$

A sequence $\{f_k\}_{k=1}^\infty$ of r.v.'s is said to be *independent* if for each $n \in \mathbb{N}$ the set $\{f_k\}_{k=1}^n$ is independent.

We recall that an r.v. f is said to be *symmetrically distributed* if the r.v.'s f and $-f$ are identically distributed. As is traditional in probability theory, we shall denote by $\mathscr{F}_f(x)$ the distribution function and by $\theta_f(t)$ the characteristic function of an r.v. f, that is, $\theta_f(t) = \int_{-\infty}^{\infty} e^{itx}\, d\mathscr{F}_f(x)$.

Let $(\Omega, \Sigma, \mathsf{P})$ be a probability space, and let f be an integrable r.v. defined on this space. If $\mathfrak{R}$ is a σ-subalgebra of the σ-algebra Σ, then there exists a unique (up to a set of measure zero) $\mathfrak{R}$-measurable and integrable r.v. $\mathbb{E}_{\mathfrak{R}} f$ which satisfies the following equality for every $B \in \mathfrak{R}$:

$$\int_B f(\omega)\, d\mathsf{P}(\omega) = \int_B \mathsf{E}_{\mathfrak{R}} f(\omega)\, d\mathsf{P}(\omega) \tag{5}$$

([15], Theorem 1.1). The random variable $\mathsf{E}_{\mathfrak{R}} f$ is called the *conditional expectation* of f with respect to the σ-algebra $\mathfrak{R}$. In addition, as usual, $\mathsf{E} f = \int_\Omega f(\omega)\, d\mathsf{P}(\omega)$.

Concepts and results from the theory of symmetric spaces will be intensively used throughout. Therefore, we focus more on this theory, restricting it mostly to function spaces defined on $[0, 1]$ (see details in the monographs [16]–[18]).

Let $J = [0, 1]$ or $(0, \infty)$. For a function $x = x(t) \in S(J)$, $x \geqslant 0$, we consider the usual distribution function from the theory of functions: $n_x(\tau) := \lambda\{t \in J : x(t) > \tau\}$ $(\tau > 0)$. It is non-negative, non-increasing, and right continuous. Non-negative functions $x, y \in S(J)$ are said to be *equimeasurable* if $n_x(\tau) = n_y(\tau)$ $(\tau > 0)$. In particular, any function $x \in S(J)$ with $x \geqslant 0$ is equimeasurable with its non-increasing and left-continuous *rearrangement*

$$x^*(t) := \inf\{\tau \geqslant 0 : n_x(\tau) < t\} \qquad (t \in J).$$

For an arbitrary $x(t) \in S(J)$ let $x^*(t)$ denote the rearrangement of the function $|x(t)|$.

Definition 2. A Banach function space $X \subset S(J)$ is called a *symmetric space* if: 1) from the conditions that $x \in X$, $y \in S(J)$, and $|y(t)| \leqslant |x(t)|$ almost everywhere it follows that $y \in X$ and $\|y\|_X \leqslant \|x\|_X$; 2) from the conditions that $x \in X$, $y \in S(J)$, and $|x(t)|$ and $|y(t)|$ are equimeasurable it follows that $y \in X$ and $\|y\|_X = \|x\|_X$.

Any symmetric space X on J is linearly and continuously embedded in the Hausdorff linear topological space $S(J)$ ([19], Theorem 4.3.1). This means that if $\|x_n - x\|_X \to 0$ $(x_n, x \in X)$, then $x_n \to x$ with respect to the measure λ on sets of finite measure in J. If X and Y are Banach spaces, then the notation $X \subset Y$ throughout means that X is linearly and continuously embedded in Y; in particular, for some constant $C > 0$ and all $x \in X$ we have $\|x\|_Y \leqslant C\|x\|_X$. Without loss of generality, we shall assume that $\|\chi_{[0,1]}\|_X = 1$. In this case, it is not difficult to check ([16], Theorem 2.4.1) that for an arbitrary symmetric space X on $[0,1]$ the following holds: $L_\infty \subset X \subset L_1$, $\|x\|_X \leqslant \|x\|_{L_\infty}$ $(x \in L_\infty)$, and $\|x\|_{L_1} \leqslant \|x\|_X$ $(x \in X)$. We shall often use the following statement.

Proposition 1 ([16], Corollary 2.4.2). *Suppose that X is a symmetric space on J, $x \in X$, and $y \in S(J)$. If for some constant $C > 0$ and all $\tau > 0$ the inequality $n_{|y|}(\tau) \leqslant C n_{|x|}(\tau)$ holds, then $y \in X$ and $\|y\|_X \leqslant \max\{1, C\}\|x\|_X$.*

If X is a symmetric space on J, then the *dual* (or *Köthe dual*) space $X^\times$ consists of all $y \in S(J)$ such that

$$\|y\|_{X^\times} = \sup\biggl\{\int_J x(t)y(t)\,dt : \|x\|_X \leqslant 1\biggr\} < \infty.$$

The space $X^\times$ is also symmetric. It is isometrically embedded in the conjugate space X^*, and $X^\times = X^*$ if and only if X is separable. Any symmetric space X is continuously embedded in its second dual space, which we denote by $X^{\times\times}$. A symmetric space X *has the Fatou property* (or is said to be *maximal*) if from the conditions $x_n \in X$ $(n = 1, 2, \dots)$, $\sup_{n=1,2,\dots} \|x_n\|_X < \infty$, $x \in S(J)$, and $x_n \to x$ almost everywhere on J it follows that $x \in X$ and $\|x\|_X \leqslant \liminf_{n\to\infty} \|x_n\|_X$. If a symmetric space X has the analogous property under the additional condition that $x \in X$, then we say that X has an *order-semicontinuous norm.* A norm on X is order-semicontinuous if and only if X is embedded in $X^{\times\times}$ isometrically ([19], Theorem 6.1.6). The space X has the Fatou property if and only if its natural embedding into $X^{\times\times}$ is a surjective isometry ([19], Theorem 6.1.7). If a symmetric space has the Fatou property or it is separable, then its norm is order-semicontinuous.

Let us consider examples of symmetric spaces. First is the space L_p $(1 \leqslant p \leqslant \infty)$, with

$$\|x\|_p = \begin{cases} \biggl(\displaystyle\int_0^1 |x(t)|^p\,dt\biggr)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{t\in[0,1]} |x(t)|, & p = \infty, \end{cases}$$

and also its generalization, the family of $L_{p,q}$-spaces $(1 < p < \infty,\ 1 \leqslant q \leqslant \infty)$, with

$$\|x\|_{p,q} = \begin{cases} \biggl[\dfrac{q}{p}\displaystyle\int_0^1 \bigl(t^{1/p}x^*(t)\bigr)^q\,\frac{dt}{t}\biggr]^{1/q}, & 1 \leqslant q < \infty, \\ \operatorname*{ess\,sup}_{t\in[0,1]} \bigl(t^{1/p}x^*(t)\bigr), & q = \infty, \end{cases}$$

which play an important role in the interpolation theory of operators. Although the functional $\|x\|_{p,q}$ is not subadditive, it is equivalent to the norm $\|x\|'_{p,q} = \|x^{**}\|_{p,q}$, where $x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds$. It is not difficult to check ([16], Lemma 2.6.5) that $L_{p,q_1} \subset L_{p,q_2}$ if $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, and that $L_{p,p} = L_p$.

Another natural generalization of L_p-spaces is given by the Orlicz spaces [20]. If $M(u)$ is an Orlicz function, that is, an increasing convex function on $[0,\infty)$ with $M(0) = 0$, then the *Orlicz space* L_M consists of all $x \in S(0,1)$ such that there exists a $u > 0$ for which $\int_0^1 M(|x(t)|/u)\,dt \leqslant 1$. As a norm in this space we will use the

Luxemburg norm $\|x\|_{L_M} = \inf u$, where the infimum is taken over all u such that the last inequality holds. The so-called exponential Orlicz spaces are of particular interest for us. It is easy to check that every function

$$N_p(t) := e^{t^p} - \sum_{k=0}^{[1/p]} \frac{t^{kp}}{k!} \quad (0 < p < 1) \quad \text{and} \quad N_p(t) := e^{t^p} - 1 \quad (p \geqslant 1)$$

is an Orlicz function on $[0, \infty)$ (below, $[z]$ denotes the integer part of a number z). The corresponding Orlicz space L_{N_p} is frequently denoted by $\exp L_p$. In addition, we set $\exp L_\infty := L_\infty$.

Recall that a non-negative function $\varphi(t)$ on $[0,1]$ is said to be *quasi-concave* if $\varphi(0) = 0$, $\varphi(t)$ is increasing, and $\varphi(t)/t$ is decreasing. In particular, any non-negative increasing function $\psi(t)$ concave on $[0,1]$ and satisfying $\psi(0) = 0$ is quasi-concave ([16], § 2.1, p. 49). On the other hand, any quasi-concave function $\varphi(t)$ is equivalent to its least concave majorant $\overline{\varphi}(t)$ ([16], Theorem 2.1.1); more precisely, $\overline{\varphi}(t)/2 \leqslant \varphi(t) \leqslant \overline{\varphi}(t)$ $(0 \leqslant t \leqslant 1)$.

An important role below is played by the set Ψ of all functions $\psi(t)$ that are quasi-concave on $[0,1]$ and satisfy the conditions $\psi(0) = \psi(+0) = 0$ and $\psi(1) = 1$. If $\psi \in \Psi$, then the Lorentz space Λ_ψ and the Marcinkiewicz space M_ψ consist of all measurable functions $x(t)$ on $[0,1]$ satisfying

$$\|x\|_{\Lambda_\psi} := \int_0^1 x^*(t)\, d\bar{\psi}(t) < \infty \quad \text{and} \quad \|x\|_{M_\psi} := \sup_{0<t\leqslant 1} \frac{1}{\psi(t)} \int_0^t x^*(s)\, ds < \infty,$$

respectively.

By $X_\circ$ we shall denote the *separable part* of a symmetric space X, that is, the closure of L_∞ in X. We note that $X_\circ$ is also a separable symmetric space if $X \neq L_\infty$ ([16], Lemma 2.4.4). In particular, the space $(M_\psi)_\circ$ can be described as the set of all functions $x \in M_\psi$ such that $\lim_{t\to 0+} \frac{1}{\psi(t)} \int_0^t x^*(s)\, ds = 0$. It is not difficult to show that $(X^{\times\times})_\circ = X_\circ$ for any symmetric space X.

The space Λ_ψ is separable and has the Fatou property, $(M_\psi)_\circ$ is separable but does not have the Fatou property, whereas the space M_ψ has the Fatou property but is not separable. The following duality relations hold: $(\Lambda_\psi)^\times = M_\psi$ and $(M_\psi)^\times = ((M_\psi)_\circ)^\times = \Lambda_\psi$ ([16], Theorems 2.5.2 and 2.5.4).

An important characteristic of a symmetric space X is its *fundamental function* $\phi_X(s) := \|\chi_{(0,s)}\|_X$ $(0 \leqslant s \leqslant 1)$. In particular,

$$\phi_{L_{p,q}}(s) = s^{1/p} \qquad (1 \leqslant q \leqslant \infty),$$
$$\phi_{\Lambda_\varphi}(s) = \overline{\varphi}(s), \quad \text{and} \quad \phi_{M_\varphi}(s) = \widetilde{\varphi}(s), \quad \text{where } \widetilde{\varphi}(s) = \frac{s}{\varphi(s)}\,.$$

For every symmetric space X the function ϕ_X is quasi-concave on $[0,1]$ ([16], Theorem 2.4.7), and if ϕ is the fundamental function of a symmetric space X, then $\Lambda_\phi \subset X \subset M_{\tilde{\phi}}$ ([16], Theorems 2.5.5 and 2.5.7).

In any symmetric space X on $[0,1]$ the *dilation operator*

$$\sigma_\tau x(t) := x\left(\frac{t}{\tau}\right) \chi_{[0,\min\{1,\tau\}]}(t) \qquad (\tau > 0)$$

is continuous and $\|\sigma_\tau\|_{X\to X}\leqslant\max\{1,\tau\}$ ([16], Theorem 2.4.5). Using this operator, we can define the *lower and upper Boyd indices* of a space X as follows:

$$\alpha_X:=\lim_{\tau\to0+}\frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln\tau}\quad\text{and}\quad\beta_X:=\lim_{\tau\to+\infty}\frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln\tau}.$$

From the above estimate of the norm of the dilation operator it follows that always $0\leqslant\alpha_X\leqslant\beta_X\leqslant1$. In particular, $\alpha_{L_{p,q}}=\beta_{L_{p,q}}=1/p$, and the Boyd indices of the Lorentz and Marcinkiewicz spaces coincide with the corresponding dilation indices of their fundamental function ([16], Chap. II, § 2, p. 102).

Further, we shall also need the notion of a symmetric space of random variables (r.v.'s) defined on a probability space $(\Omega,\Sigma,\mathsf{P})$. Namely, if X is a symmetric space on $[0,1]$, then the corresponding space $X(\Omega):=X(\Omega,\Sigma,\mathsf{P})$ consists of all r.v.'s $f\colon\Omega\to\mathbb{R}$ such that $f^*\in X$, and $\|f\|_{X(\Omega)}:=\|f^*\|_X$. Here as above,

$$f^*(t)=\inf\bigl\{\tau\geqslant0:\mathsf{P}\{\omega\in\Omega:|f(\omega)|>\tau\}<t\bigr\}\qquad(0<t\leqslant1).$$

If the space $(\Omega,\Sigma,\mathsf{P})$ is isomorphic to the interval $[0,1]$ equipped with the Lebesgue measure, then we can (and shall) identify the symmetric spaces $X(\Omega)$ and X.

The concepts and methods of the interpolation theory of operators will play an important role below. A system $\vec{X}=(X_0,X_1)$ of two Banach spaces X_0 and X_1 is called a *Banach couple* if both are linearly and continuously embedded in the same Hausdorff linear topological space. For a Banach couple we can define the *intersection* $X_0\cap X_1$ and the *sum* X_0+X_1 as Banach spaces with the norms

$$\|x\|_{X_0\cap X_1}=\max\bigl\{\|x\|_{X_0},\|x\|_{X_1}\bigr\}$$

and

$$\|x\|_{X_0+X_1}=\inf\bigl\{\|x_0\|_{X_0}+\|x_1\|_{X_1}:x=x_0+x_1,\ x_i\in X_i,\ i=0,1\bigr\}$$

respectively. For example, any two symmetric spaces X_0 and X_1 on J ($J=[0,1]$ or $J=(0,\infty)$) form a Banach couple, since $X_i\subset S(J)$ $(i=0,1)$.

We say that a Banach space X is an *interpolation* space with respect to a Banach couple (X_0,X_1) (written $X\in\operatorname{Int}(X_0,X_1)$) if $X_0\cap X_1\subset X\subset X_0+X_1$ and for every linear operator T which is bounded on X_0 and on X_1 we have $T\colon X\to X$. In this case, by the closed graph theorem, there exists a constant $C>0$ such that for any $T\colon X_i\to X_i$ $(i=0,1)$ we have $\|T\|_{X\to X}\leqslant C\max_{i=0,1}\|T\|_{X_i\to X_i}$. If the last inequality holds with $C=1$, then X is called *an interpolation space with constant* 1. In particular, by the classical Riesz–Thorin theorem ([21], Theorem 1.1.1), L_p is an interpolation space with constant 1 with respect to the couple (L_{p_0},L_{p_1}) for any $1\leqslant p_0\leqslant p\leqslant p_1\leqslant\infty$.

One of the most important ways to construct interpolation spaces is to use the *Peetre $\mathscr{K}$-functional* defined on a Banach couple (X_0,X_1) as follows:

$$\mathscr{K}(t,x;X_0,X_1)=\inf\bigl\{\|x_0\|_{X_0}+t\|x_1\|_{X_1}:x=x_0+x_1,x_i\in X_i\bigr\},$$

where $x\in X_0+X_1$ and $t>0$. One can easily show that for any Banach couple $\mathscr{K}(t,x;X_0,X_1)$ is a continuous non-negative increasing concave function of the variable $t > 0$ for a fixed $x \in X_0 + X_1$ ([21], Lemma 3.1.1). If $X_0 \subset X_1$ and $\|x\|_{X_1} \leqslant \|x\|_{X_0}$ $(x \in X_0)$ ($X_1 \subset X_0$ and $\|x\|_{X_0} \leqslant \|x\|_{X_1}$ $(x \in X_1)$, respectively), then $\mathscr{K}(t, x; X_0, X_1) = t\|x\|_{X_1}$ for $0 \leqslant t \leqslant 1$ ($\mathscr{K}(t, x; X_0, X_1) = \|x\|_{X_0}$ for $t \geqslant 1$, respectively).

Non-negative functions $f(t)$ and $g(t)$ are said to be *equivalent on the set* T(we write: $f \asymp g$ $(t \in T)$) if for some constant $C > 0$ and all $t \in T$ we have $C^{-1} f(t) \leqslant g(t) \leqslant C f(t)$.

Sometimes the $\mathscr{K}$-functional admits a quite simple equivalent expression. In particular, for an arbitrary space with a σ-finite measure and for any $p \geqslant 1$

$$\mathscr{K}(t, x; L_p, L_\infty) \asymp \left(\int_0^{t^p} (x^*(s))^p \, ds\right)^{1/p} \qquad (t > 0),$$

with equivalence constants not depending on $x \in L_p + L_\infty$ and $t > 0$ (for $p = 1$, we can replace this equivalence with equality).

Further, if F is a Banach lattice of two-sided real sequences, then for $u_k > 0$ $(k = 0, \pm 1, \pm 2, \dots)$ we denote by $F(u_k)$ the *weighted space* consisting of all sequences $a = (a_k)_{k=-\infty}^{\infty}$ such that the norm $\|a\|_{F(u_k)} := \|(a_k u_k)\|_F$ is finite. Assume that $F \supset \ell_\infty \cap \ell_\infty(2^{-k})$. If (X_0, X_1) is an arbitrary Banach couple, then *the space* $(X_0, X_1)_F^{\mathscr{K}}$ *of the real* $\mathscr{K}$*-method* consists of all $x \in X_0 + X_1$ such that $(\mathscr{K}(2^k, x; X_0, X_1))_k \in F$, with the norm $\|x\| := \|(\mathscr{K}(2^k, x; X_0, X_1))_k\|_F$. One can easily check that $(X_0, X_1)_F^{\mathscr{K}}$ is an interpolation space with respect to the couple (X_0, X_1) with constant 1 [22]. In the special case when $F = \ell_p(2^{-k\theta})$ $(0 < \theta < 1$, $1 \leqslant p \leqslant \infty)$, we obtain the classical spaces $(X_0, X_1)_{\theta,p}$, whose properties are presented in detail in the monograph [21].

The importance of the real method of interpolation stems, in particular, from the fact that for a sufficiently large class of Banach couples it gives all interpolation spaces. A Banach couple $\vec{X} = (X_0, X_1)$ is said to be $\mathscr{K}$*-monotone* (or a *Calderón–Mityagin couple*) if the inequality

$$\mathscr{K}(t, y; X_0, X_1) \leqslant \mathscr{K}(t, x; X_0, X_1) \qquad (t > 0)$$

for some $x \in X_0 + X_1$ and $y \in X_0 + X_1$ implies that there exists an operator U bounded on X_0 and on X_1 such that $y = Ux$. In view of the Brudnyi–Krugljak theorem, which is one of the main results in interpolation theory ([22], Theorem 15.1), any interpolation space X with respect to a $\mathscr{K}$-monotone couple (X_0, X_1) can be represented as $X = (X_0, X_1)_F^{\mathscr{K}}$ for some Banach lattice F.

The second name for a $\mathscr{K}$-monotone couple comes from the fact that the first result in this direction was the theorem, proved independently and almost simultaneously by Calderón [23] and Mityagin [24], that the Banach couple (L_1, L_∞) is $\mathscr{K}$-monotone. Below we shall assume that a symmetric space X on $[0, 1]$ is an interpolation space with respect to this couple with constant 1. In view of the Calderón–Mityagin theorem this is equivalent to the following condition: if $x \in X$, $y \in L_1$, and $y \prec x$, then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$, where $y \prec x$ denotes the Hardy–Littlewood semi-ordering, which means that $\int_0^t y^*(s)\,ds \leqslant \int_0^t x^*(s)\,ds$ for all $0 \leqslant t \leqslant 1$. The class of such spaces is quite extensive: it includes all spaces with the Fatou property and all separable spaces, in particular, Orlicz, Lorentz, and Marcinkiewicz spaces. In view of [17], Theorem 2.a.4, the conditional expectation operator $\mathbb{E}_{\mathscr{B}}$ corresponding to a σ-subalgebra $\mathscr{B}$ of some probability space is bounded on L_1 and on L_∞ with constant 1. Therefore, the following statement holds, and it will be used often below.

Proposition 2. *Every conditional expectation operator $\mathbb{E}_{\mathscr{B}}$ is bounded on any symmetric space on $[0,1]$ and has norm 1.*

And finally, the following interpolation construction introduced by Calderón [25] will sometimes be useful for us. Let X_0 and X_1 be Banach lattices of measurable functions defined on the same measure space $(\mathscr{M},m)$. For every $\theta\in(0,1)$ the space $X_0^{1-\theta}X_1^\theta$ consists of all measurable functions f on $(\mathscr{M},m)$ such that for some $\lambda>0$ and $f_i\in X_i$ with $\|f_i\|_{X_i}\leqslant 1$ $(i=0,1)$ the following inequality holds:

$$|f(x)|\leqslant\lambda|f_0(x)|^{1-\theta}|f_1(x)|^\theta,\qquad x\in\mathscr{M}.$$

The norm in this space is equal to the infimum of all λ for which the preceding relation holds. Although this construction is not an interpolation functor on the set of all couples of Banach lattices [26], it is still very useful in interpolation theory. In particular, if (X_0,X_1) and (Y_0,Y_1) are two couples of Banach lattices of measurable functions on measure spaces $(\mathscr{M},m)$ and $(\mathscr{M}',m')$, respectively, then any *positive* operator A acting from $S(\mathscr{M},m)$ to $S(\mathscr{M}',m')$ which is bounded from X_0 to Y_0 and from X_1 to Y_1, is also bounded from the space $X_0^{1-\theta}X_1^\theta$ to the space $Y_0^{1-\theta}Y_1^\theta$, and $\|A\|_{X_0^{1-\theta}X_1^\theta\to Y_0^{1-\theta}Y_1^\theta}\leqslant\|A\|_{X_0\to Y_0}^{1-\theta}\|A\|_{X_1\to Y_1}^\theta$ for every $\theta\in(0,1)$ (see, for example, [17], Proposition 1.d.2(i), p. 43).

A detailed exposition of the results cited above and also of many other results and facts from the interpolation theory of operators can be found in the monographs [16], [18], [21], [22].

In §§ 6 and 7 we shall discuss (quasi-)Banach symmetric sequence spaces. If $\xi=(\xi_n)_{n=1}^\infty$ is a bounded sequence of real numbers, then its *non-increasing rearrangement* $\xi^*=(\xi_n^*)_{n=1}^\infty$ is defined by $\xi_n^*=\inf_{\operatorname{card}A=n-1}\sup_{k\in\mathbb{N}\setminus A}|\xi_k|$. A quasi-Banach sequence space E is said to be *symmetric* if from the assumptions $a\in E$ and $b^*\leqslant a^*$ it follows that $b\in E$ and $\|b\|_E\leqslant\|a\|_E$. In such a space the quasi-norm $\|\cdot\|_E$ satisfies a generalized triangle inequality

$$\|a+b\|_E\leqslant L(\|a\|_E+\|b\|_E),\qquad a,b\in E,$$

where $L\geqslant 1$. If the last inequality holds with $L=1$, then E is called a (*Banach*) *symmetric sequence space*. Without loss of generality we will assume throughout that $\|e_k\|_E=1$ $(k=1,2,\dots)$, where the vectors e_k are the vectors of the standard basis in sequence spaces.

3. L_p-norm estimates of sums of independent functions

In this section we shall consider L_p-norm estimates of sums of independent functions generalizing and specializing the classical Khintchine and Marcinkiewicz–Zygmund inequalities. Here we can distinguish two main directions. In the first direction, arbitrary systems of non-negative and symmetrically distributed functions are considered rather than uniformly bounded systems. The gist of the second main direction consists in replacing the ℓ_2-norm of the sequence of coefficients (or more generally, the sum of the variances of the terms) by a more 'flexible' quantity which can be used to obtain two-sided estimates with constants independent of p.

3.1. Rosenthal inequalities. While studying complemented subspaces of the space $L_p = L_p[0,1]$, Rosenthal proved in 1970 [1] the following remarkable inequalities, of which the second can be viewed as a far-reaching generalization of the Khintchine inequalities. They show that, up to equivalence, the L_p-norm of a sum of independent functions is determined by the L_q-norms of the terms. As we shall see below, a similar phenomenon expressed in the terms of so-called 'disjointification' can be observed also in general symmetric spaces.

Theorem 6 (Rosenthal inequalities).

(i) *If $1 < p < \infty$, then for any sequence $\{f_k\}_{k=1}^\infty \subset L_p$ of non-negative independent functions and any $n \in \mathbb{N}$*

$$\max\biggl\{\biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p}, \sum_{k=1}^n \|f_k\|_1\biggr\} \leqslant \biggl\|\sum_{k=1}^n f_k\biggr\|_p \leqslant 2^p \max\biggl\{\biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p}, \sum_{k=1}^n \|f_k\|_1\biggr\}. \qquad (6)$$

(ii) *If $p > 2$, then there exists a constant $K_p > 0$ such that for any sequence $\{f_k\}_{k=1}^\infty \subset L_p$ of independent functions satisfying $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$ and for any $n \in \mathbb{N}$*

$$\frac{1}{2}\max\biggl\{\biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p}, \biggl(\sum_{k=1}^n \|f_k\|_2^2\biggr)^{1/2}\biggr\} \leqslant \biggl\|\sum_{k=1}^n f_k\biggr\|_p \leqslant K_p \max\biggl\{\biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p}, \biggl(\sum_{k=1}^n \|f_k\|_2^2\biggr)^{1/2}\biggr\}. \qquad (7)$$

For the proof we need two lemmas.

Lemma 1. *Let $1 < p < \infty$. If $\{f_k\}_{k=1}^\infty \subset L_p$ is an arbitrary sequence of independent functions, then for any $n \in \mathbb{N}$*

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_p \leqslant 2^p \max\biggl\{\biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p}, \sum_{k=1}^n \|f_k\|_1\biggr\}.$$

Proof. We can assume that $f_k \geqslant 0$ for all $k = 1, 2, \dots$. Let

$$N_p := \biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p}.$$

In view of the independence of the functions f_1 and $f_2+\cdots+f_n$ we have

$$\begin{aligned}\int_0^1 (f_1+\cdots+f_n)^{p-1} f_1\,dt &\leqslant 2^{p-1}\int_0^1 \big(f_1^{p-1}+(f_2+\cdots+f_n)^{p-1}\big) f_1\,dt\\ &= 2^{p-1}\bigg(\int_0^1 f_1^p\,dt+\int_0^1 (f_2+\cdots+f_n)^{p-1}\,dt\int_0^1 f_1\,dt\bigg)\\ &\leqslant 2^{p-1}\bigg(\int_0^1 f_1^p\,dt+\int_0^1 (f_1+\cdots+f_n)^{p-1}\,dt\int_0^1 f_1\,dt\bigg).\end{aligned}$$

Similarly,

$$\int_0^1 (f_1+\cdots+f_n)^{p-1} f_k\,dt \leqslant 2^{p-1}\bigg(\int_0^1 f_k^p\,dt+\Big\|\sum_{i=1}^n f_i\Big\|_{p-1}^{p-1}\int_0^1 f_k\,dt\bigg)$$

for all $k=1,2,\dots,n$. Summing the latter inequalities with respect to k, we get that

$$\Big\|\sum_{k=1}^n f_k\Big\|_p^p \leqslant 2^{p-1}\bigg(N_p^p+\Big\|\sum_{k=1}^n f_k\Big\|_p^{p-1} N_1\bigg)\leqslant 2^p\max\bigg\{N_p^p,\Big\|\sum_{k=1}^n f_k\Big\|_p^{p-1} N_1\bigg\},$$

whence

$$\Big\|\sum_{k=1}^n f_k\Big\|_p \leqslant \max\{2N_p, 2^p N_1\}\leqslant 2^p\max\{N_p,N_1\}.$$

The proof is complete.

Lemma 2. *Let $1\leqslant p<\infty$ and let $\{f_k\}_{k=1}^\infty \subset L_p$ be an arbitrary sequence of independent functions such that $\int_0^1 f_k(t)\,dt=0$ $(k=1,2,\dots)$.*

(a) *If $\theta_k=\pm1$ $(k=1,2,\dots)$, then for any $n\in\mathbb{N}$*

$$\Big\|\sum_{k=1}^n \theta_k f_k\Big\|_p \leqslant 2\Big\|\sum_{k=1}^n f_k\Big\|_p.$$

(b) *For any $n\in\mathbb{N}$*

$$\Big\|\sum_{k=1}^n f_k\Big\|_p \leqslant 2\bigg(\sum_{k=1}^n \|f_k\|_p^p\bigg)^{1/p} \qquad \text{if } p\leqslant 2,$$

and

$$\Big\|\sum_{k=1}^n f_k\Big\|_p \geqslant \frac12\bigg(\sum_{k=1}^n \|f_k\|_p^p\bigg)^{1/p} \qquad \text{if } p> 2.$$

Proof. Let the functions f and g be independent and let $\mathsf{E}f=0$. In this case if E_g is the conditional expectation operator with respect to the σ-algebra generated by the function g, then $\|\mathsf{E}_g\|_{L_p\to L_p}=1$ $(1\leqslant p\leqslant\infty)$ (see Proposition 2). Moreover, since for every g-measurable set B the functions f and χ_B are independent, the equality (5) implies that

$$\int_B \mathsf{E}_g f\,du=\int_B f\,du=\int_0^1 f\chi_B\,du=\mathsf{E}\,f\cdot\lambda(B)=0.$$

Therefore, $\mathsf{E}_g f = 0$ and

$$\|g\|_p = \|\mathsf{E}_g\, g\|_p = \|\mathsf{E}_g(f+g)\|_p \leqslant \|f+g\|_p.$$

For given $\theta_k = \pm 1$ $(k=1,\dots,n)$ we set

$$f = \sum_{k:\theta_k=1} f_k \quad \text{and} \quad g = \sum_{k:\theta_k=-1} f_k.$$

Then

$$f+g = \sum_{k=1}^n f_k \quad \text{and} \quad f-g = \sum_{k=1}^n \theta_k f_k.$$

By assumption, f and g are independent and $\int_0^1 f(t)\,dt = \int_0^1 g(t)\,dt = 0$. Hence in view of the observation made above, we have $\|f\|_p \leqslant \|f+g\|_p$ and $\|g\|_p \leqslant \|f+g\|_p$. Therefore, $\|f-g\|_p \leqslant 2\|f+g\|_p$ and (a) is proved.

Passing to the proof of (b), we assume first that the functions f_k $(k=1,2,\dots)$ are symmetrically distributed. Then the functions f_1-f_2 and f_1+f_2 are identically distributed, and hence $\|f_1+f_2\|_p^p = (\|f_1+f_2\|_p^p + \|f_1-f_2\|_p^p)/2$. If $1\leqslant p\leqslant 2$, then $|a+b|^p + |a-b|^p \leqslant 2(|a|^p+|b|^p)$, and so

$$\|f_1+f_2\|_p^p \leqslant \|f_1\|_p^p + \|f_2\|_p^p.$$

In the case of $p>2$ the opposite inequalities

$$|a+b|^p + |a-b|^p \geqslant 2(|a|^p+|b|^p) \quad \text{and} \quad \|f_1+f_2\|_p^p \geqslant \|f_1\|_p^p + \|f_2\|_p^p$$

hold. Hence, if the functions f_k are independent and symmetrically distributed, then for every $n\in\mathbb{N}$

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_p \leqslant \biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p} \quad \text{if } p\leqslant 2,$$

and

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_p \geqslant \biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p} \quad \text{if } p>2.$$

In the general case we use symmetrization and consider the differences $f_i - f_i'$, where f_i' is an independent copy of f_i $(i=1,2,\dots)$. They are independent and symmetrically distributed. Therefore if, for example, $1\leqslant p\leqslant 2$, then from the reasoning at the beginning of the proof and the Minkowski inequality we have

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_p \leqslant \biggl\|\sum_{k=1}^n (f_k-f_k')\biggr\|_p \leqslant \biggl(\sum_{k=1}^n \|f_k-f_k'\|_p^p\biggr)^{1/p} \leqslant 2\biggl(\sum_{k=1}^n \|f_k\|_p^p\biggr)^{1/p}.$$

The case of $p>2$ is considered similarly.

Proof of Theorem 6. To prove (i) it is enough to apply Lemma 1 and the elementary inequality $\|(b_k)\|_p \leqslant \|(b_k)\|_1$ $(p \geqslant 1,\ b_k \in \mathbb{R})$.

In proving (ii) we use the notation introduced in the proof of Lemma 1. Since the f_k are independent, we have $\int_0^1 f_i f_j\,dt = \int_0^1 f_i\,dt \int_0^1 f_j\,dt = 0$ $(i \neq j)$, that is, the system $\{f_k\}_{k=1}^\infty$ is orthogonal. Therefore,

$$N_2 = \biggl\|\sum_{k=1}^n f_k\biggr\|_2,$$

and so the left-hand estimate in (7) follows from Lemma 2 (b) and the inequality

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_2 \leqslant \biggl\|\sum_{k=1}^n f_k\biggr\|_p.$$

We prove the right-hand estimate in (7). By Lemma 2 (a), for every $u \in [0,1]$ we have

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_p^p \leqslant 2^p \int_0^1 \biggl|\sum_{k=1}^n r_k(u) f_k(t)\biggr|^p\,dt,$$

where the functions $r_k(u)$ are the Rademacher functions. Integrating the last inequality with respect to u, changing the order of integration, and using the Khintchine inequalities (1), we get that

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_p^p \leqslant 2^p B_p^p \int_0^1 \biggl(\sum_{k=1}^n f_k(t)^2\biggr)^{p/2} dt.$$

By assumption, the functions f_k^2 are non-negative independent functions in $L_{p/2}$. Since $p/2 > 1$, it follows from Lemma 1 that

$$\biggl(\int_0^1 \biggl(\sum_{k=1}^n f_k(t)^2\biggr)^{p/2} dt\biggr)^{2/p} \leqslant 2^{p/2} \max\{N_p^2, N_2^2\}.$$

As a result, the right-hand estimate in (7) with $K_p = B_p 2^{p/4+1}$ follows from the last two inequalities.

3.2. Hitczenko inequalities. The Khintchine inequalities (1) for the L_p-norms of Rademacher polynomials contain constants A_p and B_p depending on p, so on the basis of the right-hand estimate in (1) it is possible to assert only that for some constant $C = C(a)$

$$\biggl\|\sum_{k=1}^\infty a_k r_k\biggr\|_p \leqslant C\sqrt{p} \quad (p \geqslant 1) \quad \text{for } a = (a_k) \in \ell_2.$$

In 1993, P. Hitczenko proved a more precise statement making it possible in principle to determine the order of the norm $\bigl\|\sum_{k=1}^\infty a_k r_k\bigr\|_p$ as $p \to \infty$ for any coefficient sequence $(a_k)_{k=1}^\infty \in \ell_2$ [27]. The central role here is played by the Peetre $\mathscr{K}$-functional $\kappa_a(t) := \mathscr{K}(t, a; \ell_1, \ell_2)$ corresponding to the Banach couple (ℓ_1, ℓ_2) (see § 2).

Theorem 7. *There is a universal constant $\gamma > 0$ such that for any $t \geqslant 1$ and $a = (a_k)_{k=1}^\infty \in \ell_2$*

$$\gamma \kappa_a(\sqrt{t}\,) \leqslant \biggl\| \sum_{k=1}^\infty a_k r_k \biggr\|_t \leqslant \kappa_a(\sqrt{t}\,). \tag{8}$$

For the proof we need some equivalent expressions for the functional $\kappa_a(t)$. First of all, we shall prove one version of a known inequality due to Holmstedt (see [28] or [21], Exercise 5.7.3).

Proposition 3. *For any $a = (a_k)_{k=1}^\infty \in \ell_2$ and $t > 0$,*

$$\frac{1}{4}\biggl\{ \sum_{i=1}^{[t^2]} a_i^* + t\biggl(\sum_{i=[t^2]+1}^\infty (a_i^*)^2 \biggr)^{1/2} \biggr\} \leqslant \kappa_a(t) \leqslant \sum_{i=1}^{[t^2]} a_i^* + t\biggl(\sum_{i=[t^2]+1}^\infty (a_i^*)^2 \biggr)^{1/2}, \tag{9}$$

where $(a_i^*)_{i=1}^\infty$ is the non-increasing rearrangement of the sequence $(|a_n|)_{n=1}^\infty$.

Proof. We first prove the following functional analogue of (9): for any function $f \in L_1(0,\infty) + L_2(0,\infty)$ and any $t > 0$

$$\begin{aligned}\frac{1}{4}\biggl\{ \int_0^{t^2} f^*(s)\,ds + t\biggl(\int_{t^2}^\infty f^*(s)^2\,ds \biggr)^{1/2} \biggr\} &\leqslant \mathscr{K}(t, f; L_1, L_2) \\ &\leqslant \int_0^{t^2} f^*(s)\,ds + t\biggl(\int_{t^2}^\infty f^*(s)^2\,ds \biggr)^{1/2}.\end{aligned} \tag{10}$$

Without loss of generality we may assume that $f = f^*$. For any $t > 0$ we define $f_0 = f\chi_{(0,t^2)}$ and $f_1 = f\chi_{(t^2,\infty)}$. Since

$$\|f_0\|_{L_1} + t\|f_1\|_{L_2} = \int_0^{t^2} f^*(s)\,ds + t\biggl(\int_{t^2}^\infty f^*(s)^2\,ds \biggr)^{1/2},$$

the right-hand inequality in (10) follows from the definition of the $\mathscr{K}$-functional.

To prove the left-hand inequality, let us consider an arbitrary representation $f = g + h$ with $g \in L_1$ and $h \in L_2$. Then in view of an elementary inequality for rearrangements ([16], (2.23) on p. 67) we get that

$$f(u) \leqslant g^*\biggl(\frac{u}{2}\biggr) + h^*\biggl(\frac{u}{2}\biggr), \qquad u > 0. \tag{11}$$

Therefore, by the Cauchy–Schwarz–Bunyakovskii inequality,

$$\begin{aligned}\|g\|_{L_1} + t\|h\|_{L_2} &\geqslant \frac{1}{2}\biggl(\int_0^{t^2} g^*\biggl(\frac{u}{2}\biggr)\,du + t\biggl(\int_0^{t^2} \biggl(h^*\biggl(\frac{u}{2}\biggr)\biggr)^2\,du \biggr)^{1/2} \biggr) \\ &\geqslant \frac{1}{2}\biggl(\int_0^{t^2} g^*\biggl(\frac{u}{2}\biggr)\,du + \int_0^{t^2} h^*\biggl(\frac{u}{2}\biggr)\,du \biggr) \geqslant \frac{1}{2}\int_0^{t^2} f^*(u)\,du.\end{aligned} \tag{12}$$

Moreover,

$$t^2\int_{t^2}^{\infty}\Bigl(g^*\Bigl(\frac{u}{2}\Bigr)\Bigr)^2\,du\leqslant\int_0^{t^2}g^*\Bigl(\frac{u}{2}\Bigr)\,du\int_{t^2}^{\infty}g^*\Bigl(\frac{u}{2}\Bigr)\,du\leqslant\biggl(\int_0^{\infty}g^*\Bigl(\frac{u}{2}\Bigr)\,du\biggr)^2=4\|g\|_{L_1}^2,$$

and so in view of (11),

$$\|g\|_{L_1}+t\|h\|_{L_2}\geqslant\frac{t}{2}\biggl(\int_{t^2}^{\infty}\Bigl(g^*\Bigl(\frac{u}{2}\Bigr)\Bigr)^2\,du\biggr)^{1/2}+\frac{t}{2}\biggl(\int_{t^2}^{\infty}\Bigl(h^*\Bigl(\frac{u}{2}\Bigr)\Bigr)^2\,du\biggr)^{1/2}\geqslant\frac{t}{2}\biggl(\int_{t^2}^{\infty}(f^*(u))^2\,du\biggr)^{1/2}.$$

From this and (12) we get the left-hand inequality in (10) by the definition of the $\mathscr{K}$-functional.

Further, for an arbitrary sequence $a=(a_k)_{k=1}^{\infty}\in\ell_2$ we define the step-function

$$f_a(t)=\sum_{k=1}^{\infty}a_k\chi_{(k-1,k]}(t)\qquad(t>0).\tag{13}$$

Standard arguments show that for any $a=(a_k)_{k=1}^{\infty}\in\ell_2$ and $t>0$

$$\mathscr{K}(t,f_a;L_1,L_2)=\kappa_a(t).\tag{14}$$

Now we turn to the proof of the inequality (9). First of all, if $0<t<1$, then in view of the inequality $\|a\|_2\leqslant\|a\|_1$ we have $\kappa_a(t)=t\|a\|_2$, which implies (9).

Let $t\geqslant 1$. Since the spaces ℓ_1 and ℓ_2 are symmetric, we may assume that $a_k=a_k^*$. Let $b^t=(b_k^t)$, where $b_k^t=a_k$ if $k\leqslant[t^2]$ and $b_k^t=0$ if $k>[t^2]$. Let $c^t=a-b^t$. Then $\kappa_a(t)\leqslant\|b^t\|_1+t\|c^t\|_2$, and therefore the right-hand inequality in (9) is a direct consequence of the definition of the $\mathscr{K}$-functional. The left-hand inequality follows from (14) and (10). Indeed, if f_a is defined by (13), then

$$\begin{aligned}\kappa_a(t)&\geqslant\kappa_a\bigl(\sqrt{[t^2]}\,\bigr)=\mathscr{K}\bigl(\sqrt{[t^2]}\,,f_a;L_1,L_2\bigr)\\&\geqslant\frac14\biggl\{\int_0^{[t^2]}f_a^*(s)\,ds+t\biggl(\int_{[t^2]}^{\infty}f_a^*(s)^2\,ds\biggr)^{1/2}\biggr\}\\&=\frac14\biggl\{\sum_{i=1}^{[t^2]}a_i^*+t\biggl(\sum_{i=[t^2]+1}^{\infty}(a_i^*)^2\biggr)^{1/2}\biggr\}.\end{aligned}$$

The proof is complete.

Next, we need one more approximation of $\kappa_a(t)$, obtained in [29]. For any $a=(a_k)_{k=1}^{\infty}\in\ell_2$ and $t\in\mathbb{N}$ we define the norm

$$\|a\|_{Q(t)}=\sup\biggl\{\sum_{j=1}^{t}\biggl(\sum_{n\in A_j}a_n^2\biggr)^{1/2}\biggr\},\tag{15}$$

where the supremum is taken over all partitions $\{A_j\}_{j=1}^{t}$ of $\mathbb{N}$.

Proposition 4. *If* $a = (a_k)_{k=1}^\infty \in \ell_2$ *and* $t^2 \in \mathbb{N}$, *then*

$$\|a\|_{Q(t^2)} \leqslant \kappa_a(t) \leqslant \sqrt{2}\,\|a\|_{Q(t^2)}. \tag{16}$$

Proof. First of all, the inequalities

$$\|a\|_{Q(t^2)} \leqslant \|a\|_1 \quad \text{and} \quad \|a\|_{Q(t^2)} \leqslant t\|a\|_2$$

follow from the definition of $\|\cdot\|_{Q(t)}$. Indeed, the first one is obvious, and to prove the second one it suffices to note that, by the Cauchy–Schwarz–Bunyakovskii inequality, for any partition $\{A_j\}_{j=1}^{t^2}$ of $\mathbb{N}$

$$\sum_{j=1}^{t^2}\biggl(\sum_{n\in A_j} a_n^2\biggr)^{1/2} \leqslant t\biggl(\sum_{j=1}^{t^2}\sum_{n\in A_j} a_n^2\biggr)^{1/2} = t\|a\|_2.$$

Hence

$$\begin{aligned}\kappa_a(t) &= \inf\{\|b\|_1 + t\|c\|_2 : b+c=a,\ b\in\ell_1,\ c\in\ell_2\}\\ &\geqslant \inf\{\|b\|_{Q(t^2)} + \|c\|_{Q(t^2)} : b+c=a,\ b\in\ell_1,\ c\in\ell_2\} \geqslant \|a\|_{Q(t^2)},\end{aligned}$$

and the left-hand inequality in (16) is proved.

Let us prove the opposite inequality. By standard arguments it is not difficult to show that for every $t>0$ the space $\ell_2\cap\ell_\infty$ with the norm $\max\{\|a\|_\infty, t^{-1}\|a\|_2\}$ is the (Banach) conjugate space of the space $\ell_1+\ell_2$ equipped with the norm $\kappa_a(t)$.

Therefore, from the Hahn–Banach theorem it follows that

$$\kappa_a(t) = \max\biggl\{\sum_{k=1}^\infty a_k b_k : b=(b_k)_{k=1}^\infty \in \ell_2,\ \max\{\|b\|_\infty,\ t^{-1}\|b\|_2\} \leqslant 1\biggr\}. \tag{17}$$

Hence, there exists a sequence $b\in\ell_2$ such that

$$\kappa_a(t) = \sum_{k=1}^\infty a_k b_k \quad \text{and} \quad \max\{\|b\|_\infty, t^{-1}\|b\|_2\} = 1.$$

Let us choose $n_0, n_1, n_2, \dots, n_{t^2} \in \{0,1,\dots,\infty\}$ by induction in the following way: if $0 = n_0 < n_1 < \dots < n_m$ have already been found, then

$$n_{m+1} = 1 + \max\biggl\{k : \sum_{n=n_m+1}^{k} b_n^2 \leqslant 1\biggr\}.$$

Since $\|b\|_\infty \leqslant 1$, we have $\sum_{n=n_m+1}^{n_{m+1}} b_n^2 \leqslant 2$. From $\|b\|_2 \leqslant t$ it follows that $n_{t^2} = \infty$. Thus,

$$\kappa_a(t) = \sum_{k=1}^\infty a_k b_k \leqslant \sum_{m=1}^{t^2}\biggl(\sum_{n=n_{m-1}+1}^{n_m} b_n^2\biggr)^{1/2}\biggl(\sum_{n=n_{m-1}+1}^{n_m} a_n^2\biggr)^{1/2} \leqslant \sqrt{2}\,\|a\|_{Q(t^2)},$$

and the statement is proved.

Proof of Theorem 7. The right-hand inequality in (8) is an immediate consequence of the definition of the $\mathscr{K}$-functional and the Khintchine inequalities. Indeed, for an arbitrary $\varepsilon > 0$ we select $b \in \ell_1$ and $c \in \ell_2$ such that $\|b\|_1 + \sqrt{t}\,\|c\|_2 \leqslant \kappa_a(\sqrt{t}\,) + \varepsilon$. Then by (1) (with the estimate for the constant) we have

$$\biggl\|\sum_{k=1}^{\infty} a_k r_k\biggr\|_t \leqslant \biggl\|\sum_{k=1}^{\infty} b_k r_k\biggr\|_t + \biggl\|\sum_{k=1}^{\infty} c_k r_k\biggr\|_t \leqslant \sum_{k=1}^{\infty} |b_k| + \sqrt{t}\,\biggl(\sum_{k=1}^{\infty} c_k^2\biggr)^{1/2} \leqslant \kappa_a(\sqrt{t}\,) + \varepsilon.$$

First, let us prove the left-hand inequality in the case $t \in \mathbb{N}$. By the definition of $Q(t)$, choose pairwise disjoint subsets $A_1, \dots, A_t$ of the set of positive integers such that

$$\|a\|_{Q(t)} \leqslant \frac{3}{2}\biggl\{\sum_{j=1}^{t}\biggl(\sum_{k\in A_j} a_k^2\biggr)^{1/2}\biggr\}. \tag{18}$$

We define $\xi_j = \sum_{k\in A_j} a_k r_k$ $(j = 1, \dots, t)$. Obviously,

$$\sum_{k=1}^{\infty} a_k r_k = \sum_{j=1}^{t} \xi_j.$$

Let us introduce the sets $B_j = \{t : \xi_j(t) \geqslant \|\xi_j\|_2/2\}$. Since $\sum_{j=1}^{t} \xi_j \geqslant \frac{1}{2}\sum_{j=1}^{t} \|\xi_j\|_2$ on the set $B = \bigcap_{j=1}^{t} B_j$, we have

$$\biggl\|\sum_{j=1}^{t} \xi_j\biggr\|_t \geqslant \frac{1}{2}\sum_{j=1}^{t} \|\xi_j\|_2 \lambda(B)^{1/t}. \tag{19}$$

In view of the independence and symmetry of the distribution of the functions ξ_j and also the Paley–Zygmund inequality in the case of the Rademacher functions (see, for example, [30]), we have

$$\lambda(B) = \prod_{j=1}^{t} \lambda(B_j) = 2^{-t}\prod_{j=1}^{t} \lambda\biggl\{t : \xi_j(t)^2 \geqslant \frac{\|\xi_j\|_2^2}{4}\biggr\} \geqslant 32^{-t}.$$

Therefore, using the inequalities (18), (19) and Proposition 4, we get that

$$\biggl\|\sum_{k=1}^{\infty} a_k r_k\biggr\|_t = \biggl\|\sum_{j=1}^{t} \xi_j\biggr\|_t \geqslant \frac{1}{64}\sum_{j=1}^{t} \|\xi_j\|_2 \geqslant \frac{1}{96}\,\|a\|_{Q(t)} \geqslant \frac{1}{96\sqrt{2}}\,\kappa_a(\sqrt{t}\,).$$

If $t \geqslant 1$ is arbitrary, then by the concavity of the $\mathscr{K}$-functional and the elementary inequality $\sqrt{t} \leqslant 2\sqrt{[t]}$, we have

$$\frac{1}{192\sqrt{2}}\,\kappa_a(\sqrt{t}\,) \leqslant \frac{1}{96\sqrt{2}}\,\kappa_a\bigl(\sqrt{[t]}\,\bigr) \leqslant \biggl\|\sum_{k=1}^{\infty} a_k r_k\biggr\|_{[t]} \leqslant \biggl\|\sum_{k=1}^{\infty} a_k r_k\biggr\|_t,$$

and Theorem 7 is proved.

Example 1. By (9) and (8) it is not difficult to show that, for example,

$$\Big\|\sum_{k=1}^{\infty}\frac{r_k}{k}\Big\|_t \asymp \ln(1+t) \quad \text{and} \quad \Big\|\sum_{k=1}^{n} r_k\Big\|_t \asymp \sqrt{n\min(n,t)} \quad (n\in\mathbb{N}).$$

The next relation, which essentially refines the Khintchine inequalities (1), follows from Theorem 7 and Proposition 3.

Corollary 1. *There exists a universal constant $\gamma' > 0$ such that for any $p \geqslant 1$ and any $a = (a_k)_{k=1}^{\infty} \in \ell_2$*

$$\gamma'\bigg\{\sum_{i\leqslant p} a_i^* + \sqrt{p}\,\bigg(\sum_{i>p}^{\infty}(a_i^*)^2\bigg)^{1/2}\bigg\} \leqslant \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_p \leqslant \sum_{i\leqslant p} a_i^* + \sqrt{p}\,\bigg(\sum_{i>p}^{\infty}(a_i^*)^2\bigg)^{1/2}, \tag{20}$$

where $(a_i^*)_{i=1}^{\infty}$ is the non-increasing rearrangement of the sequence $(|a_n|)_{n=1}^{\infty}$.

3.3. Latala theorem. As it turns out, an analogue of the Hitczenko inequalities holds for general systems of independent functions, although in this case the $\mathscr{K}$-functional for the couple (ℓ_1, ℓ_2) has to be replaced by a special Orlicz functional. We need some definitions for the precise formulation of this interesting general result proved by Latala [31] in 1997.

If $p > 0$, then $\phi_p(t) = |1+t|^p$ $(t \in \mathbb{R})$, and $\Phi_p(f) = \mathsf{E}\,\phi_p(f)$ for any r.v. f. In addition, if $\{f_k\}_{k=1}^{\infty}$ is a sequence of non-negative (or symmetrically distributed) r.v.'s, then

$$\big|\|(f_k)\|\big|_p := \inf\bigg\{t>0 : \sum_{k=1}^{\infty}\ln\Phi_p\Big(\frac{f_k}{t}\Big) \leqslant p\bigg\}.$$

Theorem 8. (a) *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of independent non-negative r.v.'s. Then for any $n \in \mathbb{N}$*

$$\frac{e-1}{2e^2}\big|\|(f_k)\|\big|_p \leqslant \Big\|\sum_{k=1}^{n} f_k\Big\|_p \leqslant e\big|\|(f_k)\|\big|_p \quad \textit{for } p \geqslant 1, \tag{21}$$

and

$$\frac{(e^p-1)^{1/p}}{2e^2}\big|\|(f_k)\|\big|_p \leqslant \Big\|\sum_{k=1}^{n} f_k\Big\|_p \leqslant e\big|\|(f_k)\|\big|_p \quad \textit{for } 0<p\leqslant 1. \tag{22}$$

(b) *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of independent symmetrically distributed r.v.'s. Then for $p \geqslant 2$ and any $n \in \mathbb{N}$*

$$\frac{e-1}{2e^2}\big|\|(f_k)\|\big|_p \leqslant \Big\|\sum_{k=1}^{n} f_k\Big\|_p \leqslant e\big|\|(f_k)\|\big|_p. \tag{23}$$

We prove only the first two inequalities of this theorem, relating to non-negative r.v.'s. The proof of (23) is similar, but slightly more difficult (see [31]). We begin with the following simple lemmas.

Lemma 3. *If r. v's $f_1, \dots, f_n$ are non-negative and independent, then*

$$\Phi_p(f_1+\dots+f_n) \leqslant \Phi_p(f_1)\cdots\Phi_p(f_n).$$

Proof. Clearly, it is enough to prove the statement for $n = 2$. But in this case it follows from the inequality $\phi_p(u+v) \leqslant \phi_p(u)\phi_p(v)$ $(u, v \geqslant 0)$.

Lemma 4. *If r.v.'s f and g are non-negative and independent, then*

$$\Phi_p\bigl(2f + \Phi_p(f)^{2/p} g\bigr) \geqslant \Phi_p(f)\Phi_p(g).$$

Proof. First of all, it is easy to show (raising to the power $1/p$) that $\phi_p(uv) \geqslant u^{p/2}\phi_p(v)$ if $u \geqslant 1$ and $v \geqslant 1$. Therefore, taking into account that f is non-negative, we get that

$$\mathsf{E}\,\phi_p\bigl(2f + \Phi_p(f)^{2/p} g\bigr)\chi_{\{g\geqslant 1\}} \geqslant \mathsf{E}\,\phi_p\bigl(\Phi_p(f)^{2/p} g\bigr)\chi_{\{g\geqslant 1\}} \geqslant \Phi_p(f)\mathsf{E}\,\phi_p(g)\chi_{\{g\geqslant 1\}}.$$

At the same time, since $\Phi_p(f) \geqslant 1$, in the case $0 \leqslant u < 1$ and $v \geqslant 0$ we have

$$\phi_p\bigl(2v + \Phi_p(f)^{2/p} u\bigr) \geqslant \phi_p\bigl((1+u)v + u\bigr) = \phi_p(u)\phi_p(v).$$

Hence,

$$\mathsf{E}\,\phi_p\bigl(2f + \Phi_p(f)^{2/p} g\bigr)\chi_{\{g<1\}} \geqslant \Phi_p(f)\mathsf{E}\,\phi_p(g)\chi_{\{g<1\}},$$

and the assertion follows from the relations above.

Finally, we prove a statement which is in a certain sense the converse of the assertion of Lemma 3.

Lemma 5. *Let $f_1, \dots, f_n$ be non-negative independent r.v.'s satisfying the condition $\Phi_p(f_1)\cdots\Phi_p(f_n) \leqslant e^p$. Then*

$$\Phi_p\bigl(2e^2(f_1 + \dots + f_n)\bigr) \geqslant \Phi_p(f_1)\cdots\Phi_p(f_n).$$

Proof. Let $g_k := 2\bigl(\Phi_p(f_1)\cdots\Phi_p(f_k)\bigr)^{2/p}(f_1 + \dots + f_k)$. The functional Φ_p is monotonically increasing, so in view of the assumption of the lemma it is enough to show that for every $k = 1, 2, \dots$

$$\Phi_p(g_k) \geqslant \Phi_p(f_1)\cdots\Phi_p(f_k).$$

We prove this by induction. The validity of the inequality for $k = 1$ follows from the obvious estimate $\Phi_p(2\Phi_p(f_1)^{2/p} f_1) \geqslant \Phi_p(f_1)$. Assume that it holds for some k. Then again in view of the monotonicity of Φ_p and the previous lemma, we have

$$\begin{aligned}
\Phi_p(g_{k+1}) &\geqslant \Phi_p\bigl\{2\bigl(\Phi_p(f_1)\cdots\Phi_p(f_{k+1})\bigr)^{2/p}(f_1 + \dots + f_k) \\
&\qquad\qquad + 2\bigl(\Phi_p(f_1)\cdots\Phi_p(f_k)\bigr)^{2/p} f_{k+1}\bigr\} \\
&\geqslant \Phi_p\bigl(2f_{k+1} + \Phi_p(f_{k+1})^{2/p} g_k\bigr) \\
&\geqslant \Phi_p(f_{k+1})\Phi_p(g_k) \geqslant \Phi_p(f_1)\cdots\Phi_p(f_{k+1}).
\end{aligned}$$

The proof is complete.

Proof of Theorem 8. Suppose that

$$\sum_{k=1}^{\infty} \ln \Phi_p(f_k/t) = p,$$

which is equivalent to $\Phi_p(f_1/t)\cdots\Phi_p(f_n/t)=e^p$. Then on the one hand, it follows from Lemma 3 that

$$\Phi_p\biggl(\frac{f_1+\cdots+f_n}{t}\biggr)\leqslant e^p.$$

Since $\phi_p(u)\geqslant u^p$ if $u\geqslant 0$, we have $\Phi_p(h)\geqslant \|h\|_p^p$ for an arbitrary non-negative r.v. h. Hence, from the previous inequality it follows that $\|f_1+\cdots+f_n\|_p\leqslant et$, and the upper estimates in (21) and (22) are proved.

On the other hand, Lemma 5 implies that

$$\Phi_p\biggl(\frac{2e^2(f_1+\cdots+f_n)}{t}\biggr)\geqslant e^p. \tag{24}$$

Moreover, if $p\geqslant 1$, then $\Phi_p(h)\leqslant(1+\|h\|_p)^p$, but if $p\leqslant 1$, then $\phi_p(u)\leqslant 1+|u|^p$, and this means that $\Phi_p(h)\leqslant 1+\|h\|_p^p$. The lower estimates in (21) and (22) follow from the last relations and (24). This proves the theorem in the case of non-negative r.v.'s.

Remark 1. By standard randomization with respect to an independent sequence of Rademacher functions, it is not difficult to show (for details see [31], Remark 2) that the inequality (23) also holds for any sequence $\{f_k\}_{k=1}^\infty$ of mean-zero independent r.v.'s.

Remark 2. The inequalities (21), (22), and (23) may be regarded as a summary of the preceding investigations concerned with estimates of moments of sums of independent r.v.'s. First of all, in the case $f_k=a_kr_k$, where the functions r_k are the Rademacher functions on $[0,1]$ and $a_k\in\mathbb{R}$ $(k=1,2,\dots)$, (23) results in the inequalities (20) (albeit, with worse constants).

Now we consider a more general case. Let f_k be a symmetrically distributed r.v. whose tail distribution is logarithmically concave, that is, $\mathsf{P}\{|f_k|\geqslant\tau\}=e^{-N_k(\tau)}$ $(\tau\geqslant 0)$, where the function $N_k\colon[0,\infty)\to[0,\infty]$ is convex. Define $N_k^*(t)=\sup\{ts-N_k(s):s>0\}$. Then from (23) (see details in [31]) it is not difficult to obtain the following generalization of the main result of the paper [32]:

$$\biggl\|\sum_{k=1}^\infty a_kf_k\biggr\|_p\asymp\inf\biggl\{t>0:\sum_{i\leqslant p}N_i^*\biggl(\frac{pa_i^*}{t}\biggr)\leqslant p\biggr\}+\sqrt{p}\,\biggl(\sum_{i>p}(a_i^*)^2\biggr)^{1/2},$$

where $p\geqslant 2$ and the sequence $\{f_k\}$ of independent and symmetrically distributed r.v.'s is normalized in such a way that $\inf\{\tau:\mathsf{P}\{|f_k|\geqslant\tau\}\leqslant e^{-1}\}=1$.

In conclusion we consider a situation which is opposite in a certain sense. Let f_k be a symmetrically distributed r.v. whose tail distribution is logarithmically convex, that is, $\mathsf{P}\{|f_k|\geqslant\tau\}=e^{-M_k(\tau)}$ $(\tau\geqslant 0)$, where the function $M_k\colon[0,\infty)\to[0,\infty)$ is concave. Then by (23) (see [31]) we obtain

$$\biggl\|\sum_{k=1}^\infty f_k\biggr\|_p\asymp\biggl(\sum_{k=1}^\infty\mathsf{E}\,|f_k|^p\biggr)^{1/p}+\sqrt{p}\,\biggl(\sum_{k=1}^\infty\mathsf{E}\,f_k^2\biggr)^{1/2}\qquad(p\geqslant 2),$$

which was proved earlier in the paper [33].

4. Kruglov operator in symmetric spaces

4.1. Rosenthal inequalities and disjoint sums. We return to the Rosenthal inequalities (6) and (7). The expressions with which the norm $\big\| \sum_{k=1}^n f_k \big\|_p$ is compared in these inequalities can be represented somewhat differently. For a given sequence $\{f_k\}_{k=1}^\infty$ of functions on $[0,1]$ let $\{\mathbf{f}_k\}_{k=1}^\infty$ be an arbitrary sequence of disjointly supported functions on $[0,\infty)$ such that for any $k=1,2,\dots$ the functions $\mathbf{f}_k$ and f_k are identically distributed. For example, we can set $\mathbf{f}_k(t) = f_k(t-k+1)\chi_{[k-1,k)}(t)$ $(t>0)$. Then the function $\mathbf{f} = \sum_{k=1}^\infty \mathbf{f}_k$, which we call a *disjoint sum* of the functions f_k $(k=1,2,\dots)$, obviously has the following property:

$$\lambda\{t>0 : \mathbf{f}(t) > \tau\} = \sum_{k=1}^n \lambda\{t \in [0,1] : f_k(t) > \tau\} \qquad (\tau > 0), \tag{25}$$

where λ is the Lebesgue measure. Thus, the inequalities (6) and (7) mean that the norm $\big\| \sum_{k=1}^n f_k \big\|_p$ is equivalent to the norm $\|\mathbf{f}\|_{L_p \cap L_1[0,\infty)}$ if the functions f_k are independent and non-negative, and to the norm $\|\mathbf{f}\|_{L_p \cap L_2[0,\infty)}$ if these functions are independent and have mean zero, respectively. Here, as usual,

$$\|g\|_{L_p \cap L_q[0,\infty)} = \max\bigl\{\|g\|_{L_p[0,\infty)}, \|g\|_{L_q[0,\infty)}\bigr\}.$$

In other words, with constants depending only on p, the sequence $\{f_k\}_{k=1}^\infty$ in $L_p[0,1]$ is equivalent to the sequence $\{\mathbf{f}_k\}_{k=1}^\infty$ either in the space $L_p \cap L_1[0,\infty)$ or in the space $L_p \cap L_2[0,\infty)$, depending on the conditions satisfied by the sequence $\{f_k\}_{k=1}^\infty$.

This reformulation of the Rosenthal inequalities leads to the following natural question: What function spaces, besides the L_p-spaces, satisfy similar inequalities? A number of papers published in the 1980s were devoted to the problem of comparing sums of independent and disjoint functions in various spaces and to the application of these relations to the study of the geometry of these spaces. In particular, in 1988 Carothers and Dilworth [34] extended the inequalities (7) to the Lorentz spaces $L_{p,q}$ $(1 \leqslant p < \infty,\ 0 < q \leqslant \infty)$ (see their definition in § 2). In view of the equality (25), symmetric spaces are the most probable candidates for satisfying generalized Rosenthal inequalities.

Further, using the Hoffman-Jørgensen inequality [36], Johnson and Schechtman [35] extended both the inequalities (6) and (7) to the case of general symmetric spaces (we note in passing that an interesting relation similar to (6) was proved earlier for the spaces $L_{p,\infty}$ by Marcus and Pisier [37]). In order to state their theorem we introduce the following useful construction (see [38] or [17], 2.f). If X is a symmetric space on $[0,1]$ and $1 \leqslant p \leqslant \infty$, then the set Z_X^p consists of all measurable functions f on $(0,\infty)$ for which

$$\|f\|_{Z_X^p} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_{L_p[1,\infty)} < \infty.$$

The functional $\|\cdot\|_{Z_X^p}$ is only a quasi-norm (the triangle inequality holds with some constant). Nevertheless, for any symmetric space X this functional is equivalent to an appropriate symmetric norm. Indeed, since $X \subset L_1[0,1]$, the well-known relation

$$\|f\|_{(L_1+L_p)(0,\infty)} \asymp \int_0^1 f^*(x)\,dx + \left(\int_1^\infty (f^*(x))^p\,dx\right)^{1/p}, \qquad f \in (L_1+L_p)(0,\infty),$$

(see, for example, [28] or [21], Exercise 5.7.3, and also, in the case $p=2$, the inequalities (10) with $t=1$) implies that the functional $\|f\|_{Z_X^p}$ is equivalent to the norm

$$\|f\|'_{Z_X^p} := \|f^*\chi_{[0,1]}\|_X + \|f\|_{(L_1+L_p)(0,\infty)}.$$

Thus, Z_X^p is a symmetric space on the semi-axis $[0,\infty)$.

Theorem 9 (Johnson–Schechtman, [35]). *Let X be an arbitrary symmetric space on $[0,1]$. Then there exists a $C>0$ such that for any sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent functions with $\int_0^1 f_k(t)\,dt = 0$ $(k=1,2,\dots)$ and any sequence $\{g_k\}_{k=1}^\infty \subset X$ of non-negative independent functions*

$$\|\mathbf{f}\|_{Z_X^2} \leqslant C\biggl\|\sum_{k=1}^\infty f_k\biggr\|_X \quad \text{and} \quad \|\mathbf{g}\|_{Z_X^1} \leqslant C\biggl\|\sum_{k=1}^\infty g_k\biggr\|_X, \tag{26}$$

where $\mathbf{f}$ and $\mathbf{g}$ are disjoint sums for the respective sequences $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$.

If in addition $X \supset L_p$ for some $p<\infty$, then there exists a $C_1 = C_1(p) > 0$ such that for arbitrary sequences with the same properties the opposite inequalities hold:

$$\biggl\|\sum_{k=1}^\infty f_k\biggr\|_X \leqslant C_1\|\mathbf{f}\|_{Z_X^2} \tag{27}$$

and

$$\biggl\|\sum_{k=1}^\infty g_k\biggr\|_X \leqslant C_1\|\mathbf{g}\|_{Z_X^1}. \tag{28}$$

As we will see, the condition of 'remoteness' of the space X from the space L_∞ in the second part of the preceding theorem ($X\supset L_p$ for some $p<\infty$) is not necessary for the inequalities (27) and (28) to hold. They are also valid for some symmetric spaces which are on the 'other side' with respect to the L_p-scale ($p<\infty$). This has been established using another approach based on the use of the so-called Kruglov property.

4.2. Kruglov property and Kruglov operator: preliminaries. The Kruglov property was introduced by Braverman [39] while comparing sums of independent and disjoint functions in symmetric spaces. The terminology stems from some related probabilistic constructions of Kruglov [40] used in the study of infinitely divisible distributions and, in particular, in the analysis of the classical Lévy–Khintchine formula.

Let f be a measurable function (a random variable) on $[0,1]$, and let $\pi(f)$ denote the r.v. $\sum_{i=1}^N f_i$, where the functions f_i are independent copies of f and N is an r.v. that has a Poisson distribution with parameter 1 and is independent with respect to the sequence $\{f_i\}$. In other words, the distribution of $\pi(f)$ is a mixture of the discrete probability distribution $p_m = 1/(em!)$, $m=0,1,2,\dots$, with the family of m-fold convolutions $\mathscr{F}_f^{*m}(x)$ of the distribution function $\mathscr{F}_f(x)$ of f. In probability theory such a distribution is usually called a *compound Poisson* *distribution* ([41], Chap. 12). Direct calculations show that the distribution function and the characteristic function of the r.v. $\pi(f)$ are given by

$$\mathscr{F}_{\pi(f)}(x) = \frac{1}{e}\biggl(\chi_{(0,\infty)}(x) + \mathscr{F}_f(x) + \sum_{l=2}^{\infty} \frac{1}{l!}\,\mathscr{F}_f^{*l}(x)\biggr) \qquad (x \in \mathbb{R}) \tag{29}$$

and

$$\theta_{\pi(f)}(t) = \exp\biggl(\int_{-\infty}^{\infty} (e^{itx} - 1)\, d\mathscr{F}_f(x)\biggr) = \exp\bigl(\theta_f(t) - 1\bigr) \qquad (t \in \mathbb{R}), \tag{30}$$

where $\mathscr{F}_f$ and θ_f are the distribution function and the characteristic function of the r.v. f, respectively.

The following statement was proved in [40].

Theorem 10 (Kruglov). *Suppose that $\Phi(x)$ is a continuous function on $\mathbb{R}$ with $\Phi(x) \geqslant 0$ and $\Phi(0) = 0$, and suppose that it satisfies one of the following conditions*:

$$\text{(a) } \Phi(x+y) \leqslant B\Phi(x)\Phi(y) \quad \text{or} \quad \text{(b) } \Phi(x+y) \leqslant B\bigl(\Phi(x) + \Phi(y)\bigr)$$

for some $B > 0$ and all $x, y \in \mathbb{R}$. Then for an arbitrary r.v. f the relations $\mathsf{E}\,\Phi(f) < \infty$ and $\mathsf{E}\,\Phi(\pi(f)) < \infty$ are equivalent.

Proof. Assume, for example, that the condition (a) holds. Let

$$\mathsf{E}\Phi(f) = \int_0^1 \Phi(f(x))\, dx := A < \infty,$$

and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of independent functions which are identically distributed with f. Then by assumption, for any $n \in \mathbb{N}$

$$\mathsf{E}\Phi\biggl(\sum_{k=1}^{n} f_k\biggr) \leqslant B\mathsf{E}\biggl(\Phi\biggl(\sum_{k=1}^{n-1} f_k\biggr)\Phi(f_n)\biggr) = BA\mathsf{E}\Phi\biggl(\sum_{k=1}^{n-1} f_k\biggr) \leqslant \cdots \leqslant B^{n-1}A^n.$$

By definition, $\pi(f)$ takes values equal to $\sum_{k=1}^{n} f_k$ on some disjoint sets E_n with $\lambda(E_n) = 1/(en!)$ $(n = 1, 2, \dots)$, and the collection $\{f_1, \dots, f_n, \chi_{E_n}\}$ consists of independent functions. Therefore,

$$\begin{aligned}\mathsf{E}\Phi(\pi(f)) &= \sum_{n=1}^{\infty} \mathsf{E}\Phi\biggl(\sum_{k=1}^{n} f_k \chi_{E_n}\biggr) = \sum_{n=1}^{\infty} \mathsf{E}\Phi\biggl(\sum_{k=1}^{n} f_k\biggr)\lambda(E_n) \\ &\leqslant \sum_{n=1}^{\infty} B^{n-1}A^n \lambda(E_n) = \frac{1}{e}\sum_{n=1}^{\infty} \frac{B^{n-1}A^n}{n!} = \frac{e^{BA} - 1}{eB} < \infty.\end{aligned}$$

Conversely, from the first equality in the preceding relation it follows that

$$\mathsf{E}\Phi(\pi(f)) \geqslant \mathsf{E}\Phi(f_1\chi_{E_1}) = e^{-1}\mathsf{E}\Phi(f_1),$$

whence $\mathsf{E}\Phi(f) \leqslant e\mathsf{E}\Phi(\pi(f))$, because f_1 and f are identically distributed. The proof is complete.

In particular, if in addition Φ is an even convex function, then we can define the Orlicz space L_Φ on $[0,1]$ (see § 2), and then the last theorem implies that the r.v.'s f and $\pi(f)$ either both belong or both do not belong to the space L_Φ. In this connection, the following natural question arises: For which symmetric spaces does an analogous statement hold?

Definition 3. A symmetric space X on $[0,1]$ is said to have the *Kruglov property* ($X \in \mathbb{K}$) if $\pi(f) \in X$ for $f \in X$.

Note that the converse implication $\pi(f) \in X \implies f \in X$ always holds. Indeed, it follows from the definition that $\lambda\{|\pi(f)| \geqslant \tau\} \geqslant e^{-1}\lambda\{|f| \geqslant \tau\}$ $(\tau > 0)$, so by Proposition 1 we get that $\|f\|_X \leqslant e\|\pi(f)\|_X$.

Simplifying the situation, it may be said that the Kruglov property is possessed by spaces sufficiently 'remote' from the space L_∞ (for a detailed discussion of this see § 4.3). First, by Theorem 10 with $\Phi(x) = x^p$ we see that $L_p \in \mathbb{K}$ for $1 \leqslant p < \infty$. Furthermore, a similar property holds also for all symmetric spaces with the Fatou property (see § 2) containing L_p for some $p < \infty$ ([39], Theorem 1.2) (in particular, if the lower Boyd index of X is non-trivial, that is, $\alpha_X > 0$). At the same time, the latter assumption is not necessary; for example, the exponential Orlicz space $\exp L_p$ $(p > 0)$ has the Kruglov property if and only if $0 < p \leqslant 1$. The last assertion is a consequence of Theorem 10 and arguments from the beginning of § 2.4 in the monograph [39] (more precise results will be given in § 4.3.1).

It turns out that the Kruglov property is an extraordinarily useful tool in the study of various geometric properties of Banach spaces of measurable functions, in particular, it plays a central role in [39]. The reason for this is that, under a certain side condition on sequences of independent functions, the property guarantees that inequalities of the form (27) and (28) are valid.

Theorem 11 ([39], Lemma 1.4). *Let X be a symmetric space in $\mathbb{K}$. Then for an arbitrary sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent functions such that*

$$\sum_{k=1}^{\infty} \lambda\{f_k \neq 0\} \leqslant 1 \tag{31}$$

the inequality

$$\biggl\|\sum_{k=1}^{\infty} f_k\biggr\|_X \leqslant C\|\mathbf{f}\|_X \tag{32}$$

holds for some $C > 0$ depending only on X.

Remark 3. In view of (31), $\{\mathbf{f} \neq 0\} \subset [0,1]$. Therefore, in comparison with the inequalities (27) and (28), the right-hand side of the generalized Rosenthal inequality simplifies considerably, and for a proof of the inequality it is sufficient to consider only non-negative functions.

It is important to point out that for symmetric spaces with the Fatou property the converse statement also holds: if for an arbitrary sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent functions satisfying (31) the inequality (32) holds, then $X \in \mathbb{K}$ ([39], Lemma 1.6). These results suggest how to obtain necessary and sufficient conditions for a symmetric space X to satisfy the inequalities (27) and (28). The main problem arising here is to extend Theorem 11 to general sequences of independent functions (non-negative or with mean zero), that is, to get rid of the condition (31). This problem was resolved by the present authors in their papers [42] and [43] using an operator approach. On the space of all almost everywhere finite Lebesgue-measurable functions on $[0,1]$, we introduced a positive linear operator $\mathscr{K}$ closely linked with the Kruglov property (and therefore also called the Kruglov operator). First, this notion made it possible to use many advantages of the operator setting, for example, the interpolation theory of operators. Second, it became possible to consider a more general situation in which the norms of the left-hand side and the right-hand side in (27) and (28) are taken in different spaces. Finally, it was possible in a natural and simple way to resolve the question of sharpness of the constants in the Rosenthal-type inequalities [44]. In the conclusion of the present subsection we give a detailed exposition of this construction, and using it we obtain in the next chapter definitive results on comparison of sums of independent and disjoint functions in symmetric spaces.

We first define a subsidiary operator $\mathscr{K}_1$ with values in the space $S(\Omega,\mathscr{P})$, where $(\Omega,\mathscr{P}) := \prod_{k=0}^{\infty}([0,1],\lambda_k)$ (λ_k is the Lebesgue measure on $[0,1]$, $k=0,1,\dots$). Let $\{E_n\}$ be a sequence of pairwise disjoint subsets of $[0,1]$ with $\lambda(E_n)=1/(en!)$ ($n\in\mathbb{N}$). For an arbitrary $f\in S([0,1],\lambda)$ we set

$$\mathscr{K}_1 f(\omega_0,\omega_1,\omega_2,\dots) := \sum_{n=1}^{\infty}\sum_{k=1}^{n} f(\omega_k)\chi_{E_n}(\omega_0).$$

It is well known that there exists a measure-preserving transformation $\delta\colon(\Omega,\mathscr{P})\to([0,1],\lambda)$. For every $g\in S(\Omega,\mathscr{P})$ define $R(g)(x) := g(\delta^{-1}x)$, $x\in[0,1]$. It is easy to see that

$$\lambda\{x\in[0,1] : R(g)(x)<t\} = \mathscr{P}\{\omega\in\Omega : g(\omega)<t\} \qquad (t\in\mathbb{R}).$$

Hence $\mathscr{K} := R\mathscr{K}_1$ is a positive linear operator acting in the space $S([0,1],\lambda)$, and in addition, for any measurable function f on $[0,1]$ the functions $\mathscr{K}f$ and $\mathscr{K}_1 f$ are identically distributed.

Since we are considering symmetric spaces, the only matter of concern here is the distribution of the function $\mathscr{K}f$, so we can look at the definition of the operator $\mathscr{K}$ from a somewhat different (and frequently more convenient) point of view. If $f\in S([0,1],\lambda)$ and $\{f_{n,k}\}_{k=1}^{n}$ ($n\in\mathbb{N}$) is a sequence of collections of measurable functions on $[0,1]$ such that for every $n\in\mathbb{N}$

(i) $f_{n,1},\dots,f_{n,n},\ \chi_{E_n}$ is a sequence of independent r.v.'s,

(ii) $\mathscr{F}_{f_{n,k}} = \mathscr{F}_f$, $k=1,\dots,n$,

then we set

$$\mathscr{K}'f(x) := \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{n,k}(x)\chi_{E_n}(x), \qquad x\in[0,1]. \tag{33}$$

One can check that the r.v.'s $\mathscr{K}f$ and $\mathscr{K}'f$ are identically distributed. Therefore, the operator $\mathscr{K}$ can in this sense be identified with $\mathscr{K}'$.

The first simple result involving the operator $\mathscr{K}$ is an immediate consequence of the closed graph theorem.

Lemma 6. *If X and Y are symmetric spaces on $[0,1]$ such that $\mathscr{K}f \in Y$ for every $f \in X$, then $\|\mathscr{K}f\|_Y \leqslant C\|f\|_X$ for some $C > 0$.*

Since $\mathscr{F}_{\mathscr{K}'f}(x) = \mathscr{F}_{\pi(f)}(x)$ $(x \in \mathbb{R})$ for an arbitrary $f \in S(0,1)$, it follows from (30) that

$$\theta_{\mathscr{K}f}(t) = \theta_{\mathscr{K}'f}(t) = \theta_{\pi(f)}(t) = \exp\bigl(\theta_f(t) - 1\bigr) \qquad (t \in \mathbb{R}). \tag{34}$$

The last remark and the definition of the Kruglov property yield the following statement, which will often be used below.

Lemma 7. *If X is an arbitrary symmetric space on $[0,1]$, then $X \in \mathbb{K}$ if and only if the operator $\mathscr{K}$ acts boundedly in X.*

4.3. Boundedness of the Kruglov operator on symmetric spaces. In the next section it will be shown that the boundedness of the Kruglov operator on a symmetric space guarantees that the generalized Rosenthal inequalities hold in this space (to a certain extent, this is already clear from Theorem 11 and Lemma 7). First of all, we shall refine the results in [39] on boundedness of the operator $\mathscr{K}$ on exponential Orlicz spaces. In addition, necessary and sufficient conditions for its boundedness on Lorentz spaces will be found. These conditions have interesting consequences. In particular, we shall show that the boundedness of the Kruglov operator is a very delicate property, which cannot be characterized in terms of embeddings as done, for example, for the Khintchine inequalities in symmetric spaces (see the beginning of § 7, or the more detailed exposition in [66]). A seemingly natural 'Rubicon', the exponential Orlicz space $\exp L_1$, in fact, fails to be such a space: the Kruglov operator $\mathscr{K}$ can be bounded on a symmetric space contained in $\exp L_1$ and yet be unbounded on a symmetric space containing $\exp L_1$.

4.3.1. *Exponential Orlicz spaces.* The action of the Kruglov operator in the spaces $\exp L_p$ (see the definition in § 2) can be studied sufficiently completely. The results obtained here will be repeatedly used below. Recall that $\exp L_\infty := L_\infty$.

If $0 < p \leqslant 1$, then in view of Theorem 10, $\exp L_p \in \mathbb{K}$. Hence, by Lemma 7 the Kruglov operator is bounded on $\exp L_p$ in this case. At the same time if $p > 1$, then $\exp(L_p)$ does not have the Kruglov property (see the arguments at the beginning of § 2.4 of the monograph [39]), and therefore the operator $\mathscr{K}$ does not act in this space. Here we shall make the last statement precise by determining the least symmetric space containing the image $\mathscr{K}(\exp L_p)$ for $p > 1$. For that we need one more family of Orlicz functions on $[0,\infty)$:

$$M_p(t) := e^{t\ln^{1/p}(e+t)} - 1 \qquad (p > 0).$$

Further, we shall use more than once the fact that for every $p > 0$ the space L_{M_p} (L_{N_p}, respectively) coincides with the Marcinkiewicz space M_{ψ_p} (M_{φ_p}, respectively), where

$$\psi_p(t) := \frac{t\ln(e/t)}{\ln^{1/p}(\ln(e^e/t))} \qquad (\varphi_p(t) := t\ln^{1/p}(e/t), \quad \text{respectively}).$$

To prove this, we need the following equivalent expressions for the norms of the Marcinkiewicz spaces M_{ψ_p} and M_{φ_p} ([16], Theorem 2.5.3):

$$\begin{aligned} \|x\|_{M_{\psi_p}} &\asymp \sup_{t\in(0,1)} \frac{t}{\psi_p(t)}\, x^*(t), \qquad (x \in M_{\psi_p}), \\ \|x\|_{M_{\varphi_p}} &\asymp \sup_{t\in(0,1)} \frac{t}{\varphi_p(t)}\, x^*(t) \qquad (x \in M_{\varphi_p}). \end{aligned} \tag{35}$$

Lemma 8. *The equalities $L_{M_p} = M_{\psi_p}$ and $(\exp L_p =)\ L_{N_p} = M_{\varphi_p}$ (with equivalence of norms) hold for every $p > 0$.*

Proof. We shall prove only the first equality, since the proof of the second one is similar (and simpler). Fix $p > 0$. It is sufficient to prove that the fundamental functions $\phi_{L_{M_p}}$ and $\phi_{M_{\psi_p}}$ are equivalent and that the function $\bar f_p(t) := \psi_p(t)/t$ belongs to the Orlicz space L_{M_p} (see (35)). Since (see [20] and [16], Chap. 2, § 5)

$$\phi_{L_{M_p}}(t) = \frac{1}{M_p^{-1}(1/t)} \quad\text{and}\quad \phi_{M_{\psi_p}}(t) = \frac{\ln^{1/p}(\ln(e^e/t))}{\ln(e/t)} \ \left(= \frac{1}{\bar f_p(t)}\right) \quad (0 < t \leqslant 1),$$

to prove the first equality it suffices to check that the functions $M_p^{-1}(1/t)$ and $\bar f_p(t)$ are equivalent in a neighbourhood of 0. It is easy to see that for every $c > 0$

$$\lim_{t\to 0} \frac{\ln^{1/p}\bigl(e + c\ln(e/t)/\ln^{1/p}(\ln(e^e/t))\bigr)}{\ln^{1/p}(\ln(e^e/t))} = 1.$$

Therefore, we can select a $\delta > 0$ such that for all $t \in (0,\delta)$

$$M_p\biggl(\frac{1}{2}\,\bar f_p(t)\biggr) \leqslant e^{2\ln(e/t)/3} \leqslant \frac{1}{t} \leqslant e^{3\ln(e/t)/2} \leqslant M_p(2\bar f_p(t)). \tag{36}$$

Hence, the functions $M_p^{-1}(1/t)$ and $\bar f_p(t)$ are equivalent on $(0,\delta)$. Finally, from the first inequality on the left in (36) it follows that $\bar f_p \in L_{M_p}$, which completes the proof.

The following statement shows the importance of the space L_{M_1} in the study of sums of uniformly bounded independent r.v.'s (in this connection, see Corollary 3.5.2 in [45]).

Theorem 12. *The space L_{M_1} is minimal among symmetric spaces Y such that the Kruglov operator $\mathscr{K}$ acts boundedly from L_∞ to Y.*

Proof. Let $g := \mathscr{K}\chi_{[0,1]}$. By the definition of $\mathscr{K}$ (also see (33)),

$$g^*(t) = k \qquad (t_k < t < t_{k-1}), \quad \text{where } t_k := \frac{1}{e}\sum_{i=k}^{\infty}\frac{1}{i!}\ (k \in \mathbb{N}) \text{ and } t_0 = 1.$$

By Lemma 8, the space L_{M_1} coincides with the Marcinkiewicz space M_{ψ_1}, whose norm satisfies (35). Therefore, it is enough to prove only that the functions $g^*(t)$ and $\bar f_1(t) = \ln(e/t)/\ln(\ln(e^e/t))$ are equivalent in a neighbourhood of zero. Since the function $\bar{f}_1$ decreases on $(0,1)$ and $g^* \equiv k$ on every interval (t_{k-1}, t_k) $(k \in \mathbb{N})$, it is sufficient to show that

$$\frac{1}{2} \leqslant \lim_{k\to\infty} \frac{\bar{f}_1(t_{k-1})}{k} \leqslant \lim_{k\to\infty} \frac{\bar{f}_1(t_k)}{k} \leqslant 1.$$

First, for every $k \in \mathbb{N}$

$$\begin{aligned} k!t_k &= \frac{1}{e}\biggl(1 + \sum_{i=1}^{\infty} \frac{1}{(k+1)(k+2)\cdots(k+i)}\biggr) \\ &\leqslant \frac{1}{e}\biggl(1 + \frac{1}{k+1} + \sum_{i=1}^{\infty} \frac{1}{(k+i)(k+i+1)}\biggr) \leqslant \frac{1}{e}\biggl(1+\frac{2}{k+1}\biggr) \leqslant \frac{2}{e}\,, \end{aligned}$$

whence $1/(ek!) \leqslant t_k \leqslant 2/(ek!)$ for all $k \in \mathbb{N}$. By the Stirling formula, $k! \asymp \sqrt{2\pi k}\, k^k e^{-k}$, so the inequalities $k^{-k} \leqslant t_k < t_{k-1} \leqslant (k-1)^{-k/2}$ hold for sufficiently large k. As a result, elementary calculations show that

$$\frac{1}{2} = \lim_{k\to\infty} \frac{\bar{f}_1((k-1)^{-k/2})}{k} \leqslant \lim_{k\to\infty} \frac{\bar{f}_1(t_{k-1})}{k} \leqslant \lim_{k\to\infty} \frac{\bar{f}_1(t_k)}{k} \leqslant \lim_{k\to\infty} \frac{\bar{f}_1(k^{-k})}{k} = 1.$$

The proof is complete.

The result of Theorem 12 extends to all values of $p \in (1,\infty)$.

Theorem 13. *For an arbitrary $p \in (1,\infty)$ the space L_{M_q}, where $1/p + 1/q = 1$, is minimal among symmetric spaces Y such that the Kruglov operator $\mathscr{K}$ acts boundedly from* $\exp L_p$ *to Y.*

Proof. It follows from Theorems 10, 12 and Lemma 7 that $\mathscr{K}$ maps L_∞ boundedly to L_{M_1} and $\exp(L_1)(= L_{N_1})$ boundedly to itself. Hence, using the real method of interpolation (see § 2 or [21], Chap. 3) we infer that

$$\mathscr{K}\colon (L_\infty, L_{N_1})_{\theta,\infty} \to (L_{M_1}, L_{N_1})_{\theta,\infty} \qquad (0<\theta<1).$$

By Lemma 8 the Orlicz spaces L_{N_1} and L_{M_1} coincide with the Marcinkiewicz spaces M_{φ_1} and M_{ψ_1}, respectively. The space L_∞ has a similar property, namely, $L_\infty = M_{\mathrm{id}}$, where $\mathrm{id}(t) = t$ $(t \in [0,1])$. Therefore, by the well-known description of the spaces of the form $(M_\varphi, M_\psi)_{\theta,\infty}$ (see [46], Example 7.1.3) we have

$$(L_\infty, L_{N_1})_{\theta,\infty} = L_{N_{\theta^{-1}}}, \qquad (L_{M_1}, L_{N_1})_{\theta,\infty} = L_{M_{(1-\theta)^{-1}}}.$$

Setting $p = \theta^{-1}$, we conclude that $\mathscr{K}$ is continuous from $\exp L_p = L_{N_p}$ to L_{M_q}. So in view of (35), the proof will be finished as soon as we show that $h_0(t) \leqslant C(\mathscr{K} g_0)^*(t)$ for some constant $C > 0$ (possibly, depending on p) and for all sufficiently small $t > 0$, where $g_0(t) := \ln^{1/p}(e/t)$ and $h_0(t) := \ln(e/t)/\ln^{1/q}(\ln(e^e/t))$. Since $h_0 = h_0^*$, the last inequality holds if and only if for some $C > 0$ and for sufficiently large $\tau > 0$ we have

$$\lambda\biggl\{\mathscr{K} g_0 > \frac{\tau}{C}\biggr\} \geqslant \lambda\{h_0 > \tau\}.$$

Since $\lambda\{h_0 > \tau\} \leqslant e^{-\tau \ln^{1/q}\tau}$ and $\lambda\{\mathscr{K} g_0 > \tau/3\} \geqslant e^{-\tau\ln^{1/q}\tau}$ for sufficiently large τ (see details of the proof of Theorem 4.5 in [42]), the theorem is proved.

Finally, we cite a useful necessary condition for boundedness of the Kruglov operator. For $n \in \mathbb{N}$ let $\ln_n u := \ln \dots \ln u$ (n times) and let $\Phi_n(u) := \exp\bigl(u \ln_n(c_n + u)\bigr) - 1$, where the constant c_n satisfies $\ln_n c_n = 1$. By arguments similar to those in the proof of Lemma 8 it is not difficult to check that the Orlicz space L_{Φ_n} coincides with the Marcinkiewicz space M_{φ_n}, where

$$\varphi_n(u) := \frac{u \ln(e/u)}{\ln_{n+1}(e^{c_n}/u)}, \qquad u \in (0,1].$$

Arguing as in the proof of Theorem 13, we obtain the following result (see details in [42], Theorem 7.2).

Theorem 14. *If the operator $\mathscr{K}$ is bounded on a symmetric space X, then $L_{\Phi_n} \subset X$ for every $n \in \mathbb{N}$.*

As a consequence, we prove the following statement, which will be repeatedly used below.

Theorem 15. *Let $\mathscr{K}$ be bounded on a symmetric space X. Then $\mathscr{K}$ is bounded on the separable part $X_\circ$ of X (see § 2), and $\|\mathscr{K}\|_{X_\circ \to X_\circ} \leqslant \|\mathscr{K}\|_{X \to X}$.*

In particular, if X is separable and $\mathscr{K}$ is bounded on the second dual space $X^{\times\times}$, then $\mathscr{K}$ is bounded on X and $\|\mathscr{K}\|_{X \to X} \leqslant \|\mathscr{K}\|_{X^{\times\times} \to X^{\times\times}}$.

Proof. Since without loss of generality we can assume that $X \neq L_\infty$, it follows that $X_\circ$ is separable. Let $x \in X_\circ$. Then there exists a sequence $\{x_n\}_{n=1}^\infty \subset L_\infty$ such that $\|x_n - x\|_X \to 0$. We have

$$\|\mathscr{K}x_n - \mathscr{K}x\|_X = \|\mathscr{K}(x_n - x)\|_X \leqslant \|\mathscr{K}\|_{X \to X} \|x_n - x\|_X,$$

whence

$$\|\mathscr{K}x_n - \mathscr{K}x\|_X \to 0 \quad \text{as } n \to \infty. \tag{37}$$

On the one hand, by Theorem 12 it follows that $\mathscr{K}x_n \in L_{M_1}$ ($n = 1, 2, \dots$). On the other hand, it is easy to check that $L_{M_1} \subset (L_{\Phi_2})_\circ$. Since $L_{\Phi_2} \subset X$ by Theorem 14, we have $X_\circ \supset (L_{\Phi_2})_\circ \supset L_{M_1}$. Therefore, $\mathscr{K}x_n \in X_\circ$ ($n = 1, 2, \dots$). Hence $\mathscr{K}x \in X_\circ$ in view of (37). Since $X_\circ$ is a subspace of X, $\|\mathscr{K}\|_{X_\circ \to X_\circ} \leqslant \|\mathscr{K}\|_{X \to X}$.

The second statement of the theorem follows from the first one, taking into account the equalities $X = X_\circ = (X^{\times\times})_\circ$.

4.3.2. *Lorentz spaces.* The following theorem proved in [42] completely characterizes Lorentz spaces (see the definition in § 2) on which the Kruglov operator is bounded. Recall that Ψ is the set of all quasi-concave functions $\psi(t)$ on $[0,1]$ such that $\psi(0) = \psi(+0) = 0$ and $\psi(1) = 1$.

Theorem 16. *Let $\psi \in \Psi$. The operator $\mathscr{K}$ acts boundedly in the Lorentz space Λ_ψ if and only if for some constant $C > 0$*

$$\sum_{k=1}^\infty \psi\biggl(\frac{u^k}{k!}\biggr) \leqslant C\psi(u) \qquad (0 < u \leqslant 1). \tag{38}$$

Proof. Let $f := \chi_{(0,u]}$ $(u \in (0,1])$. Assume that the sequences $\{f_{n,k}\}_{k=1}^n$ $(n \in \mathbb{N})$ and $\{\chi_{E_k}\}_{k=1}^\infty$ satisfy conditions (i) and (ii) in the definition of the operator $\mathscr{K}'$ (see (33)). Then the function $f_n := \sum_{k=1}^n f_{n,k}$ has the binomial distribution with parameter u, that is,

$$\lambda\{f_n = k\} = \binom{n}{k} u^k (1-u)^{n-k}, \qquad k = 0,1,\dots,n, \quad n \in \mathbb{N},$$

where $\binom{n}{k} = n!/(k!(n-k)!)$. Hence $\mathscr{K} f(s) = \sum_{k=1}^\infty k\chi_{A_k}(s)$ and

$$\begin{aligned}\lambda(A_k) &= \sum_{n=k}^\infty \binom{n}{k} u^k (1-u)^{n-k} \lambda(E_n) = \frac{1}{e}\sum_{n=k}^\infty \frac{n!}{k!\,(n-k)!}\, u^k (1-u)^{n-k} \frac{1}{n!} \\ &= \frac{1}{e}\, \frac{u^k}{(1-u)^k k!} \sum_{n=k}^\infty \frac{(1-u)^n}{(n-k)!} = \frac{1}{e}\, \frac{u^k (1-u)^k}{(1-u)^k k!} \sum_{n=0}^\infty \frac{(1-u)^n}{n!} = e^{-u}\frac{u^k}{k!}\,.\end{aligned}$$

This equality shows that the function $\mathscr{K} f$ has the Poisson distribution with parameter u, which coincides with the distribution of the function

$$h(s) := \sum_{k=1}^\infty \chi_{(0,\tau_k]}(s), \qquad s \in [0,1],$$

where $\tau_k := e^{-u} sum_{i=k}^\infty u^i/i!$ $(k \in \mathbb{N})$. Therefore, denoting by $\bar{\psi}$ the least concave majorant of the function ψ (see § 2), we get that

$$\|\mathscr{K} f\|_{\Lambda_\psi} = \|h\|_{\Lambda_\psi} = \sum_{k=1}^\infty \int_0^{\tau_k} d\bar{\psi}(s) \leqslant 2\sum_{k=1}^\infty \psi(\tau_k). \tag{39}$$

Since $\mathscr{K}$ is a positive operator acting boundedly from Λ_ψ to the space $S(0,1)$, and since the extreme points of the positive part of unit sphere of the space Λ_ψ are precisely the normalized characteristic functions of measurable subsets of $[0,1]$, the boundedness of $\mathscr{K}$ on Λ_ψ is equivalent to its boundedness on the set of all such functions (see Corollary 1 of Lemma 4.5.2 in [16]). So it follows from (39) that $\mathscr{K}$ acts boundedly on Λ_ψ if and only if for some constant $C > 0$

$$\sum_{k=1}^\infty \psi(\tau_k) \leqslant C\psi(u) \qquad (0 < u \leqslant 1).$$

It is not difficult to show (see the proof of Theorem 12) that $e^{-1}u^k/k! \leqslant \tau_k \leqslant 2u^k/k!$ $(k \in \mathbb{N})$. Thus, since the function ψ is quasi-concave, the preceding inequality is equivalent to (38).

The proof of the following statement is similar and therefore we omit it (see Remark 5.2 in [42]).

Theorem 17. *Let $\psi \in \Psi$. The operator $\mathscr{K}$ acts boundedly from the Lorentz space Λ_ψ to the Marcinkiewicz space $M_{t/\psi(t)}$ (or, in terms of the interpolation theory of operators ([16], § 2.6), the operator $\mathscr{K}$ is an operator of weak type (ψ,ψ)) if and only if*

$$\sup_{u\in(0,1],\, k\in\mathbb{N}} \frac{k\psi(u^k/k!)}{\psi(u)} < \infty. \tag{40}$$

Next, we give a number of consequences of Theorems 16 and 17. First of all, as already mentioned, a natural conjecture in view of Theorem 14 is that the space $\exp L_1$ (more precisely, its separable part) is the minimal symmetric space in which the Kruglov operator acts boundedly. In § 4.3 we show that this is not the case: we present an example of a Marcinkiewicz space that has this property and is strictly smaller than $\exp L_1$. Nevertheless, we show that all Lorentz spaces with the Kruglov property contain $\exp L_1$ ([5], Theorem 4).

Theorem 18. *Let $\psi \in \Psi$. If $\Lambda_\psi \in \mathbb{K}$, then $\Lambda_\psi \supset \exp L_1$.*

We first prove the following lemma.

Lemma 9. *If a function $\psi \in \Psi$ satisfies condition (38), then*

$$\sum_{k=1}^{\infty} \psi(2^{-k}) \leqslant A\psi(1), \tag{41}$$

where $A > 0$ depends only on the constant C in (38).

Proof. In view of (38) we have $\sum_{j=1}^{\infty} \psi(2^{-ij} j^{-j}) \leqslant C\psi(2^{-i})$ $(i \in \mathbb{N})$, that is,

$$\sum_{j=1}^{\infty} \psi\bigl(2^{-j(i+[\log_2 j])}\bigr) \leqslant C\psi(2^{-i}) \qquad (i \in \mathbb{N}). \tag{42}$$

Direct estimates show that the quantity

$$\alpha_n := \operatorname{card}\{(i,j) \in \mathbb{N}^2 : j(i + [\log_2 j]) \leqslant n\}$$

satisfies the relation $\lim_{n\to\infty} n^{-1}\alpha_n = \infty$. Therefore, $\alpha_n \geqslant (C+1)n$ for some $m \in \mathbb{N}$ and all $n \geqslant m$. Using this, the inequality (42), and the monotonicity of the function ψ, we get that for any $l > m$

$$(C+1)\sum_{n=m}^{l} \psi(2^{-n}) \leqslant \sum_{i=1}^{l}\sum_{j=1}^{\infty} \psi\bigl(2^{-j(i+[\log_2 j])}\bigr) \leqslant C\sum_{i=1}^{l} \psi(2^{-i}),$$

whence

$$\sum_{n=m}^{l} \psi(2^{-n}) \leqslant C\sum_{i=1}^{m-1} \psi(2^{-i}).$$

Since $l > m$ is arbitrary and m does not depend on ψ, we get the inequality (41).

Proof of Theorem 18. By assumption, $\Lambda_\psi \in \mathbb{K}$, so in view of Theorem 16 the function ψ satisfies (38). Using Lemma 9, we deduce (41). Moreover, by Lemma 8 and (35), we have

$$\|x\|_{\exp L_1} \asymp \sup_{0<t\leqslant 1} x^*(t)\log_2^{-1}\biggl(\frac{2}{t}\biggr).$$

Thus, the theorem will be proved if we show that $\log_2(2/t) \in \Lambda_\psi$. In fact,

$$\begin{aligned}\left\|\log_2 \frac{2}{t}\right\|_{\Lambda_\psi} &= \int_0^1 \log_2 \frac{2}{t}\, d\bar{\psi}(t) = \sum_{k=1}^\infty \int_{2^{-k}}^{2^{-k+1}} \log_2 \frac{2}{t}\, d\bar{\psi}(t) \\ &\leqslant \sum_{k=1}^\infty (k+1)\bigl(\bar{\psi}(2^{-k+1}) - \bar{\psi}(2^{-k})\bigr) \\ &\leqslant 4 + 2\sum_{k=1}^\infty \psi(2^{-k}) < \infty.\end{aligned}$$

The proof is complete.

Using Theorem 17, we can give a direct (and rather short) proof of the fact that, under the condition $X \supset L_p$ for some $p < \infty$, a symmetric space X which is separable or has the Fatou property also has the Kruglov property. In the case of spaces with the Fatou property this result was known earlier (see [39], Theorem 1.2), but its original proof uses the Johnson–Schechtman Theorem 9.

Theorem 19. *If a symmetric space X contains the space L_p for some $p \in [1,\infty)$, then the operator $\mathscr{K}$ acts boundedly in the space $X^{\times\times}$. In particular, if X is separable or has the Fatou property, then $\mathscr{K}$ acts boundedly in X.*

Here we need the following technical lemma.

Lemma 10 ([47], Lemma 11). *If ψ is an increasing concave function on $[0,1]$ such that $\psi(0) = \psi(+0) = 0$, $\psi(1) = 1$, and $\psi(t) \leqslant at^{1/p}$ for every $t \in [0,1]$ and some $p, a \in [1,\infty)$, then*

$$\sup_{0<u\leqslant 1} \frac{1}{\psi(u)} \sum_{k=1}^\infty \psi\left(\frac{u^k}{k!}\right) \leqslant 5ap.$$

Proof of Theorem 19. By the definition of $X^{\times\times}$ we have

$$X^{\times\times} = \bigcap \Lambda_{\psi_y} \quad \text{and} \quad \|x\|_{X^{\times\times}} = \sup \|x\|_{\Lambda_{\psi_y}}, \tag{43}$$

where the intersection and the supremum are taken over all functions ψ_y such that

$$\psi_y(t) = \int_0^t y^*(s)\, ds \qquad (0 < t \leqslant 1), \quad y \in X^\times, \quad \|y\|_{X^\times} \leqslant 1.$$

Following [47] we set

$$\theta_y(t) := \frac{\psi_y(t) + t}{\|y\|_1 + 1} \qquad (0 < t \leqslant 1), \quad y \in X^\times, \quad \|y\|_{X^\times} \leqslant 1. \tag{44}$$

Clearly, for every $y \in X^\times$, θ_y is an increasing concave function on $[0,1]$ with $\theta_y(0) = \theta_y(+0) = 0$ and $\theta_y(1) = 1$. If $\|y\|_{X^\times} \leqslant 1$, then by duality

$$\|y\|_1 = \int_0^1 |y(t)|\, dt \leqslant \phi_X(1)\|y\|_{X^\times} \leqslant 1,$$

and then $\|x\|_{\Lambda_{\psi_y}} \leqslant 2\|x\|_{\Lambda_{\theta_y}}$ for all $x \in X^{\times\times}$. On the other hand, in view of (43) and the equality $\phi_{X^\times}(x) = 1/\phi_X(x)$ ([16], (2.4.39)),

$$\|x\|_{\Lambda_{\theta_y}} \leqslant \|x\|_{\Lambda_{\psi_y}} + \|x\|_1 \leqslant \|x\|_{\Lambda_{\psi_y}} + \phi_{X^\times}(1)\|x\|_{X^{\times\times}}$$
$$\leqslant \biggl(1 + \frac{1}{\phi_X(1)}\biggr)\|x\|_{X^{\times\times}} = 2\|x\|_{X^{\times\times}}.$$

From the last estimates and (43) it follows that

$$X^{\times\times} = \bigcap \Lambda_{\theta_y}, \qquad \|x\|_{X^{\times\times}} \asymp \sup \|x\|_{\lambda_{\theta_y}}, \tag{45}$$

where the intersection and the supremum are taken over all the functions θ_y as in (44).

Further, it follows from the assumption $L_p \subset X$ that $X^\times \subset L_p^\times = L_q$, where $q = p/(p-1)$. Hence, for some $a > 0$

$$\theta_y(t) \leqslant \int_0^t y^*(s)\,ds + t \leqslant \|y\|_q t^{1/p} + t \leqslant a\|y\|_{X^\times} t^{1/p} + t \leqslant (a+1)t^{1/p} \qquad (0 < t \leqslant 1).$$

From Lemma 10 and Theorem 16 with its proof, it now follows that the operator $\mathscr{K}$ is bounded on the Lorentz space Λ_{θ_y} if $\|y\|_{X^\times} \leqslant 1$, and in addition,

$$\sup_{\|y\|_{X^\times} \leqslant 1} \|\mathscr{K}\|_{\Lambda_{\theta_y} \to \Lambda_{\theta_y}} = C < \infty.$$

Thus, in view of (45) we get that for any $x \in X^{\times\times}$

$$\|\mathscr{K}x\|_{X^{\times\times}} \asymp \sup_{\|y\|_{X^\times} \leqslant 1} \|\mathscr{K}x\|_{\Lambda_{\theta_y}} \leqslant C \sup_{\|y\|_{X^\times} \leqslant 1} \|x\|_{\Lambda_{\theta_y}} \asymp \|x\|_{X^{\times\times}},$$

which implies the first statement of the theorem. If X has the Fatou property, then the second statement is obvious, and if X is separable, then it suffices to use Theorem 15.

As already mentioned, the converse of the last theorem fails: there exist symmetric spaces with the Kruglov property which do not contain an L_p-space with $p < \infty$. The first examples are the exponential Orlicz spaces $\exp L_p$ with $0 < p \leqslant 1$. Moreover, the interpolation theory of operators gives a rather general method for obtaining such examples.

Proposition 5. *Suppose that ρ is a quasi-concave function on $[0,1]$, and let $\psi(t) := \rho(t\ln(e/t))$. If $\tilde{\psi}(t) := t/\psi(t)$ $(0 < t \leqslant 1)$, then the Marcinkiewicz space $M_{\tilde{\psi}}$ has the Kruglov property.*

Proof. Let $\ell_\infty(1/\rho)$ be the space of all two-sided sequences $a = \{a_k\}_{k=-\infty}^\infty$, with the norm

$$\|a\|_{l_\infty(1/\rho)} := \biggl\|\biggl\{\frac{a_k}{\rho(2^k)}\biggr\}_{k=-\infty}^\infty\biggr\|_{\ell_\infty}.$$

Since, in view of Theorem 10, $\mathscr{K}$ is bounded on L_1 and on L_{N_1}, it is bounded on the space $(L_1, L_{N_1})^{\mathscr{K}}_{l_\infty(1/\rho)}$ of the real $\mathscr{K}$-method of interpolation (see § 2). Regarding L_1 and L_{N_1} as Marcinkiewicz spaces and using Example 7.1.3 in [46] (cf. the proof of Theorem 13), we get that $(L_1, L_{N_1})^{\mathscr{K}}_{l_\infty(1/\rho)} = M_{\tilde{\psi}}$, which completes the proof.

We provide one more consequence of Theorem 17 which shows that the boundedness of the Kruglov operator on a symmetric space cannot be characterized in terms of embeddings. More precisely, there does not exist a symmetric space X such that the operator $\mathscr{K}$ is bounded on a symmetric space Y if and only if $Y \supseteq X$.

Theorem 20. *Let $\varphi \in \Psi$ be such that*

$$\sup_{t\in(0,1]} t^{-\alpha}\varphi(t) = \infty$$

for every $\alpha \in (0,1]$. Then there exists a $\psi \in \Psi$ with $\psi \leqslant \varphi$ such that the operator $\mathscr{K}$ is unbounded on any symmetric space X with the fundamental function ψ.

We begin with the following lemma, which will also be used later.

Lemma 11. *Let $\varphi \in \Psi$ have the property that for every $\psi \in \Psi$ with $\psi \leqslant \varphi$ the condition (40) holds. Then there exist $a \geqslant 1$ and $\alpha \in (0,1]$ for which $\varphi(t) \leqslant at^{\alpha}$ $(0 \leqslant t \leqslant 1)$.*

Proof. We use an idea from the proof of Theorem 14 in [47]. Assume that the statement of the lemma is wrong, that is, $\sup_{0<t\leqslant 1}\varphi(t)t^{-1/n} = \infty$ for every $n = 1, 2, \dots$. Then there is a sequence $t_n \downarrow 0$ such that $t_1 = 1$ and

$$\varphi(t_n) \geqslant \frac{n\varphi(t_{n-1})}{t_{n-1}}\, t_n^{1/n} \qquad (n = 2, 3\dots).$$

Therefore,

$$\biggl(\frac{t_{n-1}}{\varphi(t_{n-1})}\,\varphi(t_n)\biggr)^n \geqslant n^n t_n > n!\, t_n. \tag{46}$$

If $s_n := t_{n-1}\varphi(t_n)/\varphi(t_{n-1})$, then $t_n < s_n < t_{n-1}$ for all $n = 2, 3, \dots$. We define the function

$$\psi(t) = \begin{cases} \varphi(t_n) & \text{if } t_n \leqslant t \leqslant s_n,\ n = 1, 2, 3, \dots, \\ \dfrac{t\varphi(t_{n-1})}{t_{n-1}} & \text{if } s_n \leqslant t \leqslant t_{n-1},\ n = 2, 3, \dots, \\ 0 & \text{if } t = 0. \end{cases}$$

It is quasi-concave on $[0,1]$ and $\psi \leqslant \varphi$. In view of (46), $\psi(s_n^n/n!) = \psi(t_n) = \psi(s_n)$, whence condition (40) does not hold for this function.

Proof of Theorem 20. By Lemma 11 there exists a function $\psi \in \Psi$ with $\psi \leqslant \varphi$ which does not satisfy (40). Therefore, by Theorem 17 the operator $\mathscr{K}$ is not bounded from the Lorentz space Λ_ψ to the Marcinkiewicz space $M_{t/\psi(t)}$. Since the first of these spaces is minimal and the second one is maximal among symmetric spaces with the fundamental function ψ (see § 2 or [16], Theorems 2.5.5 and 2.5.7), we obtain the required assertion.

Remark 4. A direct application of Theorem 20 to Lorentz spaces shows that the converse of Theorem 18 does not hold in general.

4.3.3. *Marcinkiewicz spaces 'close' to* $\exp L_1$. It was already mentioned that the Orlicz space $\exp L_1$ constructed from the function $e^t - 1$ has the Kruglov property. By Theorem 15 the same holds also for its separable part $(\exp L_1)_\circ$. We observe that all the symmetric spaces X with the Kruglov property which we have considered up to this point contain the latter space. This is true, for example, for every Lorentz space with the Kruglov property (see Theorem 18). In view of Theorem 14 it would at first glance seem natural to conjecture that $(\exp L_1)_\circ$ is the minimal symmetric space on which the Kruglov operator is bounded. Nevertheless, in [5] it was shown that this assertion fails. Moreover, it was proved there that for every symmetric space $X \in \mathbb{K}$ there exists a Marcinkiewicz space $M_\psi \overset{\neq}{\subset} X$ also having the Kruglov property. The main role will be played here by a family of Marcinkiewicz spaces defined in the following theorem.

Theorem 21. *There exists a family* $\{M_{\psi_\varepsilon}\}_{0<\varepsilon<1}$ *of Marcinkiewicz spaces with* $M_{\psi_\varepsilon} \subseteq M_{\psi_\delta}$ $(0 < \varepsilon \leqslant \delta < 1)$ *that satisfies the following conditions*:

1) $M_{\psi_\varepsilon} \in \mathbb{K}$, $0 < \varepsilon < 1$;
2) *if* $X \in \mathbb{K}$ *is symmetric, then* $M_{\psi_\varepsilon} \subset X$ *for all sufficiently small* ε;
3) *the functions* ψ_ε *are not pairwise equivalent, or more precisely*,

$$\lim_{t\to 0} \frac{\psi_\varepsilon(t)}{\psi_\delta(t)} = 0 \quad \text{if } 0 < \varepsilon < \delta < 1; \tag{47}$$

4) $M_{\psi_\varepsilon} \overset{\neq}{\subset} (\exp L_1)_\circ$ *for sufficiently small* $\varepsilon > 0$.

For the proof the following simple statement will be needed.

Lemma 12. *If* $f \in L_1[0,1]$, *then*

$$\lim_{n\to\infty} \lambda(\operatorname{supp} \mathscr{K}^n f) = 0.$$

Proof. Since $\mathscr{K}$ is positive, we can assume that $f \geqslant 0$ and $\lambda(\operatorname{supp} f) = 1$. If $a_n := \lambda\{t : \mathscr{K}^n f(t) = 0\}$ $(n \in \mathbb{N})$, then by the definition of $\mathscr{K}$ (see (33)) $a_1 = 1/e$ and

$$a_{n+1} = \frac{1}{e} + \frac{1}{e}\sum_{k=1}^{\infty} \frac{a_n^k}{k!} = e^{a_n - 1} \qquad (n = 1, 2, \dots).$$

It is easy to see that the sequence $\{a_n\}$ increases and $a_n \in (0,1)$. Since $f(x) := e^{x-1} - x$ decreases on $[0,1]$, the function e^{x-1} has a unique fixed point $x = 1$ on this interval. Hence $\lim_{n\to\infty} a_n = 1$, and the statement is proved.

Proof of Theorem 21. Consider the functions $h_n := (\mathscr{K}^n \chi_{[0,1]})^*$, $n \geqslant 0$. Since $\mathscr{K}$ maps equimeasurable functions to equimeasurable functions,

$$(\mathscr{K} h_n)^* = h_{n+1}. \tag{48}$$

Further, by Lemma 12, $\lambda(\operatorname{supp} h_n) \to 0$ as $n \to \infty$, and this means that the series

$$g_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n h_n \tag{49}$$

converges everywhere on $(0,1]$ for every $\varepsilon>0$ and that the function g_ε is decreasing. In addition, from the definition of $\mathscr{K}$ (see (33)) it follows that $\|\mathscr{K}\|_{L_1\to L_1}=1$. Therefore if $0<\varepsilon<1$, then the series (49) converges in L_1 and $g_\varepsilon\in L_1$. We now show that the statements of the theorem hold for the family $\{M_{\psi_\varepsilon}\}_{\varepsilon>0}$, where $\psi_\varepsilon(t)=\int_0^t g_\varepsilon(s)\,ds$ $(0\leqslant t\leqslant 1)$.

1) We shall prove that $\mathscr{K}$ is bounded on M_{ψ_ε}. Since all extreme points of the unit ball of this space are equimeasurable with the function g_ε [48], [49], it is enough to show that $\mathscr{K} g_\varepsilon\in M_{\psi_\varepsilon}$. In view of the boundedness of $\mathscr{K}$ on L_1 we have

$$\mathscr{K} g_\varepsilon=\sum_{n=0}^\infty \varepsilon^n\mathscr{K} h_n\prec\sum_{n=0}^\infty\varepsilon^n h_{n+1}\leqslant\frac{1}{\varepsilon}\sum_{n=0}^\infty\varepsilon^n h_n=\frac{1}{\varepsilon}\,g_\varepsilon,$$

where the first inequality follows from (48) and the well-known Hardy–Littlewood property of submajorization (see § 2 and [16], § 2.2). Therefore, $\mathscr{K} g_\varepsilon\in M_{\psi_\varepsilon}$.

2) Assume now that $X\in\mathbb{K}$. Then by Lemma 7 and Theorem 15, $\mathscr{K}\colon X_\circ\to X_\circ$. Hence, $C=\|\mathscr{K}\|_{X_\circ\to X_\circ}<\infty$, and then $\|h_n\|_X\leqslant C^n\|\chi_{[0,1]}\|_X=C^n$. Therefore, for every $\varepsilon<C^{-1}$ the series (49) converges in the space $X_\circ$, and $g_\varepsilon\in X_\circ$. In addition, since $X_\circ$ is separable, the conditions $x\in X_\circ$ and $y\prec x$ imply that $y\in X_\circ$ and $\|y\|_{X_\circ}\leqslant\|x\|_{X_\circ}$ (see § 2 or [16], Theorems 2.4.3 and 2.4.10). Thus, the unit ball of the Marcinkiewicz space M_{ψ_ε} (which includes g_ε) is contained in $X_\circ$, and this guarantees that $M_{\psi_\varepsilon}\subset X_\circ\subset X$.

3) As above, let the function g_ε be defined by (49) and let $0<\varepsilon<\delta$. It is not difficult to check (see the proof of Theorem 7.2 in [42]) that

$$\lim_{t\to0}\frac{h_{n+1}(t)}{h_n(t)}=\infty.$$

Therefore, for every $m=1,2,\dots$

$$\begin{aligned}\limsup_{t\to0}\frac{g_\varepsilon(t)}{g_\delta(t)}&=\limsup_{t\to0}\biggl(\sum_{n=1}^\infty\varepsilon^n h_n(t)\biggr)\biggl(\sum_{n=1}^\infty\delta^n h_n(t)\biggr)^{-1}=\cdots\\&=\limsup_{t\to0}\biggl(\sum_{n=m}^\infty\varepsilon^n h_n(t)\biggr)\biggl(\sum_{n=m}^\infty\delta^n h_n(t)\biggr)^{-1}\leqslant\Bigl(\frac{\varepsilon}{\delta}\Bigr)^m.\end{aligned}$$

Thus, $\lim_{t\to0}g_\varepsilon(t)/g_\delta(t)=0$, yielding (47).

4) As mentioned at the beginning of this section, the operator $\mathscr{K}$ acts boundedly in the space $(\exp L_1)_\circ$. Therefore, it is enough to use 2) and 3) proved above.

Corollary 2. *For any symmetric space $X\in\mathbb{K}$ there exists a symmetric space $Y\in\mathbb{K}$ such that $Y\overset{\neq}{\subset}X$.*

5. Comparing sums of independent and disjoint functions in symmetric spaces

The next theorem refines Theorem 1 in [44] (see also Theorems 3.5 and 6.1 in [42]). The assertion of its item (ii) strengthens and supplements the inequality (28) of Johnson and Schechtman, whereas the assertion of (i) shows the definitive character of this strengthening. As above, we denote by $\mathbf{f}$ a disjoint sum corresponding to the sequence $\{f_k\}_{k=1}^{\infty}$ and by L_{M_p} the Orlicz space generated by the function $M_p(t) := e^{t\ln^{1/p}(e+t)} - 1$ $(p>0)$.

Theorem 22. *Let X and Y be symmetric spaces on $[0,1]$ such that $X \subseteq Y$.*

(i) *Suppose that the inequality*

$$\biggl\|\sum_{i=1}^{n} f_i\biggr\|_Y \leqslant C\|\mathbf{f}\|_{Z_X^1} \quad (= C\|\mathbf{f}\|_X) \tag{50}$$

holds for some $C>0$ and for any sequence $\{f_k\}_{k=1}^{n} \subset X$ of independent functions satisfying condition (31). Then the Kruglov operator $\mathscr{K}$ is bounded from X to $Y^{\times\times}$, and $\|\mathscr{K}\|_{X\to Y^{\times\times}} \leqslant C$.

(ii) *If $\mathscr{K}$ acts boundedly from X to Y, then for any sequence $\{f_k\}_{k=1}^{n} \subset X$ of independent functions*

$$\biggl\|\sum_{i=1}^{n} f_i\biggr\|_Y \leqslant 8\|\mathscr{K}\|_{X\to Y}\|\mathbf{f}\|_{Z_X^1}. \tag{51}$$

In the proof of (i) we will use a property of weak convergence of distributions. Recall that one of the equivalent conditions for weak convergence of r.v.'s f_n to an r.v. f is the pointwise convergence of the corresponding characteristic functions $\theta_{f_n}(t) \to \theta_f(t)$ for every $t \in \mathbb{R}$ (see, for example, [50]).

Lemma 13 ([39], Proposition 5). *Let X be a symmetric space on $[0,1]$ with the Fatou property and let $\{f_n\}_{n=1}^{\infty} \subset X$. If the functions f_n converge weakly to f and*

$$\sup_{n=1,2,\dots} \|f_n\|_X = C < \infty,$$

then $f \in X$ and $\|f\|_X \leqslant C$.

In the proof of (ii) we need an analogue of the well-known Prohorov inequality [51] for non-negative functions. Let a sequence $\{f_k\}_{k=1}^{\infty}$ of independent functions be defined on $[0,1]$. Let $\{h_k\}_{k=1}^{\infty}$ be a sequence of independent functions such that $\mathscr{F}_{h_k}(x) = \mathscr{F}_{\mathscr{K} f_k}(x)$ $(x \in \mathbb{R})$ for every $k = 1,2,\dots$.

Lemma 14. *If the f_k are independent and non-negative, then for any $n \in \mathbb{N}$*

$$\lambda\biggl\{\sum_{k=1}^{n} f_k \geqslant x\biggr\} \leqslant 2\lambda\biggl\{\sum_{k=1}^{n} h_k \geqslant \frac{x}{2}\biggr\} \qquad (x>0). \tag{52}$$

Proof. Without loss of generality we can assume that the functions f_i are defined on the space $\prod_{i=1}^{n}([0,1],\lambda_i)$ by the equality $f_i(t_1,\dots,t_n) = f_i(t_i)$ $(i=1,2,\dots,n)$. Since $f_i \geqslant 0$, it follows from the definition of the Kruglov operator that $\mathscr{K} f_i \geqslant g_i$ for some r.v. g_i which is identically distributed with $\sigma_{1/2} f_i$. Hence,

$$\sum_{i=1}^{n} \mathscr{K} f_i(t_i) \geqslant \sum_{i=1}^{n} g_i(t_i). \tag{53}$$

On the other hand, it is clear that for every $i = 1, 2, \dots, n$ the function f_i is identically distributed with the sum $g_i + g_i'$, where g_i and g_i' are disjoint and identically distributed. Therefore if $\lambda^n = \prod_{i=1}^n \lambda_i$, then in view of independence and the inequality (53), for every $x > 0$

$$\lambda^n\biggl\{\sum_{i=1}^n f_i(t_i) > x\biggr\} = \lambda^n\biggl\{\sum_{i=1}^n \bigl(g_i(t_i) + g_i'(t_i)\bigr) > x\biggr\}$$
$$\leqslant 2\lambda^n\biggl\{\sum_{i=1}^n g_i(t_i) > \frac{x}{2}\biggr\} \leqslant 2\lambda^n\biggl\{\sum_{i=1}^n \mathscr{K} f_i(t_i) > \frac{x}{2}\biggr\}.$$

Since $\mathscr{K} f_i$ and h_i are identically distributed and the functions h_i are independent, (52) is proved.

Proof of Theorem 22. (i) We follow the idea of the proof of Lemma 1.6 in [39]. Let $n \in \mathbb{N}$. For an arbitrary function $f \in X$ we choose a function $h \in X$ which is identically distributed with it and such that h and $\chi_{[0,1/n]}$ are independent. Then it is easy to check that the function $h_n := h\chi_{[0,1/n]}$ is identically distributed with the function $\sigma_{1/n} f$. We consider a collection $\{\chi_{[0,1/n]}, h_{n,k}\}_{k=1}^n$ consisting of independent functions with $\mathscr{F}_{h_{n,k}}(x) = \mathscr{F}_{h_n}(x)\,(x \in \mathbb{R})$ for $1 \leqslant k \leqslant n$. Since the functions $\bigl|\sum_{k=1}^n \mathbf{h}_{n,k}\bigr|$ and $|h|$ have the same distribution function, the same holds also for the functions $\bigl|\sum_{k=1}^n \mathbf{h}_{n,k}\bigr|$ and $|f|$. Moreover, the collection $\{h_{n,k}\}_{k=1}^n$ satisfies the inequality (31). Hence by the assumption,

$$\biggl\|\sum_{k=1}^n h_{n,k}\biggr\|_Y \leqslant C\biggl\|\sum_{k=1}^n \mathbf{h}_{n,k}\biggr\|_X = C\|f\|_X. \tag{54}$$

Direct calculations show that $\theta_{h_n}(t) = n^{-1}\theta_f(t) + (1 - n^{-1})$ for all $t \in \mathbb{R}$. Therefore, the characteristic function of the sum $H_n := \sum_{k=1}^n h_{n,k}$ is given by

$$\theta_{H_n}(t) = \bigl(n^{-1}(\theta_f(t) - 1) + 1\bigr)^n \qquad (t \in \mathbb{R}).$$

This implies that $\lim_{n\to\infty} \theta_{H_n}(t) = \exp(\theta_f(t) - 1) = \theta_{\mathscr{K} f}(t)\ \ (t \in \mathbb{R})$, and therefore H_n converges weakly to $\mathscr{K} f$. Since the space $Y^{\times\times}$ has the Fatou property, it follows from (54) and Lemma 13 that $\|\mathscr{K} f\|_{Y^{\times\times}} \leqslant C\|f\|_X$.

(ii) First of all, it is clearly enough to consider only non-negative functions. We assume that a sequence $\{f_k\}_{k=1}^\infty \subset X$ consists of non-negative independent functions satisfying (31). As before, let the h_k be independent functions such that $\mathscr{F}_{h_k}(x) = \mathscr{F}_{\mathscr{K} f_k}(x)\,(x \in \mathbb{R})$ for $k = 1, 2, \dots$. Then by Lemma 14 we have (52), and from this and Proposition 1 we infer that

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_Y \leqslant 4\biggl\|\sum_{k=1}^n h_k\biggr\|_Y. \tag{55}$$

Since the functions $\mathbf{f}_k\ \ (k = 1, \dots, n)$ are pairwise disjoint, we get by the definition of a disjoint sum that

$$\int_{-\infty}^{\infty} (e^{itx} - 1)\, d\mathscr{F}_{\mathbf{f}}(x) = \sum_{k=1}^n \int_{-\infty}^{\infty} (e^{itx} - 1)\, d\mathscr{F}_{f_k}(x),$$

whence

$$\theta_{\mathscr{K}\mathbf{f}}(t) = \prod_{k=1}^{n} \theta_{\mathscr{K} f_k}(t) = \prod_{k=1}^{n} \theta_{h_k}(t) = \theta_{\sum_{k=1}^{n} h_k}(t) \qquad (t \in \mathbb{R}).$$

Thus, the functions $\sum_{k=1}^{n} h_k$ and $\mathscr{K}\mathbf{f}$ are identically distributed. Hence by (55),

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_Y \leqslant 4\Big\|\sum_{k=1}^{n} h_k\Big\|_Y = 4\|\mathscr{K}\mathbf{f}\|_Y \leqslant 4\|\mathscr{K}\|_{X\to Y}\|\mathbf{f}\|_X,$$

and the inequality (51) is proved in this case.

Now let $\{f_k\}_{k=1}^{\infty} \subset X$ be an arbitrary sequence of non-negative independent functions. Without loss of generality we can assume that $\lambda\{f_i = \tau\} = 0$ for every $\tau \geqslant 0$ and all $i = 1, 2, \dots, n$. Then there exist $0 = t_n < t_{n-1} < \dots < t_1 < t_0 = \infty$ such that

$$\sum_{i=1}^{n} \lambda\{t_j \leqslant f_i < t_{j-1}\} = 1 \qquad (j = 1, \dots, n).$$

Note that condition (31) holds for the sequence $\{f_i\chi_{\{f_i>t_1\}}\}_{i=1}^{n}$. Therefore, as already proved,

$$\Big\|\sum_{i=1}^{n} f_i\chi_{\{f_i>t_1\}}\Big\|_Y \leqslant 4\|\mathscr{K}\|\,\Big\|\sum_{i=1}^{n} \mathbf{f}_i\chi_{\{\mathbf{f}_i>t_1\}}\Big\|_X = 4\|\mathscr{K}\|\,\bigg\|\bigg(\sum_{i=1}^{n} \mathbf{f}_i\bigg)^{*}\chi_{[0,1]}\bigg\|_X. \tag{56}$$

Each sequence $\big\{f_i\chi_{\{t_j<f_i<t_{j-1}\}}\big\}_{i=1}^{n}$ $(j = 2, \dots, n)$ also satisfies (31). Thus, since $\|\chi_{[0,1]}\|_X = 1$, we get that

$$\begin{aligned}
\Big\|\sum_{i=1}^{n} f_i\chi_{\{f_i<t_1\}}\Big\|_Y &\leqslant \sum_{j=2}^{n}\Big\|\sum_{i=1}^{n} f_i\chi_{\{t_j<f_i<t_{j-1}\}}\Big\|_Y \\
&\leqslant 4\|\mathscr{K}\|\sum_{j=2}^{n}\Big\|\sum_{i=1}^{n} \mathbf{f}_i\chi_{\{t_j<\mathbf{f}_i<t_{j-1}\}}\Big\|_X \leqslant 4\|\mathscr{K}\|\sum_{j=2}^{n} t_{j-1} \\
&\leqslant 4\|\mathscr{K}\|\bigg(\Big\|\sum_{i=1}^{n} \mathbf{f}_i\chi_{\{\mathbf{f}_i>t_1\}}\Big\|_X + \sum_{j=2}^{n-1}\Big\|\sum_{i=1}^{n} \mathbf{f}_i\chi_{\{t_j<\mathbf{f}_i<t_{j-1}\}}\Big\|_1\bigg) \\
&= 4\|\mathscr{K}\|\bigg(\bigg\|\bigg(\sum_{i=1}^{n} \mathbf{f}_i\bigg)^{*}\chi_{[0,1]}\bigg\|_X + \sum_{j=2}^{n-1}\bigg\|\bigg(\sum_{i=1}^{n} \mathbf{f}_i\bigg)^{*}\chi_{[j-1,j]}\bigg\|_1\bigg) \\
&= 4\|\mathscr{K}\|\,\Big\|\sum_{i=1}^{n} \mathbf{f}_i\Big\|_{Z_X^1}.
\end{aligned}$$

The inequality (51) follows from this and (56), and the theorem is proved.

Clearly, if Y has the Fatou property, then the assertion (i) of the last theorem implies that $\mathscr{K}$ is bounded from X to Y and $\|\mathscr{K}\|_{X\to Y} \leqslant C$. This result can be extended to the case when $X = Y$ and X is separable. Thus, we obtain a partial affirmative answer to a question asked in [39], § 1.6, p. 16.

Theorem 23. *Suppose that X is a separable symmetric space on $[0,1]$ such that*

$$\Big\|\sum_{i=1}^{n} f_i\Big\|_X \leqslant C\|\mathbf{f}\|_{Z_X^1} \quad (= C\|\mathbf{f}\|_X) \tag{57}$$

for some $C>0$ and for any sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of independent functions satisfying condition (31). Then the operator $\mathscr{K}$ is bounded on X and $\|\mathscr{K}\|_{X\to X} \leqslant C$.

Proof. Since $X^{\times\times}$ has the Fatou property, standard arguments using cut-offs of the functions f_k give an extension of the inequality (57) to infinite sums. Therefore, by Theorem 22, $\mathscr{K}$ acts boundedly in $X^{\times\times}$ and $\|\mathscr{K}\|_{X^{\times\times}\to X^{\times\times}} \leqslant C$. Theorem 15 now yields the required assertion.

Remark 5. However, if $X \neq Y$, then the assertion (i) of Theorem 22 cannot be strengthened in a similar way, even in the case of separable Y. To see this, it is enough to consider the spaces $X = L_\infty$ and $Y = (L_{M_1})_\circ$.

Using Theorem 22 and the results obtained in the previous section on boundedness of the Kruglov operator on symmetric spaces, we can prove the inequality (51) in various concrete spaces. Here we cite only one such case. Let $p \in (1,\infty]$ and $q = p/(p-1)$. Since $(L_{M_q})_\circ$ is minimal among all symmetric spaces Y with the property $Y^{\times\times} \supseteq L_{M_q}$ and since $\exp L_p \subset (L_{M_q})_\circ$, we obtain the following assertion from Theorems 22, 12, and 13.

Corollary 3. *For any $p \in (1,\infty]$ there exists a constant $C_p>0$ such that for any $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^{n} \subset \exp L_p$ of independent functions*

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_{(L_{M_q})_\circ} \leqslant C_p\|\mathbf{f}\|_{Z^1_{\exp L_p}}.$$

Moreover, if Y is a symmetric space and the inequality

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_{Y} \leqslant C\|\mathbf{f}\|_{\exp L_p}$$

holds for some $C>0$ and for any sequence $\{f_k\}_{k=1}^{n} \subset \exp L_p$ of independent functions satisfying condition (31), then $(L_{M_q})_\circ \subseteq Y$.

The following theorem, proved in [42], is (in a sense) a converse of the second statement in the Johnson–Schechtman Theorem 9.

Theorem 24. *Suppose that a symmetric space X has the following property: for every symmetric space Y with the Fatou property for which $Y \supset X$, there exists a constant $C>0$ such that for any $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^{n} \subset Y$ of independent functions satisfying condition (31)*

$$\Big\|\sum_{i=1}^{n} f_i\Big\|_{Y} \leqslant C\|\mathbf{f}\|_{Y},$$

where $\mathbf{f}$ is a disjoint sum for the sequence $\{f_k\}_{k=1}^{n}$. Then $X \supset L_p$ for some $p<\infty$.

Proof. By Theorem 2.5.7 in [16], $X \subseteq M_{t/\phi_X}$, where ϕ_X is the fundamental function of X. Hence, for every function $\psi \in \Psi$ such that $\psi \leqslant \phi_X$ we have $X \subseteq M_{t/\psi(t)}$. Thus by the assumption and by Theorem 22 (i), the operator $\mathscr{K}$ acts boundedly in the space $M_{t/\psi(t)}$. Then in view of Theorem 17, condition (40) holds for each such function ψ. By Lemma 11 we get that $\phi_X(t) \leqslant at^\alpha$ $(0 \leqslant t \leqslant 1)$ for some $a \geqslant 1$ and $\alpha \in (0,1]$. The last inequality together with Theorem 2.5.5 in [16] guarantees that

$$X \supseteq \Lambda_{\phi_X} \supseteq \Lambda_{t^\alpha}.$$

Since the Lorentz space Λ_{t^α} contains the space L_p if $p > 1/\alpha$, the assertion follows.

Remark 6. It is clear from the proof that the last assertion still holds if we replace the class of all symmetric spaces with the Fatou property by the class of all Marcinkiewicz spaces.

We now turn to the case of independent mean-zero functions. The original version of the following theorem, which strengthens Johnson–Schechtman inequality (27), was proved in [52] by using a technique from the theory of infinitely divisible distributions. Here the proof is substantially simplified [53].

Theorem 25. *Let X and Y be symmetric spaces on $[0,1]$ with $X \subset Y$. If the operator $\mathscr{K}$ acts boundedly from X to Y, then there exists a universal constant $\alpha > 0$ such that for any sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent functions with $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$ and any $n \in \mathbb{N}$*

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_Y \leqslant \alpha \|\mathscr{K}\|_{X\to Y} \|\mathbf{f}\|_{Z_X^2}. \tag{58}$$

A key role in the proof of this theorem will be played by the Prokhorov inequality in Theorem 5. Below we need the following two auxiliary statements.

Proposition 6. *There is a universal constant $\beta > 0$ such that for any sequence $\{f_k\}_{k=1}^\infty$ of bounded symmetrically distributed independent functions*

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_{L_{M_1}} \leqslant \beta \|\mathbf{f}\|_{L_2 \cap L_\infty} \qquad (n \in \mathbb{N}).$$

Proof. It is easily seen to be sufficient to prove the statement for sequences of r.v.'s of the form $\{f(k-1+t_k)\cdot r_k(s)\}_{k=1}^\infty$ defined on the probability space $\Omega \times [0,1]$, where f is a measurable function on $(0,\infty)$ and, as above, $\Omega = \prod_{k=1}^\infty [0,1]$ with the probability measure $\mathscr{P} = \prod_{k=1}^\infty \lambda_k$ (λ_k is the Lebesgue measure). We define the following sequence of linear operators:

$$U_n \colon L_2 \cap L_\infty \to L_{M_1}(\Omega, \mathscr{P}), \quad U_n f(s, t_1, \dots, t_n, \dots) = \sum_{k=1}^n f(k-1+t_k)\cdot r_k(s).$$

It is obvious that $\|U_n\|_{L_2\cap L_\infty \to L_{M_1}} \leqslant n$. Let the functions $\mathbf{f}_k$ be disjoint copies of the functions $f(k-1+t_k)$ $(k = 1, 2, \dots)$ and let $\mathbf{f}^{(n)} = \sum_{k=1}^n \mathbf{f}_k$ be a disjoint sum of the first n functions. Since the functions $f(k-1+t_k)\cdot r_k(s)$ are symmetrically distributed, by Theorem 5 we have

$$\mathscr{P}\biggl\{\sum_{k=1}^{n} f(k-1+t_k)\cdot r_k(s) > \tau\biggr\} \leqslant \exp\biggl(-\frac{\tau}{2\|\mathbf{f}^{(n)}\|_\infty}\operatorname{arcsinh}\frac{\tau\|\mathbf{f}^{(n)}\|_\infty}{2\|\mathbf{f}^{(n)}\|_2^2}\biggr) \quad (\tau>0). \tag{59}$$

We recall that the function $\bar{f}_1(t) := \ln(e/t)/\ln(\ln(e^e/t))$ is 'maximal with respect to rearrangement' in the space L_{M_1} (see the first relation in (35) and Lemma 8). Moreover, it is not difficult to check that for some $C\geqslant 1$ and for all sufficiently large $\tau>0$

$$e^{-C^{-1}\tau\ln\tau} \leqslant \lambda\{t\in(0,1] : \bar{f}_1(t)>\tau\} \leqslant e^{-\tau\ln\tau}.$$

Assume that $f\in L_2\cap L_\infty$, $\|f\|_{L_2\cap L_\infty}=1$, and $n_0\in\mathbb{N}$ is chosen so that $f\chi_{(0,n_0]}\neq 0$. It is easy to see that the right-hand side in (59) is not greater than

$$\exp\biggl(-\frac{\tau}{2}\operatorname{arcsinh}\frac{\tau\|f\chi_{(0,n_0]}\|_\infty}{2}\biggr)$$

for every $n\geqslant n_0$ and $\tau>0$. Therefore, since $\operatorname{arcsinh} t\sim\ln t$ as $t\to+\infty$, we get that $\sup_{n=1,2,\dots}\|U_nf\|_{L_{M_1}}<\infty$ for every fixed $f\in L_2\cap L_\infty$. Thus, in view of the uniform boundedness principle, the norms $\|U_n\|_{L_2\cap L_\infty\to L_{M_1}}$ $(n\in\mathbb{N})$ are uniformly bounded and the assertion follows.

Lemma 15. *Let X and Y be symmetric spaces on $[0,1]$ and let $\mathscr{K}\colon X\to Y$. Then $L_{M_1}\subset Y$, and in addition, there exists a universal constant $\gamma>0$ such that for any $x\in L_{M_1}$*

$$\|x\|_Y\leqslant\gamma\|\mathscr{K}\|_{X\to Y}\|x\|_{L_{M_1}}.$$

Proof. If the function $\bar{f}_1$ is the same as in the proof of the preceding statement, then as pointed out there, for any $x\in L_{M_1}$ and any $0<t\leqslant 1$ the inequality $x^*(t)\leqslant\gamma\bar{f}_1(t)\|x\|_{L_{M_1}}$ holds with some universal constant $\gamma>0$. Moreover, $\bar{f}_1\leqslant\mathscr{K}\chi_{[0,1]}$ (see the proof of Theorem 12), and hence

$$\|\bar{f}_1\|_Y\leqslant\|\mathscr{K}\chi_{[0,1]}\|_Y\leqslant\|\mathscr{K}\|_{X\to Y}\|\chi_{[0,1]}\|_X=\|\mathscr{K}\|_{X\to Y}.$$

As a result, we get that

$$\|x\|_Y\leqslant\gamma\|\bar{f}_1\|_Y\|x\|_{L_{M_1}}\leqslant\gamma\|\mathscr{K}\|_{X\to Y}\|x\|_{L_{M_1}}.$$

Proof of Theorem 25. First we assume that the functions f_k $(k=1,2,\dots)$ are symmetrically distributed. Let $t_0:=\mathbf{f}^*(1)$, where, as above, $\mathbf{f}$ is a corresponding disjoint sum of the first n functions. Then

$$\sum_{k=1}^{n}\lambda\{s\in[0,1] : |f_k(s)|>t_0\}=\lambda\{s>0 : |\mathbf{f}(s)|>t_0\}\leqslant 1.$$

If $u_k:=f_k\chi_{\{|f_k|>t_0\}}$ and $v_k:=f_k\chi_{\{|f_k|\leqslant t_0\}}$ $(k=1,\dots,n)$, then $\{u_k\}$ and $\{v_k\}$ are sequences of independent symmetrically distributed functions, and $f_k=u_k+v_k$. Since $\sum_{k=1}^n\lambda(\operatorname{supp} u_k)\leqslant 1$, by Theorem 22 (see also its proof) we have

$$\biggl\|\sum_{k=1}^{n}u_k\biggr\|_Y\leqslant 4\|\mathbf{u}\|_X=4\|\mathbf{f}^*\chi_{[0,1]}\|_X. \tag{60}$$

At the same time, using Lemma 15 and Proposition 6 and taking into account that $\|\chi_{[0,1]}\|_X = 1$, we get that

$$\begin{aligned}\biggl\|\sum_{k=1}^n v_k\biggr\|_Y &\leqslant \gamma\|\mathscr{K}\|_{X\to Y}\biggl\|\sum_{k=1}^n v_k\biggr\|_{L_{M_1}} \leqslant \gamma\beta\|\mathscr{K}\|_{X\to Y}\|\mathbf{v}\|_{L_2\cap L_\infty} \\ &\leqslant \gamma\beta\|\mathscr{K}\|_{X\to Y}\|\mathbf{f}^*\chi_{[1,\infty)}\|_{L_2\cap L_\infty} \leqslant \gamma\beta\|\mathscr{K}\|_{X\to Y}(t_0 + \|\mathbf{f}^*\chi_{[1,\infty)}\|_{L_2}) \\ &\leqslant \gamma\beta\|\mathscr{K}\|_{X\to Y}(\|\mathbf{f}^*\chi_{[0,1]}\|_X + \|\mathbf{f}^*\chi_{[1,\infty)}\|_{L_2}) = \gamma\beta\|\mathscr{K}\|_{X\to Y}\|\mathbf{f}\|_{Z_X^2}.\end{aligned}$$

This relation and (60) imply that the inequality (58) holds with the constant $4+\gamma\beta$ for symmetrically distributed functions.

Consider the general case, when it is known only that $\int_0^1 f_k(t)\,dt = 0$ $(k = 1,2,\dots)$. Proceeding in the standard way, we use symmetrization. Let $\{f_k'\}_{k=1}^n$ be a sequence of independent copies of the functions f_k $(k = 1,2,\dots,n)$, and in addition let $h_k := f_k - f_k'$ $(k = 1,\dots,n)$. Then $\{h_k\}_{k=1}^n$ is a sequence of independent symmetrically distributed r.v.'s, for which the inequality (58) has already been established.

Let $\mathscr{B}$ be the σ-subalgebra generated by the sequence $\{f_k\}_{k=1}^n$ and let $\mathsf{E}_{\mathscr{B}}$ be the conditional expectation operator with respect to it. In view of Proposition 2, $\|\mathsf{E}_{\mathscr{B}}\|_{Y\to Y} = 1$. Hence

$$\biggl\|\sum_{k=1}^n f_k\biggr\|_Y = \biggl\|\mathsf{E}_{\mathscr{B}}\biggl(\sum_{k=1}^n h_k\biggr)\biggr\|_Y \leqslant \biggl\|\sum_{k=1}^n h_k\biggr\|_Y.$$

Since $\lambda\{t>0 : |\mathbf{h}(t)| > x\} \leqslant 2\lambda\{t>0 : |\mathbf{f}(t)| > x\}$ $(x \geqslant 0)$, by Proposition 1 the inequality (58) holds in this case with the constant $\alpha = 8 + 2\gamma\beta$. The proof is complete.

6. Rosenthal-type inequalities for symmetric (quasi-)norms on sequences of independent functions

Let E be a symmetric (quasi-)Banach sequence space (see § 2), let $\{e_k\}_{k=1}^\infty \subset E$ be the standard basis in E, and let X be a symmetric function space on $[0,1]$. For an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of non-negative independent functions we consider the quantity

$$\biggl\|\biggl\|\sum_{k=1}^n f_k e_k\biggr\|_E\biggr\|_X.$$

The main objective of this section is to find estimates as precise as possible for this quantity in terms of disjoint copies of the functions in the given sequence. Results of this type have found numerous applications in the study of various geometrical questions, for example, in the local theory of convex bodies [54], [55]. The origin of this quantity is obvious: in the case $E = l_1$ and $X = L_p$ $(1 \leqslant p < \infty)$ it appears in the Rosenthal inequalities (6). We shall consider upper and lower estimates separately, since the assumptions under which they hold and the methods used in their proofs are different.

6.1. Lower estimates. We begin with a simple, but important statement proved in [56], Proposition 1 (see a close result in [35], Lemma 3).

Proposition 7. *Let $\{f_k\}_{k=1}^\infty$ be a sequence of non-negative independent functions defined on $[0,1]$ and let $\mathbf{f}$ be a corresponding disjoint sum. For every $t>0$*

$$\frac{1}{2}\,\lambda\{\mathbf{f}^*\chi_{[0,1]}>t\}\leqslant\lambda\Big\{\sup_{k=1,2,\dots}f_k>t\Big\}\leqslant\lambda\{\mathbf{f}^*\chi_{[0,1]}>t\}.$$

Proof. Let $t>\mathbf{f}^*(1)$ be fixed (it is enough to consider only this case) and let $a_k:=\lambda\{f_k>t\}$ $(k=1,2,\dots)$. Then in view of (25) and the independence of the functions f_k we have

$$\lambda\{\mathbf{f}^*\chi_{[0,1]}>t\}=\sum_{k=1}^\infty a_k\leqslant 1\quad\text{and}\quad\lambda\Big\{\sup_{k=1,2,\dots}f_k>t\Big\}=1-\prod_{k=1}^\infty(1-a_k).$$

It is not difficult to show that

$$1-\sum_{k=1}^\infty a_k\leqslant\prod_{k=1}^\infty(1-a_k)\leqslant 1-\frac{\sum_{k=1}^\infty a_k}{1+\sum_{k=1}^\infty a_k}\,.$$

For example, the right-hand inequality is a consequence of the estimates

$$\prod_{k=1}^\infty(1-a_k)\bigg(1+\sum_{k=1}^\infty a_k\bigg)\leqslant\prod_{k=1}^\infty(1-a_k)(1+a_k)=\prod_{k=1}^\infty(1-a_k^2)\leqslant 1.$$

Therefore,

$$\frac{1}{2}\,\lambda\{\mathbf{f}^*\chi_{[0,1]}>t\}\leqslant\frac{\lambda\{\mathbf{f}^*\chi_{[0,1]}>t\}}{1+\lambda\{\mathbf{f}^*\chi_{[0,1]}>t\}}\leqslant\lambda\Big\{\sup_{k=1,2,\dots}f_k>t\Big\}\leqslant\lambda\{\mathbf{f}^*\chi_{[0,1]}>t\},$$

and the proposition is proved.

The following statement was obtained in [57] (see also [58] and [59]).

Theorem 26. *Let X be a symmetric function space and E a Banach symmetric sequence space. Then for any $n\in\mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n\subset X$ of non-negative independent functions*

$$\|\mathbf{f}^*\chi_{[0,1]}\|_X+\bigg\|\sum_{k=1}^n\mathbf{f}^*(k)e_k\bigg\|_E\leqslant 6\bigg\|\,\bigg\|\sum_{k=1}^n f_ke_k\bigg\|_E\bigg\|_X.\tag{61}$$

Proof. We first prove this inequality in the particular case when $X=L_1$ and E is the sequence space $\mathbf{s}_r$ $(r=1,2,\dots)$ with the norm $\|(a_k)_{k=1}^\infty\|_{\mathbf{s}_r}:=\sum_{k=1}^r a_k^\sharp$, where $(a_k^\sharp)_{k=1}^\infty$ is the non-increasing permutation of the sequence $(|a_k|)_{k=1}^\infty$.

Let $A(t) := \mathbf{f}^*(t)$ and $B(t) := \max_{1\leqslant k\leqslant n} f_k(t)$. Then in view of Proposition 7,

$$\frac{1}{2}\int_0^1 A(t)\,dt \leqslant \int_0^1 B(t)\,dt \leqslant \int_0^1 A(t)\,dt. \tag{62}$$

Further, let $\{g_k\}_{k=1}^n$ be a sequence of independent r.v.'s such that $\lambda\{g_k=1\}=1/r$ and $\lambda\{g_k=0\}=(r-1)/r$ $(k\geqslant 1)$, where $r\in\mathbb{N}$ and $1<r\leqslant n$. Assume that this sequence is independent with respect to the sequence $\{f_k\}_{k=1}^n$ and that all the functions are defined on the square $[0,1]\times[0,1]$ in the following way: $f_k(s,t)=f_k(t)$ and $g_k(s,t)=g_k(s)$. Consider the sequence $\{f_kg_k\}_{k=1}^n$ and denote by $\mathscr{A}$ and $\mathscr{B}$ the corresponding quantities analogous to A and B (defined above for the sequence $\{f_k\}_{k=1}^n$). Applying (62) to this sequence, we get that

$$\frac{1}{2}\int_0^1 \mathscr{A}(u)\,du \leqslant \int_0^1\int_0^1 \mathscr{B}(s,t)\,ds\,dt \leqslant \int_0^1 \mathscr{A}(u)\,du.$$

Since $(f_kg_k)^*=(\sigma_{1/r}f_k)^*$ (see the proof of Theorem 22), where, as usual, $\sigma_{1/r}f_k(t) = f_k(rt)$, we have $\mathscr{A}(u)=\sigma_{1/r}A(u)$ $(u>0)$. Hence,

$$\int_0^1 \mathscr{A}(u)\,du = \int_0^1 \sigma_{1/r}A(u)\,du = \int_0^1 A(ru)\,du = \frac{1}{r}\biggl(\int_0^1 A(u)\,du + \int_1^r A(u)\,du\biggr),$$

and in view of the inequalities $\sum_{k=2}^r A(k) \leqslant \int_1^r A(u)\,du \leqslant \sum_{k=1}^r A(k)$, the preceding relations imply that

$$\begin{aligned}\frac{1}{4r}\biggl(\int_0^1 A(u)\,du + \sum_{k=1}^r A(k)\biggr) &\leqslant \int_0^1\int_0^1 \mathscr{B}(s,t)\,ds\,dt\\ &\leqslant \frac{1}{r}\biggl(\int_0^1 A(u)\,du + \sum_{k=1}^r A(k)\biggr).\end{aligned} \tag{63}$$

We shall show that (61) follows from the last inequality in the case when $X=L_1$ and $E=\mathbf{s}_r$. Consider the conditional expectation operator $\mathsf{E}_{\mathscr{M}}$ with respect to the σ-subalgebra $\mathscr{M}$ of the σ-algebra of Lebesgue-measurable subsets of $[0,1]\times[0,1]$ which consists of all sets of the form $e\times[0,1]$, where e is an arbitrary measurable subset of $[0,1]$. If f is integrable on $[0,1]\times[0,1]$, then $\mathsf{E}_{\mathscr{M}}f(t)=\int_0^1 f(t,s)\,ds$. In particular,

$$\mathsf{E}_{\mathscr{M}}\mathscr{B}(t) = \int_0^1 \max_{1\leqslant k\leqslant n} f_k(t)g_k(s)\,ds.$$

If now $\mathscr{A}_t$ is the quantity analogous to A corresponding to the sequence $\{f_k(t)g_k(\,\cdot\,)\}_{k=1}^n$, then for every $t\in[0,1]$

$$\int_0^1 \mathscr{A}_t(u)\,du = \int_0^1 \sum_{k=1}^r (f_k(t))^\sharp \chi_{((k-1)/r,k/r)}(s)\,ds = \frac{1}{r}\sum_{k=1}^r (f_k(t))^\sharp.$$

Hence, applying (62) to the sequence $\{f_k(t)g_k(\cdot)\}_{k=1}^n$, we get that

$$\frac{1}{2r}\sum_{k=1}^r (f_k(t))^\sharp \leqslant \mathsf{E}_{\mathscr{M}}\mathscr{B}(t) \leqslant \frac{1}{r}\sum_{k=1}^r (f_k(t))^\sharp, \qquad 0<t\leqslant 1. \tag{64}$$

Since $\int_0^1\int_0^1 \mathscr{B}(s,t)\,ds\,dt = \int_0^1 \mathsf{E}_{\mathscr{M}}\mathscr{B}(t)\,dt$, it follows from (64) and (63) that

$$\frac{1}{4}\biggl(\int_0^1 A(u)\,du + \sum_{k=1}^r A(k)\biggr) \leqslant \int_0^1 \sum_{k=1}^r (f_k(t))^\sharp\,dt \leqslant 2\biggl(\int_0^1 A(u)\,du + \sum_{k=1}^r A(k)\biggr), \tag{65}$$

and the inequality (61) is proved if $X = L_1$ and $E = \mathbf{s}_r$ $(r = 1, 2, \dots)$.

Let us consider the general case. Note first that the inequality (61) is a consequence of the two estimates

$$\|A\chi_{[0,1]}\|_X \leqslant 2\biggl\|\biggl\|\sum_{k=1}^n f_k e_k\biggr\|_E\biggr\|_X \quad \text{and} \quad \biggl\|\sum_{k=1}^n A(k)e_k\biggr\|_E \leqslant 4\biggl\|\biggl\|\sum_{k=1}^n f_k e_k\biggr\|_E\biggr\|_X. \tag{66}$$

In view of the obvious inequality $\max_{1\leqslant k\leqslant n} f_k \leqslant \bigl\|\sum_{k=1}^n f_k e_k\bigr\|_E$ and Proposition 1, the first of these follows immediately from the left-hand inequality proved in Proposition 7.

To prove the second inequality in (66) we use duality. First of all,

$$\begin{aligned}\biggl\|\sum_{k=1}^n f_k e_k\biggr\|_E &= \sup_{b=\{b_k\}_{k=1}^n,\,\|b\|_{E^\times}\leqslant 1} \sum_{k=1}^n (f_k(t))^\sharp b_k^\sharp \\ &= \sup_{b=\{b_k\}_{k=1}^n,\,\|b\|_{E^\times}\leqslant 1} \sum_{r=1}^n \|(f_k(t))_{k=1}^n\|_{\mathbf{s}_r}(b_r^\sharp - b_{r+1}^\sharp), \qquad b_{n+1}^\sharp = 0,\end{aligned}$$

where the definition of the dual sequence space $E^\times$ is analogous to that in the case of function spaces. Therefore,

$$\int_0^1 \biggl\|\sum_{k=1}^n f_k(t)e_k\biggr\|_E\,dt \geqslant \sup_{b=\{b_k\}_{k=1}^n,\,\|b\|_{E^\times}\leqslant 1} \sum_{r=1}^n (b_r^\sharp - b_{r+1}^\sharp)\int_0^1 \|(f_k(t))_{k=1}^n\|_{\mathbf{s}_r}\,dt.$$

From this and (65) it follows that

$$\begin{aligned}4\int_0^1 \biggl\|\sum_{k=1}^n f_k(t)e_k\biggr\|_E\,dt &\geqslant \sup_{b=\{b_k\}_{k=1}^n,\,\|b\|_{E^\times}\leqslant 1} \sum_{r=1}^n (b_r^\sharp - b_{r+1}^\sharp)\sum_{k=1}^r A(k) \\ &= \sup_{b=\{b_k\}_{k=1}^n,\,\|b\|_{E^\times}\leqslant 1} \sum_{k=1}^n A(k)b_k^\sharp = \biggl\|\sum_{k=1}^n A(k)e_k\biggr\|_E.\end{aligned}$$

Since $\|f\|_{L_1} \leqslant \|f\|_X$ $(f \in X)$ for an arbitrary symmetric space X on $[0,1]$, we obtain the required inequality, and the theorem is proved.

The key point in the proof of the last theorem is the application of duality, and therefore it cannot be generalized to quasi-normed symmetric sequence spaces. Moreover, in the last part of the proof we use the embedding $X \subset L_1$, which in general fails for quasi-normed symmetric spaces X. At the same time, as shown in the recent paper [60], the validity of the inequality (61) is determined exclusively by probabilistic-combinatorial reasons. We shall cite several results from that paper without proofs.

Theorem 27. *Let $n \in \mathbb{N}$ and let $\{f_k\}_{k=1}^n$ be an arbitrary sequence of non-negative independent functions on* $[0,1]$. *Then*

$$\lambda\left\{t \in [0,1] : \sum_{k=1}^n f_k(t) \geqslant \frac{1}{2}\sum_{k=1}^n \mathbf{f}^*(k)\right\} \geqslant \frac{1}{8}.$$

Corollary 4. *Let X be an arbitrary quasi-normed symmetric space on* $[0,1]$. *Then there exists a constant $\alpha > 0$ depending only on X such that for any $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n$ of non-negative independent functions on* $[0,1]$

$$\|\mathbf{f}^*\chi_{[0,1]}\|_X + \sum_{k=1}^n \mathbf{f}^*(k) \leqslant \alpha\left\|\sum_{k=1}^n f_k\right\|_X.$$

Theorem 28. *Let $n \in \mathbb{N}$ and let $\{f_k\}_{k=1}^n$ be an arbitrary sequence of equimeasurable non-negative independent functions on* $[0,1]$. *Then there is a constant $c_0 > 0$ independent of n and the functions f_k such that*

$$\lambda\left\{t \in [0,1] : f_j^\sharp(t) \geqslant \mathbf{f}^*(2j-1),\ j = 1, \dots, \left[\frac{n}{2}\right]\right\} \geqslant c_0.$$

Corollary 5. *Let X and E be a quasi-normed symmetric function space on* $[0,1]$ *and a quasi-normed symmetric sequence space, respectively, and let $\{e_k\}_{k=1}^\infty$ be the standard basis in E. Then there exists a constant $C > 0$ depending only on the constant in the triangle inequality in E such that for any $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of equimeasurable non-negative independent functions*

$$\left\|\sum_{k=1}^n \mathbf{f}^*(k)e_k\right\|_E \leqslant C\left\|\left\|\sum_{k=1}^n f_k e_k\right\|_E\right\|_X.$$

6.2. Upper estimates in symmetric spaces with the Kruglov property. Now we consider the following inequality opposite to (61):

$$\left\|\left\|\sum_{k=1}^n f_k e_k\right\|_E\right\|_X \leqslant C\left(\|\mathbf{f}^*\chi_{[0,1]}\|_X + \left\|\sum_{k=1}^n \mathbf{f}^*(k)e_k\right\|_E\right), \tag{67}$$

where, as above, $\mathbf{f}$ is a disjoint sum for the sequence $\{f_k\}_{k=1}^n \subset X$.

If $E = \ell_1$ and $X = L_p$ $(1 \leqslant p < \infty)$, then this relation coincides (up to a constant) with the right-hand Rosenthal inequality in (6). In the more general case when $E = \ell_1$ and X is an arbitrary symmetric space containing L_p for some $p < \infty$, we obtain the inequality (27) proved by Johnson and Schechtman [35]. Later Montgomery-Smith ([57], Theorem 1) extended (67) on an arbitrary symmetric sequence space E (with the same condition on X).

In all the papers cited above, the proof is crucially based on use of the well-known Hoffman-Jørgensen inequality [36] or its variants. But in [42] (Theorem 6.7) a construction of the Kruglov operator and the interpolation theory of operators are used. In the special case when $E = \ell_q$ $(1 \leqslant q \leqslant \infty)$ the estimate (67) was obtained there for any symmetric space X with the Kruglov property. Here we shall prove a more general (and in fact definitive) result of this type from a later paper ([61], Theorem 1), which holds not only for normed but also for quasi-normed symmetric sequence spaces satisfying some mild side conditions.

Below we need the inequality (67) in the special case $E = \ell_q$ ($1 \leqslant q \leqslant \infty$), with a sharp constant as $q \to \infty$. As already mentioned, the original version of this result was obtained in [42]; here we present a more precise result from [44], in which the dependence of the constant C on the norm of the Kruglov operator and q is revealed (this will play a decisive role in the proof of the next theorem).

Theorem 29. *Let X and Y be symmetric spaces on $[0,1]$, let $X \subseteq Y$ with constant 1, and let the norm on Y be order-semicontinuous. If the operator $\mathscr{K}$ acts boundedly from X to Y, then there exists a universal constant $\beta > 0$ such that for any $n \in \mathbb{N}$, any sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions, and any $q \in [1,\infty]$ the following inequality holds*:

$$\big\|\|(f_k)_{k=1}^n\|_{\ell_q}\big\|_Y \leqslant \beta\|\mathscr{K}\|_{X\to Y}^{1/q}\big(\|\mathbf{f}^*\chi_{[0,1]}\|_X + \|(\mathbf{f}^*(k))_{k=1}^n\|_{\ell_q}\big). \tag{68}$$

For the proof we need the following construction from Banach space theory ([17], p. 46). If X is a symmetric space on $[0,1]$, then $\widetilde{X(\ell_p)}$ is the set of all sequences $\{f_k\}_{k=1}^\infty \subset X$ such that

$$\|\mathbf{f}\|_{\widetilde{X(\ell_p)}} := \sup_{n=1,2,\dots}\bigg\|\bigg(\sum_{k=1}^n |f_k|^p\bigg)^{1/p}\bigg\|_X < \infty$$

(with the natural modification for $p = \infty$). We shall denote by $X(\ell_p)$ the closed subspace of $\widetilde{X(\ell_p)}$ generated by all sequences $\{f_k\} \subset X$ which are eventually zero.

Proof of Theorem 29. We assume first that the space Y has the Fatou property. It is not difficult to check that the statement (ii) of Theorem 22 holds for infinite sums, and therefore for some universal constant $\alpha \geqslant 1$ we have

$$\|(f_k)_{k=1}^\infty\|_{Y(\ell_1)} := \bigg\|\sum_{k=1}^\infty |f_k|\bigg\|_Y \leqslant \alpha\|\mathscr{K}\|_{X\to Y}\|\mathbf{f}\|_{Z_X^1}.$$

Moreover, by an assumption of the theorem and by Proposition 7,

$$\|(f_k)\|_{Y(\ell_\infty)} \leqslant \|(f_k)\|_{X(l_\infty)} \leqslant \|\mathbf{f}\|_{Z_X^\infty}.$$

Next we use the interpolation property of the Calderón construction (see § 2). Let $(\Omega,\mathscr{P}) := \prod_{k=0}^\infty([0,1],\lambda_k)$ and let $\delta\colon (\Omega,\mathscr{P}) \to ([0,1],\lambda)$ be a measure-preserving map. For every function $g \in S(\Omega,\mathscr{P})$ we set $Tg(x) := g(\delta^{-1}x)$ $(x \in [0,1])$. Then the map $T\colon S(\Omega,\mathscr{P}) \to S([0,1],\lambda)$ sends each function g into an equimeasurable function, and thus $(Tg)^* = g^*$. Finally, we define a positive linear operator Q acting from $S(0,\infty)$ to $S(\Omega,\mathscr{P})$ by

$$Qf(\omega_0,\omega_1,\dots) := \{f_k(\omega_k)\}_{k=0}^\infty, \qquad f \in S(0,\infty),$$

where $f_k(\omega_k) := f(\omega_k + k)$ $(\omega_k \in [0,1])$, $k \geqslant 0$. The argument at the beginning of the proof shows that the positive operator $Q'\colon = TQ$ is bounded from Z_X^1 to

$Y(\ell_1)$ and from Z_X^∞ to $Y(\ell_\infty)$, with norms which do not exceed $\alpha\|\mathscr{K}\|_{X\to Y}$ and 1, respectively. Therefore,

$$Q'\colon (Z_X^1)^{1-\theta}(Z_X^\infty)^\theta \to (Y(\ell_1))^{1-\theta}(Y(\ell_\infty))^\theta$$

and

$$\|Q'\| \leqslant \|Q'\|^{1-\theta}_{Z_X^1\to Y(\ell_1)}\|Q'\|^{\theta}_{Z_X^\infty\to Y(\ell_\infty)} \leqslant \alpha^{1-\theta}\|\mathscr{K}\|^{1-\theta}_{X\to Y}, \qquad \theta\in(0,1).$$

If $q=1/(1-\theta)$, then for each $\theta\in(0,1)$ we have the embeddings

$$Z_X^q \subseteq (Z_X^1)^{1-\theta}(Z_X^\infty)^\theta, \qquad (Y(\ell_1))^{1-\theta}(Y(\ell_\infty))^\theta \subseteq Y(\ell_q).$$

Indeed, to check the first of them, for any $g\in Z_X^q$ with $g=g^*$ and $\|g\|_{Z_X^q}=1$ we set

$$g_1 := g\chi_{[0,1]} + g^q\chi_{[1,\infty)}, \qquad g_\infty := g\chi_{[0,1]} + \chi_{[1,\infty)}.$$

It is obvious that $g=(g_1)^{1-\theta}(g_\infty)^\theta$. Moreover, since $g(1)\leqslant 1$, g_1 is decreasing. Hence, $g_i\in Z_X^i$ and $\|g_i\|_{Z_X^i}\leqslant 2$ $(i=1,\infty)$. The second embedding with constant 1 is proved, for example, in [62], Theorem 3. As a result, we get that

$$\|Q'\|_{Z_X^q\to Y(\ell_q)} \leqslant 2\alpha\|\mathscr{K}\|^{1/q}_{X\to Y}.$$

The statement of the theorem now follows from the easily verified inequality

$$\|\mathbf{f}\|_{Z_X^q} \leqslant \|\mathbf{f}^*\chi_{[0,1]}\|_X + \|(\mathbf{f}^*(k))\|_{\ell_q} \leqslant 2\|\mathbf{f}\|_{Z_X^q},$$

which holds for an arbitrary symmetric space X and all $1\leqslant q\leqslant\infty$.

Now let the norm on Y be just order-semicontinuous. Then Y is isometrically embedded in the space $Y^{\times\times}$ having the Fatou property (see § 2). Therefore, the result follows from the case considered above and from the assumption that $X\subseteq Y$.

Remark 7. Let X be a symmetric space on $[0,1]$ for which there exists a constant $C=C_X>0$ such that the inequality (67) holds for every sequence $\{f_k\}_{k=1}^n\subset X$ $(n\in\mathbb{N})$ of independent functions. As mentioned in § 4.1, in the classical case $X=L_p$ $(1\leqslant p<\infty)$ this is the Rosenthal inequality. In Rosenthal's proof of Theorem 6 and in Burkholder's proof of a somewhat more general result in [63], the constant C_{L_p} in (67) increases exponentially as $p\to\infty$. The sharp constant $C_{L_p}\asymp p/\ln(p+1)\,(1\leqslant p<\infty)$ was found in the paper [64] (see also alternative proofs in [31] and [58]). The notion of the Kruglov operator lets us consider this question from a more general point of view. From Theorems 22 and 23 it follows that in the case of symmetric spaces that are separable or have the Fatou property the best constant C_X in (67) must be equivalent to the norm of $\mathscr{K}$ on X. Due to this fact, the sharp constants in (67) were found in [44] for various scales of symmetric spaces, in particular, for Lorentz and Marcinkiewicz spaces.

We shall now prove a statement similar to Theorem 29 for general quasi-Banach symmetric sequence spaces E satisfying some conditions. The first condition is that for every $k\in\mathbb{N}$ the dilation operator

$$\sigma_k(\{a_j\}_{j=1}^\infty) := (\underbrace{a_1,\dots,a_1}_{k},\underbrace{a_2,\dots,a_2}_{k},\dots,\underbrace{a_j,\dots,a_j}_{k},\dots)$$

is bounded on E and

$$\text{(i)} \quad \|\sigma_k\|_{E\to E} \leqslant k, \quad k \in \mathbb{N}.$$

The second condition is that the following continuous embedding holds:

$$\text{(ii)} \quad \ell_1 \subseteq E.$$

Obviously, a (Banach) symmetric sequence space E satisfies the condition (i) as well as the condition (ii).

Theorem 30. *Let X be a symmetric function space with an order-semicontinuous norm and with the Kruglov property, and let E be a quasi-Banach symmetric sequence space satisfying the conditions (i) and (ii). Then there exists a constant $C > 0$ depending on X such that (67) holds for any $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions.*

Proof. The key role in our approach is played by the 'weak' ℓ_1-space $\ell_{1,\infty}$ (see also [58], proof of Theorem 0.2). This space consists of all sequences $a = (a_k)_{k=1}^\infty$ such that

$$\|a\|_{1,\infty} := \sup_{k\geqslant 1} k a_k^* < \infty.$$

It is easy to see that $\ell_{1,\infty}$ is a quasi-Banach symmetric sequence space and that

$$\|a+b\|_{1,\infty} \leqslant 2(\|a\|_{1,\infty} + \|b\|_{1,\infty}) \qquad (a, b \in \ell_{1,\infty}). \tag{69}$$

We first prove the inequality (67) in this special case.

Proposition 8. *If a symmetric function space X has the Kruglov property, then there exists a universal constant $C_0 > 0$ such that for any $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions*

$$\biggl\|\biggl\|\sum_{k=1}^n f_k e_k\biggr\|_{\ell_{1,\infty}}\biggr\|_X \leqslant C_0\biggl(\|\mathbf{f}^*\chi_{[0,1]}\|_X + \biggl\|\sum_{k=1}^n \mathbf{f}^*(k) e_k\biggr\|_{\ell_{1,\infty}}\biggr). \tag{70}$$

Proof. Since the proof is rather lengthy, we provide here only its key points (see the details in [61]).

Without loss of generality, we can assume that a given sequence $\{f_k\}_{k=1}^n \subset X$ consists of non-negative functions. Let $t_0 := \mathbf{f}^*(1)$. Note that

$$\sum_{k=1}^n \lambda\{s \in [0,1] : f_k(s) > t_0\} = \lambda\{s > 0 : \mathbf{f}(s) > t_0\} \leqslant 1.$$

Hence if $g_k := f_k\chi_{\{f_k > t_0\}}$ and $h_k := f_k\chi_{\{f_k \leqslant t_0\}}$ $(k = 1, \dots, n)$, then $f_k = g_k + h_k$ and $\sum_{k=1}^n \lambda(\operatorname{supp} g_k) \leqslant 1$. We shall estimate the quantities

$$\biggl\|\biggl\|\sum_{k=1}^n g_k e_k\biggr\|_{\ell_{1,\infty}}\biggr\|_X \quad \text{and} \quad \biggl\|\biggl\|\sum_{k=1}^n h_k e_k\biggr\|_{\ell_{1,\infty}}\biggr\|_X$$

separately. Let $\mathbf{g}$ and $\mathbf{h}$ be disjoint sums of the sequences $\{g_k\}_{k=1}^n$ and $\{h_k\}_{k=1}^n$, respectively. Then by the definition of the functions g_k and h_k, it follows that $\mathbf{g}^* = \mathbf{g}^*\chi_{[0,1]} = \mathbf{f}^*\chi_{[0,1]}$. Since $\|a\|_{\ell_{1,\infty}} \leqslant \|a\|_{\ell_1}$ $(a \in \ell_1)$, by Theorem 22 we have

$$\Bigl\|\Bigl\|\sum_{k=1}^n g_k e_k\Bigr\|_{\ell_{1,\infty}}\Bigr\|_X \leqslant \Bigl\|\sum_{k=1}^n g_k\Bigr\|_X \leqslant 8\|\mathscr{K}\|_{X\to X}\|\mathbf{g}^*\|_X = 8\|\mathscr{K}\|_{X\to X}\|\mathbf{f}^*\chi_{[0,1]}\|_X. \tag{71}$$

It remains to prove a similar estimate for the sequence $\{h_k\}_{k=1}^n$ consisting of uniformly bounded independent functions. First we can use Theorem 29 in the case $X = L_p$ $(p \geqslant 1)$ to show that for some universal constant $\gamma > 0$

$$\Bigl\|\Bigl\|\sum_{k=1}^n h_k(u) e_k\Bigr\|_{\ell_{1,\infty}}\Bigr\|_p \leqslant \gamma \Bigl\|\sum_{k=1}^n \mathbf{f}^*(k) e_k\Bigr\|_{\ell_{1,\infty}} \frac{p}{\sqrt{\ln(p+1)}}, \qquad p \geqslant 1,$$

whence

$$\sup_{p\geqslant 1}\biggl(\frac{\sqrt{\ln(p+1)}}{p}\Bigl\|\Bigl\|\sum_{k=1}^n h_k(u) e_k\Bigr\|_{\ell_{1,\infty}}\Bigr\|_p\biggr) \leqslant \gamma \Bigl\|\sum_{k=1}^n \mathbf{f}^*(k) e_k\Bigr\|_{\ell_{1,\infty}}.$$

The quantity on the left-hand side of this inequality is equivalent to the norm of the function

$$u \mapsto \Bigl\|\sum_{k=1}^n h_k(u) e_k\Bigr\|_{\ell_{1,\infty}}$$

in the now familiar Orlicz space L_{M_2}, where $M_2(u) = e^{u\sqrt{\ln(u+e)}} - 1$ (see, for example, [65], Corollary 1 or [64], Proposition 3.6). Since X has the Kruglov property, it follows by Theorem 14 that $L_{M_2} \subseteq X$. Therefore, for some universal constant $\gamma' > 0$ we get that

$$\Bigl\|\Bigl\|\sum_{k=1}^n h_k(u) e_k\Bigr\|_{\ell_{1,\infty}}\Bigr\|_X \leqslant \gamma' \Bigl\|\sum_{k=1}^n \mathbf{f}^*(k) e_k\Bigr\|_{\ell_{1,\infty}}.$$

Together with the estimate (71) and the inequality (69) this implies (70), and the proof of the proposition is finished.

Before we continue the proof of Theorem 30, let us introduce some additional notation. Let E be a quasi-Banach symmetric sequence space. For a given $n \in \mathbb{N}$ we define the operator U_n^E acting from the set of all $n\times n$ matrices to the n^n-dimensional space of step-functions in $L_1(0,1)$ spanned by the characteristic functions of the intervals $\Delta_i := [(i-1)n^{-n}, in^{-n})$, $i = 1,\dots,n^n$. To define this operator, we consider the set $\{1,2,\dots,n\}^n$ and an arbitrary one-to-one map ϕ of this set into the set of intervals $\{\Delta_i\}_{i=1}^{n^n}$. For each matrix $\alpha = (\alpha_{i,j})_{i,j=1}^n$ its image $U_n^E(\alpha)$ is the function

$$U_n^E\alpha(t) := \sum_{j_1,\dots,j_n=1}^n \Bigl\|\sum_{k=1}^n \alpha_{k,j_k} e_k\Bigr\|_E \chi_{\Delta_{\phi(j_1,\dots,j_n)}}(t) \qquad (0 \leqslant t \leqslant 1).$$

The definition of the operator U_n^E depends, of course, on the choice of the map ϕ. Nevertheless, if X is a symmetric space on $[0,1]$, then the norm $\|U_n^E(\alpha)\|_X$ does not depend on ϕ.

Further, we recall that a matrix $\mu = (\mu_{i,j})_{i,j=1}^n$ is said to be *bistochastic* if

$$\mu_{i,j} \geqslant 0, \quad \sum_{i=1}^n \mu_{i,j} = \sum_{j=1}^n \mu_{i,j} = 1, \qquad 1 \leqslant i,j \leqslant n.$$

For such a matrix, we define a family $\xi^\mu = \{\xi_k^\mu\}_{k=1}^n$ of independent r.v.'s on the probability space $[0,1]^n$ as follows. For $k = 1,2,\ldots,n$, $i = 1,2,\ldots,n$, and $t = (t_1,t_2,\ldots,t_n) \in [0,1]^n$, let $\xi_k^\mu(t) = \xi_k^\mu(t_k) = 1/i$ on some subset of $[0,1]$ with measure $\mu_{i,k}$. Since the matrix μ is bistochastic,

$$\sum_{k=1}^n \lambda\left\{\xi_k^\mu = \frac{1}{i}\right\} = 1 \qquad (i = 1,\ldots,n). \tag{72}$$

Moreover, if $\xi_k^\mu(t) = 1/i_k$ for $t \in [0,1]^n$, where $1 \leqslant i_k \leqslant n$ $(1 \leqslant k \leqslant n)$, then

$$\begin{aligned}\|\xi_k^\mu(t)\|_{l_{1,\infty}} &= \sup_{k=1,\ldots,n} k\xi_k^\mu(t)^* = \sup_{r=1,\ldots,n} \frac{1}{r}\operatorname{card}\left\{j : \xi_j^\mu(t) \geqslant \frac{1}{r}\right\} \\ &= \sup_{r=1,\ldots,n} \frac{1}{r}\operatorname{card}\{k : j_k \leqslant r\}.\end{aligned}$$

In this notation we can state the following result, which is implicitly contained in the proof of Proposition 2.1 in [59].

Proposition 9. *Suppose that X is an arbitrary symmetric function space and E is a quasi-Banach symmetric sequence space satisfying the condition (i) (p. 1054). Then for any matrix $\alpha = (\alpha_{i,j}\}_{i,j=1}^n$ there exists a bistochastic matrix $\mu = (\mu_{i,j}\}_{i,j=1}^n$ such that*

$$\|U_n^E\alpha\|_X \leqslant 2\|\max\{1, \|\xi^\mu\|_{1,\infty}\}\|_X \left\|\sum_{k=1}^n \alpha^*_{(k-1)n+1} e_k\right\|_E,$$

where $\{\alpha_j^*\}_{j=1}^{n^2}$ is the non-increasing permutation of the sequence $\{|\alpha_{i,j}|\}_{i,j=1}^n$.

We now apply Propositions 8 and 9 to the proof of Theorem 30. Since X has the Kruglov property, by Proposition 8 we have

$$\|\,\|\xi^\mu\|_{1,\infty}\|_X \leqslant C_0\left(\|\mathbf{f}_\xi^*\chi_{[0,1]}\|_X + \left\|\sum_{k=1}^n \mathbf{f}_\xi^*(k)e_k\right\|_{\ell_{1,\infty}}\right),$$

where $\mathbf{f}_\xi$ is a disjoint sum for the sequence $\xi^\mu = \{\xi_k^\mu\}_{k=1}^n$. Since from (72) it follows immediately that

$$\mathbf{f}_\xi^*(u) = \sum_{i=1}^n \frac{1}{i}\,\chi_{[i-1,i)}(u),$$

we have

$$\|\mathbf{f}_\xi^*\chi_{[0,1]}\|_X \leqslant 1 \quad \text{and} \quad \Big\|\sum_{k=1}^n \mathbf{f}_\xi^*(k)e_k\Big\|_{1,\infty} \leqslant 1.$$

Thus, $\| \|\xi^\mu\|_{1,\infty} \|_X \leqslant 2C_0$, and by Proposition 9 we obtain the following very general statement, which can be regarded as an analogue of a well-known result of Kwapień and Schütt [6].

Corollary 6. *If a symmetric function space X has the Kruglov property, then for any $n \in \mathbb{N}$, any quasi-Banach symmetric sequence space E satisfying the condition (i), and any matrix $\alpha = (\alpha_{i,j}\}_{i,j=1}^n$*

$$\|U_n^E\alpha\|_X \leqslant K_0\Big\|\sum_{k=1}^n \alpha^*_{(k-1)n+1}e_k\Big\|_E,$$

where the constant K_0 depends only on X.

We are now ready to finish the proof of Theorem 30. Since the space E is symmetric and the functions f_k $(k = 1, 2, \dots, n)$ are independent, we can assume that the functions $f_k(s_1, s_2, \dots, s_n) = f_k(s_k)$ defined on $[0,1]^n$ are non-negative and non-increasing. We represent each of them as a sum of three disjoint terms in the following way. If $t_0 = \mathbf{f}^*(1)$, then $f_k^1 := f_k\chi_{\{f_k>t_0\}}$, $f_k^2 := (f_k - f_k^1)\chi_{[0,1/n)}$, and $f_k^3 := f - f_k^1 - f_k^2$. Note that $\sum_{k=1}^n \lambda(\operatorname{supp} f_k^i) \leqslant 1$ for $i = 1, 2$. Hence, in view of the condition (ii) (p. 1054) and Theorem 22 we have

$$\Big\|\Big\|\sum_{k=1}^n f_k^i e_k\Big\|_E\Big\|_X \leqslant \Big\|\sum_{k=1}^n f_k^i\Big\|_X \leqslant 8\|\mathscr{K}\|_{X\to X}\|\mathbf{f}^*\chi_{[0,1]}\|_X \qquad (i = 1, 2). \tag{73}$$

It remains to show that a similar estimate also holds in the case $i = 3$. For this we use Corollary 6. As in the proof of Theorem 0.2 in [58], we introduce the matrix $\alpha = (\alpha_{i,j}\}_{i,j=1}^n$, where

$$\alpha_{i,j} := \sup_{j/n<s_i\leqslant(j+1)/n} f_i^3(s_i), \quad j = 1, \dots, n-1, \qquad \alpha_{i,n} = 0.$$

Let $(j_1, \dots, j_n) \in \{1, \dots, n\}^n$. If the coordinates of a vector $s := (s_1, s_2, \dots, s_n)$ are such that $j_i/n < s_i \leqslant (j_i + 1)/n$, then

$$\Big\|\sum_{i=1}^n f_i^3(s_i)e_i\Big\|_E \leqslant \Big\|\sum_{i=1}^n \alpha_{i,j_i}e_i\Big\|_E$$

(note that $f_i^3(s_i) = 0$ if $0 \leqslant s_i \leqslant 1/n$). Since for every $(j_1, \dots, j_n) \in \{1, \dots, n\}^n$

$$\lambda\Big\{s \in [0,1]^n : \frac{j_i}{n} < s_i \leqslant \frac{j_i+1}{n},\ i = 1, \dots, n\Big\} = n^{-n},$$

we conclude from Corollary 6 that

$$\Big\|\Big\|\sum_{k=1}^n f_k^3 e_k\Big\|_E\Big\|_X \leqslant \|U_n^E\alpha\|_X \leqslant K_0\Big\|\sum_{k=1}^n \alpha^*_{(k-1)n+1}e_k\Big\|_E. \tag{74}$$

To estimate the quantity on the right-hand side of (74), we introduce a family $\{\tilde{f}_i\}_{i=1}^n$ of independent r.v.'s on $[0,1]^n$, defined using the given matrix $\alpha = (\alpha_{i,j}\}_{i,j=1}^n$ as follows:

$$\tilde{f}_i(t) = \tilde{f}_i(t_i) = \sum_{j=1}^n \alpha_{i,j}\chi_{[(j-1)/n,j/n]}(t_i), \qquad i = 1,\dots,n.$$

Since $\tilde{f}_i(t) \leqslant f_i(t)$ $(i = 1,\dots,n)$, for the corresponding disjoint sums

$$\mathbf{f}(t) := \sum_{i=1}^n f_i(t-i+1)\chi_{(i-1,i]}(t) \quad \text{and} \quad \tilde{\mathbf{f}}(t) := \sum_{i=1}^n \tilde{f}_i(t-i+1)\chi_{(i-1,i]}(t) \qquad (t > 0)$$

we have $\tilde{\mathbf{f}}(t) \leqslant \mathbf{f}(t)$ $(t > 0)$. Moreover, $\tilde{\mathbf{f}}^*(k) = \alpha^*_{kn+1}$ $(k = 0, 1, \dots, n-1)$ and

$$\alpha_1^* = \sup_{i=1,\dots,n}\ \sup_{0<s_i\leqslant 1} f_i^3(s_i) \leqslant t_0 = \mathbf{f}^*(1).$$

Hence, if L is the constant in the triangle inequality for the space E, then

$$\begin{aligned}\Big\|\sum_{k=1}^n \alpha^*_{(k-1)n+1}e_k\|_E &\leqslant L\bigg(\alpha_1^* + \Big\|\sum_{k=1}^{n-1}\alpha^*_{kn+1}e_k\Big\|_E\bigg) \leqslant L\bigg(\mathbf{f}^*(1) + \Big\|\sum_{k=1}^{n-1}\tilde{\mathbf{f}}^*(k)e_k\Big\|_E\bigg)\\ &\leqslant L\bigg(\mathbf{f}^*(1) + \Big\|\sum_{k=1}^{n-1}\mathbf{f}^*(k)e_k\Big\|_E\bigg) \leqslant 2L\Big\|\sum_{k=1}^{n-1}\mathbf{f}^*(k)e_k\Big\|_E,\end{aligned}$$

so that in view of (74)

$$\bigg\|\Big\|\sum_{k=1}^n f_k^3 e_k\Big\|_E\bigg\|_X \leqslant 2K_0L\Big\|\sum_{k=1}^{n-1}\mathbf{f}^*(k)e_k\Big\|_E. \tag{75}$$

Since for every $t \in [0,1]^n$

$$\Big\|\sum_{k=1}^n f_k(t)e_k\Big\|_E \leqslant L^2\bigg(\Big\|\sum_{k=1}^n f_k^1(t)e_k\Big\|_E + \Big\|\sum_{k=1}^n f_k^2(t)e_k\Big\|_E + \Big\|\sum_{k=1}^n f_k^3(t)e_k\Big\|_E\bigg),$$

the statement of Theorem 30 follows from (73) and (75).

Corollary 7. *Let a symmetric function space X have the Kruglov property and let E be an arbitrary (Banach) symmetric sequence space. Then there exists a constant $C > 0$ depending only on X such that for any $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions*

$$\begin{aligned}C^{-1}\bigg(\|\mathbf{f}^*\chi_{[0,1]}\|_X + \bigg\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_E\bigg\|_{L_1}\bigg) &\leqslant \bigg\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_E\bigg\|_X\\ &\leqslant C\bigg(\|\mathbf{f}^*\chi_{[0,1]}\|_X + \bigg\|\Big\|\sum_{k=1}^n f_k e_k\Big\|_E\bigg\|_{L_1}\bigg).\end{aligned} \tag{76}$$

Proof. The left-hand inequality in (76) holds for every symmetric space X, and it is an immediate consequence of Proposition 7 and the fact that $\|f\|_{L_1} \leqslant \|f\|_X$ ($f \in X$). To obtain the right-hand estimate, note that, in view of the inequality (61) in the special case $X = L_1$, we have

$$\Bigl\|\sum_{k=1}^{n} \mathbf{f}^*(k)e_k\Bigr\|_E \leqslant C' \Bigl\|\Bigl\|\sum_{k=1}^{n} f_k e_k\Bigr\|_E\Bigr\|_{L_1}.$$

The required result follows from this and the inequality (67).

Remark 8. The estimate obtained in Theorem 30 (as well as the upper estimate in Corollary 7) is sharp in the following sense. If it holds in the case $E = \ell_1$ for every $n \in \mathbb{N}$ and for an arbitrary sequence $\{f_k\}_{k=1}^{n} \subset X$ of independent functions, where X is a symmetric space on $[0,1]$ which is separable or has the Fatou property, then X has the Kruglov property (see Theorems 22 and 23).

Remark 9. If Banach spaces X and E have the Fatou property, then it is not difficult to show that the inequalities (61), (67), and (76) also hold for all infinite sequences $\{f_k\}_{k=1}^{\infty} \subset X$ of independent functions.

7. The Khintchine inequality and its generalizations

As before, let $r_k(t) = \operatorname{sgn}\sin 2^k\pi t$ $(k = 1, 2, \dots)$ be the Rademacher functions on $[0,1]$. The Khintchine inequalities (1) cited in § 1 have generated an enormous amount of research and generalizations and have found numerous applications in the most diverse parts of analysis (see, for example, [3]). In particular, in 1975 Rodin and Semenov [66] proved that the upper estimate

$$\Bigl\|\sum_{k=1}^{\infty} a_k r_k\Bigr\|_X \leqslant C\biggl(\sum_{k=1}^{\infty} a_k^2\biggr)^{1/2} \tag{77}$$

holds in a symmetric space X on $[0,1]$ if and only if X contains the space $(\exp L_2)_\circ$.

Theorem 25 proved in § 5 enables us to determine sharp conditions on a symmetric space X under which a similar inequality holds with scalar multiples of the Rademacher functions replaced by arbitrary mean-zero independent functions. In fact, we consider the question of the validity of even more general estimates, when the ℓ_2-norm on the right-hand side of (77) is replaced by the norm in an arbitrary symmetric sequence space $E \neq \ell_1$. The following theorem is proved in the paper [67] (see also [68]).

Theorem 31. *Suppose that X is a symmetric space on $[0,1]$ which is separable or has the Fatou property. Then the following conditions are equivalent*:

(a) $X \in \mathbb{K}$;

(b) *there exists a $C > 0$ such that for any sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of independent functions with $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$*

$$\Bigl\|\sum_{k=1}^{\infty} f_k\Bigr\|_X \leqslant C\biggl\|\biggl(\sum_{k=1}^{\infty} f_k^2\biggr)^{1/2}\biggr\|_X; \tag{78}$$

(c) *there exist a symmetric sequence space* $E \neq \ell_1$ *and a constant* $C > 0$ *such that for any sequence* $\{f_k\}_{k=1}^{\infty} \subset X$ *of independent functions with* $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_X \leqslant C \big\| \|(f_k)\|_E \big\|_X; \tag{79}$$

(d) *there exist a symmetric sequence space* $E \neq \ell_1$ *and a constant* $C > 0$ *such that the inequality* (79) *holds for any sequence* $\{f_k\}_{k=1}^{\infty} \subset X$ *of symmetrically distributed independent functions with*

$$\sum_{k=1}^{\infty} \lambda(\operatorname{supp} f_k) \leqslant 1. \tag{80}$$

Below we need the following two lemmas. The first of them is only stated here (see its proof in [67]).

Lemma 16. *Let* X *be a symmetric function space on* $[0, 1]$ *and let* E *be a symmetric sequence space with* $E \neq \ell_1$. *Suppose that for some* $C > 0$ *and for any sequence* $\{f_k\}_{k=1}^{\infty} \subset X$ *of symmetrically distributed independent functions satisfying condition* (80) *the inequality* (79) *holds. Then a similar inequality* (*possibly with a different constant*) *holds for an arbitrary sequence of independent functions satisfying* (80).

The second statement can be obtained by general arguments involving the theory of interpolation of operators (see, for example, [69], item 12); however, we prefer to give a direct proof of it.

Lemma 17. *Let* E *be a symmetric sequence space with* $E \neq \ell_1$. *There exists a positive first-degree homogeneous function* $\psi(u, v)$ *increasing with respect to each argument such that* $\lim_{u\to 0+} \psi(u, v) = 0$ *and such that for any* $a = (a_k)_{k=1}^{\infty} \in l_1$

$$\|a\|_E \leqslant \psi(\|a\|_{\ell_\infty}, \|a\|_{\ell_1}). \tag{81}$$

Proof. Let ϕ_E be the fundamental function of the space E, that is, $\phi_E(n) = \big\| \sum_{k=1}^{n} e_k \big\|_E$ $(n \in \mathbb{N})$, where the vectors e_k are the vectors of the standard basis in sequence spaces. Consider a piecewise linear function $\varphi(t)$ on $[0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(n) = \phi(n)$ $(n \in \mathbb{N})$. Since $\phi(n)$ is increasing and $\phi(n)/n$ is decreasing, $\varphi(t)$ is quasi-concave for $t \geqslant 0$. Therefore, if $\overline{\varphi}(t)$ is its least concave majorant, then (see § 2 or [16], Theorem 2.1.1) we have

$$\varphi(t) \leqslant \overline{\varphi}(t) \leqslant 2\varphi(t) \qquad (t > 0). \tag{82}$$

It is easy to check that $\psi(u, v) := v\overline{\varphi}(u/v)$ $(u, v > 0)$ is a positive first-degree homogeneous function increasing with respect to each argument. In addition, by (82) and the definition of φ we have $\lim_{u\to 0+} \psi(u, v) = 0$.

In conclusion we prove (81). Let $a = (a_k)_{k=1}^{\infty} \in \ell_1$. Without loss of generality, we can assume that $a_1 \geqslant a_2 \geqslant \dots \geqslant 0$. For each $n \in \mathbb{N}$ define $a^n = \sum_{k=1}^{n} a_k e_k$. Since $a^n = \sum_{k=1}^{n} (a_k - a_{k+1}) \sum_{i=1}^{k} e_i$, where $a_{n+1} = 0$, we get by the concavity of

$\overline{\varphi}(t)$, the inequalities (82), and the definition and properties of the function ψ that

$$\|a^n\|_E \leqslant \sum_{k=1}^{n}(a_k-a_{k+1})\varphi(k) \leqslant a_1\sum_{k=1}^{n}\frac{a_k-a_{k+1}}{a_1}\,\overline{\varphi}(k)$$
$$\leqslant a_1\overline{\varphi}\biggl(\sum_{k=1}^{n}\frac{a_k-a_{k+1}}{a_1}\,k\biggr)=\psi\biggl(a_1,\sum_{k=1}^{n}(a_k-a_{k+1})k\biggr)\leqslant\psi(\|a\|_{\ell_\infty},\|a\|_{\ell_1}).$$

From $\|a^n-a\|_{\ell_1}\to 0$ it follows that $\|a^n-a\|_E\to 0$. Therefore, passing to the limit in the last inequality as $n\to\infty$, we obtain (81).

Proof of Theorem 31. (a) $\Rightarrow$ (b). Let $\{f_k\}_{k=1}^\infty\subset X$ consist of independent functions with $\int_0^1 f_k(t)\,dt=0$ $(k=1,2,\dots)$. Then by Theorem 25 there exists a constant $C>0$ depending only on X such that

$$\biggl\|\sum_{k=1}^{\infty}f_k\biggr\|_X\leqslant C\biggl\|\sum_{k=1}^{\infty}\mathbf{f}_k\biggr\|_{Z_X^2}. \tag{83}$$

Recall that $\mathbf{f}_k(t)=f_k(t-k+1)\chi_{[k-1,k)}(t)$ $(t>0)$ and that Z_X^2 is the symmetric space on $(0,\infty)$ with the quasi-norm $\|g\|_{Z_X^2}:=\|g^*\chi_{[0,1]}\|_X+\|g^*\chi_{[1,\infty)}\|_{L_2[1,\infty)}$. Therefore, if $\mathbf{f}=\sum_{k=1}^\infty\mathbf{f}_k$, then by Theorem 26 with $E=\ell_2$ we get that

$$\biggl\|\sum_{k=1}^{\infty}\mathbf{f}_k\biggr\|_{Z_X^2}\leqslant\|\mathbf{f}^*\chi_{[0,1]}\|_X+\biggl(\sum_{k=1}^{\infty}\mathbf{f}^*(k)^2\biggr)^{1/2}\leqslant 6\biggl\|\biggl(\sum_{k=1}^{\infty}f_k^2\biggr)^{1/2}\biggr\|_X.$$

This and (83) immediately yield (78), so (b) is proved.

The implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a). By the assumption and Lemma 16, (31) holds for an arbitrary sequence of independent functions satisfying (80). Applying (31) to the functions $|f_k|$ $(k=1,2,\dots)$ and using (81), we have

$$\biggl\|\sum_{k=1}^{\infty}|f_k|\biggr\|_X\leqslant C\|\,\|(f_k)\|_E\,\|_X\leqslant C\bigl\|\psi\bigl(\|(f_k)\|_{l_\infty},\|(f_k)\|_{l_1}\bigr)\bigr\|_X. \tag{84}$$

Let $x,y\in X$, $x\ne 0$, $y\ne 0$. Since the properties of the function ψ imply that for arbitrary $s,t>0$

$$\psi(x,y)\leqslant\max\biggl\{\frac{x}{s},\frac{y}{t}\biggr\}\psi(s,t)\leqslant\biggl(\frac{x}{s}+\frac{y}{t}\biggr)\psi(s,t),$$

we get for s and t with $\|x\|/s=\|y\|/t$ that

$$\|\psi(x,y)\|\leqslant\biggl(\frac{\|x\|}{s}+\frac{\|y\|}{t}\biggr)\psi(s,t)\leqslant 2\max\biggl\{\frac{\|x\|}{s},\frac{\|y\|}{t}\biggr\}\psi(s,t)=2\psi(\|x\|,\|y\|).$$

Therefore, from (84) it follows that

$$\biggl\|\sum_{k=1}^{\infty}|f_k|\biggr\|_X\leqslant 2C\psi\bigl(\bigl\|\|(f_k)\|_{\ell_\infty}\bigr\|_X,\bigl\|\|(f_k)\|_{\ell_1}\bigr\|_X\bigr)$$
$$=2C\biggl\|\sum_{k=1}^{\infty}|f_k|\biggr\|_X\psi\biggl(\frac{\bigl\|\|(f_k)\|_{\ell_\infty}\bigr\|_X}{\bigl\|\|(f_k)\|_{\ell_1}\bigr\|_X},1\biggr),$$

whence

$$\psi\left(\frac{\big\|\|(f_k)\|_{\ell_\infty}\big\|_X}{\big\|\|(f_k)\|_{\ell_1}\big\|_X},1\right)\geqslant\frac{1}{2C}\,.$$

Since $\lim_{u\to 0+}\psi(u,v)=0$ by Lemma 17, it follows from the last inequality that for some $C_1>0$

$$\Big\|\sum_{k=1}^{\infty}|f_k|\Big\|_X\leqslant C_1\Big\|\sup_{k=1,2,\dots}|f_k|\Big\|_X.$$

Then in view of Proposition 7 the inequality

$$\Big\|\sum_{k=1}^{\infty}|f_k|\Big\|_X\leqslant C_1\Big\|\sum_{k=1}^{\infty}\mathbf{f}_k\Big\|_X$$

holds for an arbitrary sequence of independent functions satisfying condition (80). Since the space X is separable or has the Fatou property, Theorems 22 and 23 give us that $X\in\mathbb{K}$.

Remark 10. If the functions f_k $(k=1,2,\dots)$ are scalar multiples of Rademacher functions, then the result will be quite different. As proved in [70], for every sequence space E which is an interpolation space for the couple (ℓ_1,ℓ_2) there exists a symmetric function space X on $[0,1]$ for which the two-sided estimate $C^{-1}\|(a_k)\|_E\leqslant\big\|\sum_{k=1}^{\infty}a_kr_k\big\|_X\leqslant C\|(a_k)\|_E$ holds. Therefore, of course, the last inequality for some $E\neq\ell_1$ does not at all guarantee that a similar estimate holds for ℓ_2.

We now apply Theorem 31 to the study of vector-valued series of the form

$$F(s,t)=\sum_{k=1}^{\infty}f_k(s)r_k(t)\qquad(s,t\in[0,1]),$$

where $\{f_k\}$ is an arbitrary sequence of independent functions and the r_k are the Rademacher functions. If X is a symmetric space on $I=[0,1]$, then $X(I\times I)$ is the corresponding symmetric space on the square $I\times I$ with the norm $\|f\|_{X(I\times I)}=\|f^*\|_X$.

Theorem 32. *If a symmetric space X is separable or has the Fatou property, then the following conditions are equivalent*:

(α) $X\in\mathbb{K}$;

(β) *there exists a constant $C>0$ such that for any sequence $\{f_k\}_{k=1}^{\infty}\subset X$ of independent functions*

$$C^{-1}\Big\|\Big(\sum_{k=1}^{\infty}f_k^2\Big)^{1/2}\Big\|_X\leqslant\Big\|\sum_{k=1}^{\infty}f_k(s)r_k(t)\Big\|_{X(I\times I)}\leqslant C\Big\|\Big(\sum_{k=1}^{\infty}f_k^2\Big)^{1/2}\Big\|_X;\qquad(85)$$

(γ) *there exists a constant $C>0$ such that for any sequence $\{f_k\}_{k=1}^{\infty}\subset X$ of independent functions*

$$C^{-1}\Big\|\Big(\sum_{k=1}^{\infty}f_k^2\Big)^{1/2}\Big\|_X\leqslant\int_0^1\Big\|\sum_{k=1}^{\infty}f_k(s)r_k(t)\Big\|_X\,dt\leqslant C\Big\|\Big(\sum_{k=1}^{\infty}f_k^2\Big)^{1/2}\Big\|_X.\qquad(86)$$

Proof. We first note that the left-hand inequalities in (85) and (86) hold in any symmetric space. Let us check this, for example, for (85). Since the sequence $\{f_k(s)r_k(t)\}_{k=1}^\infty$ is unconditional in $X(I\times I)$ with constant 1 ([39], Proposition 1.14), the Khintchine inequality in (1) for L_1 with the optimal constant $A_1=1/\sqrt{2}$ [71] gives us that

$$\begin{aligned}\biggl\|\sum_{k=1}^\infty f_k(s)r_k(t)\biggr\|_{X(I\times I)}&=\int_0^1\biggl\|\sum_{k=1}^\infty f_k(s)r_k(t)r_k(u)\biggr\|_{X(I\times I)}\,du\\&\geqslant\biggl\|\int_0^1\biggl|\sum_{k=1}^\infty f_k(s)r_k(t)r_k(u)\biggr|\,du\biggr\|_{X(I\times I)}\geqslant\frac{1}{\sqrt{2}}\biggl\|\biggl(\sum_{k=1}^\infty f_k^2\biggr)^{1/2}\biggr\|_X.\end{aligned}$$

The proof of the left-hand inequality in (86) is quite similar.

Further, since the functions $f_k(s)r_k(t)\,(k=1,2,\dots)$ are independent and

$$\int_0^1\int_0^1 f_k(s)r_k(t)\,ds\,dt=0\qquad(k=1,2,\dots),$$

the right-hand inequality in (85) is an immediate consequence of (α) by Theorem 31. Let us prove the right-hand inequality in (86). Let $\{f_k\}_{k=1}^\infty\subset X$ be a sequence of independent functions. Then the functions $g_k(s)=f_k(s)-\int_0^1 f_k(t)\,dt$ $(k=1,2,\dots)$ are also independent, and $\int_0^1 g_k(t)\,dt=0$, $k=1,2,\dots$. Therefore, applying Theorem 31 to the sequence $\{g_k(\cdot)r_k(t)\}_{k=1}^\infty$ for fixed $t\in[0,1]$ and then taking into account that $\|\chi_{[0,1]}\|_X=1$ and $\|g\|_X\geqslant\int_0^1|g(s)|\,ds$, we get that

$$\begin{aligned}\int_0^1\biggl\|\sum_{k=1}^\infty f_k(s)r_k(t)\biggr\|_X\,dt&=\int_0^1\biggl\|\sum_{k=1}^\infty\biggl(g_k(s)+\int_0^1 f_k(u)\,du\biggr)r_k(t)\biggr\|_X\,dt\\&\leqslant\int_0^1\biggl\|\sum_{k=1}^\infty g_k(s)r_k(t)\biggr\|_X\,dt+\int_0^1\biggl|\sum_{k=1}^\infty\int_0^1 f_k(u)\,du\cdot r_k(t)\biggr|\,dt\\&\leqslant C\biggl\|\biggl(\sum_{k=1}^\infty g_k^2\biggr)^{1/2}\biggr\|_X+\biggl(\sum_{k=1}^\infty\biggl(\int_0^1 f_k(u)\,du\biggr)^2\biggr)^{1/2}\leqslant C\biggl\|\biggl(\sum_{k=1}^\infty f_k^2\biggr)^{1/2}\biggr\|_X\\&\qquad+(C+1)\biggl(\sum_{k=1}^\infty\biggl(\int_0^1 f_k(u)\,du\biggr)^2\biggr)^{1/2}\leqslant(2C+1)\biggl\|\biggl(\sum_{k=1}^\infty f_k^2\biggr)^{1/2}\biggr\|_X.\end{aligned}$$

Conversely, let (β) or (γ) (more precisely, the right-hand inequality in (85) or (86)) hold. To prove (α) it is enough to check that X satisfies the condition (d) of Theorem 31. We assume that the sequence $\{f_k\}_{k=1}^\infty\subset X$ consists of symmetrically distributed independent functions satisfying (80). Using Proposition 1.14 in [39] again, we conclude that $\{f_k\}$ is an unconditional sequence in X with constant 1. Therefore, if the right-hand inequality in (86) holds, then

$$\biggl\|\sum_{k=1}^\infty f_k\biggr\|_X=\int_0^1\biggl\|\sum_{k=1}^\infty f_k(s)r_k(t)\biggr\|_X\,dt\leqslant C\biggl\|\biggl(\sum_{k=1}^\infty f_k^2\biggr)^{1/2}\biggr\|_X,$$

and (d) is proved. In addition, the distribution of the function f_k on I coincides with the distribution of the function $f_k(s)r_k(t)$ on $I \times I$ $(k = 1, 2, \dots)$. Indeed, since f_k is symmetrically distributed, for every $\tau \in \mathbb{R}$ we have

$$\begin{aligned}&\lambda^2\{(s,t) \in I \times I : f_k(s)r_k(t) > \tau\}\\ &\qquad = \frac{1}{2}\left(\lambda\{s \in I : f_k(s) > \tau\} + \lambda\{s \in I : -f_k(s) > \tau\}\right) = \lambda\{s \in I : f_k(s) > \tau\}.\end{aligned}$$

Therefore, due to the independence of the functions f_k $(k = 1, 2, \dots)$ and $f_k(s)r_k(t)$ $(k = 1, 2, \dots)$, respectively, we get that

$$\left\| \sum_{k=1}^{\infty} f_k \right\|_X = \left\| \sum_{k=1}^{\infty} f_k(s)r_k(t) \right\|_{X(I\times I)},$$

and thus (d) is a consequence of the right-hand inequality in (85) as well. The proof is complete.

Remark 11. It is of interest to compare the statement of the last theorem with two well-known results. According to the first one, for every symmetric space X which is separable or has the Fatou property, the following conditions are equivalent: (i) there exists a constant $C > 0$ such that (85) holds for an *arbitrary* sequence $\{f_k\}_{k=1}^{\infty} \subset X$ and (ii) the lower Boyd index α_X is positive (the implication (ii) $\Rightarrow$ (i) is proved in [17], Proposition 2.d.1, and the opposite implication (i) $\Rightarrow$ (ii) is proved in [72]). The second result (the Maurey inequality) says that if X is a q-concave Banach lattice ([17], Definition 1.d.3) for some $q < \infty$, then there exists a constant $C > 0$ such that the inequalities (86) hold for an *arbitrary* sequence $\{f_k\}_{k=1}^{\infty} \subset X$ ([17], Theorem 1.d.6(i)). We observe that in both the cited results the coefficients of a series are *arbitrary* functions; on the other hand, the conditions guaranteeing that the inequalities (85) and (86) hold are much more restrictive than the condition (α) in Theorem 32.

In symmetric spaces 'close' to L_∞ the Maurey inequality mentioned in the paragraph above does not hold in general. Nevertheless, estimates of the norms of polynomials in systems of independent functions have recently been obtained also for such spaces. This will be discussed in the next section.

8. Estimates of norms of polynomials in systems of independent functions in spaces 'close' to L_∞

Let $\{\xi_i\}_{i=1}^n$ be a system of independent r.v.'s on a probability space $(T, \mathscr{T}, \tau)$, and let $\{f_i\}_{i=1}^n$ be a system of measurable functions on another probability space (X, Σ, μ). We consider the expectation (with respect to τ)

$$\mathsf{E}\left\| \sum_{i=1}^{n} \xi_i(t) f_i(x) \right\|_{L_\infty(\mu)}. \tag{87}$$

The classical result of Salem and Zygmund [73] asserts that for some $C > 0$ and all $n \in \mathbb{N}$ we have

$$C^{-1}(n \ln n)^{1/2} \leqslant \mathsf{E}\bigg\| \sum_{j=1}^{n} r_j(t) e^{2\pi i j x} \bigg\|_{L_\infty} \leqslant C(n \ln n)^{1/2}, \tag{88}$$

where, as above, the functions $r_i(t)$ are the Rademacher functions on $[0,1]$.

In the monographs [74] and [75] a detailed study is presented of a more general situation concerning the behavior of the quantity (87) for the case when $\{f_i\}$ is a system of characters on a locally compact Abelian group G, restricted to a compact symmetric neighbourhood V of the unit element $e \in G$. The proof of lower estimates for (87) is based on a comparison of it with the quantity

$$\mathsf{E}\bigg\| \sum_{i=1}^{n} \gamma_i(t) f_i(x) \bigg\|_{L_\infty(\mu)}, \tag{89}$$

where $\{\gamma_i\}_{i=1}^{\infty}$ is a system of independent Gaussian r.v.'s with $\mathsf{E}\gamma_i = 0$ and $\mathsf{E}\gamma_i^2 = 1$ ($i = 1, 2, \dots$). The crucial role here is played by the well-known Slepian lemma, which enables us to get lower estimates for expressions of the type

$$\mathsf{E} \max_{1 \leqslant k \leqslant m} \bigg| \sum_{i=1}^{n} \gamma_i(t) f_i(x_k) \bigg|,$$

and thereby for the quantity (89) under the assumption that points $x_1, \dots, x_m$ are selected in V in such a way that the corresponding vectors

$$W_{x_k} = \big(f_1(x_k), \dots, f_n(x_k)\big) \in \mathbb{C}^n \qquad (k = 1, \dots, m)$$

are located sufficiently far from each other in the Euclidean metric in $\mathbb{C}^n$. Thus, a lower estimate for (89) can be obtained only if the ε-entropy of the set $\{W_x\}_{x \in V}$ in the ℓ_2^n-metric can be controlled. The latter is a non-trivial problem even in relatively simple cases.

To estimate the expectation (89) from below, Kashin and Tzafriri proposed in [76] and [77] a new and interesting method based on the use of a two-dimensional version of the central limit theorem with sharp estimates of the remainder term. This method shows that it is sufficient to ensure the smallness of the *mean*

$$\frac{1}{n^2} \sum_{i,j=1}^{n} |(W_{x_i}, W_{x_j})|^2.$$

This is, of course, much easier to achieve than to guarantee that a similar condition holds for the entropy of the set $\{W_{x_i}\}_{i=1}^{n}$, that is, to guarantee the smallness of *all* the scalar products (W_{x_i}, W_{x_j}) $(1 \leqslant i \neq j \leqslant n)$. Due to this fact, it was shown that the lower estimate in (88) holds for random polynomials in *any* orthonormal bounded system in L_3. We cite the following theorem proved in [76].

Theorem 33. *For any $M > 0$ there exist constants $C_j = C_j(M) > 0$ $(j = 1, 2, 3)$ and $q = q(M) > 0$ such that for any system $\{f_i\}_{i=1}^{n}$ of functions with*

(a) $\|f_i\|_{L_2(\mu)} = 1$, $\|f_i\|_{L_3(\mu)} \leqslant M$ $(i = 1, \dots, n)$, *and*

(b) $\displaystyle\Big\|\sum_{i=1}^n a_i f_i\Big\|_{L_2(\mu)} \leqslant M\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2}$ *for all* $a_i \in \mathbb{C}$

and for any sequence $\{\xi_i\}_{i=1}^n$ *of r.v.'s defined on a probability space* $(T, \mathscr{T}, \tau)$ *with*

(c) *the* ξ_i *independent*, $\mathsf{E}\xi_i = 0$, $\mathsf{E}|\xi_i|^2 = 1$, *and* $(\mathsf{E}|\xi_i|^3)^{1/3} \leqslant M$ $(i = 1, \dots, n)$, *the following inequality holds*:

$$\tau\Big\{t \in T : \Big\|\sum_{i=1}^n \xi_i(t) f_i\Big\|_{L_\infty(\mu)} \leqslant C_1(n\ln n)^{1/2}\Big\} \leqslant C_2 n^{-q},$$

whence, in particular,

$$\mathsf{E}\Big\|\sum_{i=1}^n \xi_i f_i\Big\|_{L_\infty(\mu)} \geqslant C_3(n\ln n)^{1/2}.$$

In [77] the last theorem was generalized to the case of 'random' norms of the form

$$\Big\|\sum_{i=1}^n \xi_i(t) a_i f_i(x)\Big\|_{L_\infty(\mu)},$$

where $a_i \in \mathbb{C}$. It was noted in the same paper that in fact similar lower estimates hold not only for the L_∞-norm, but also for the so-called *integral-uniform* norm (introduced there)

$$\|f\|_{m,\infty} = \int_X \cdots \int_X \max\{|f(x_1)|, \dots, |f(x_m)|\}\, d\mu(x_1) \cdots d\mu(x_m),$$

where $m \in \mathbb{N}$ and f is an arbitrary integrable function on the probability space (X, Σ, μ). The estimates for the norms $\|\cdot\|_{m,\infty}$ are undoubtedly of interest, since these norms carry substantial information about the distribution function of f. First of all, it is not difficult to show that

$$\|f\|_{m,\infty} = \int_0^\infty \big(1 - (1 - n_f(z))^m\big)\, dz,$$

where $n_f(z) = \mu\{x \in X : |f(x)| > z\}$. In particular, it follows that $\|f\|_1 \leqslant \|f\|_{m,\infty} \leqslant \|f\|_\infty$ and $\|f\|_{m,\infty} \to \|f\|_\infty$ as $m \to \infty$. In addition, as shown in Proposition 2 of [78] and independently in Theorem 2 of [79],

$$\|f\|_{m,\infty} \asymp m \int_0^{1/m} f^*(t)\, dt, \tag{90}$$

with constants which do not depend on $m \in \mathbb{N}$,

The following generalization of the results of Kashin and Tzafriri was obtained by Grigoriev in [80], Theorem 1 (see also [81], Theorem 1).

Theorem 34. *Let sequences of functions* $\{f_i\}_{i=1}^n$ *and* $\{\xi_i\}_{i=1}^n$ *defined on probability spaces* (X, Σ, μ) *and* $(T, \mathscr{T}, \tau)$, *respectively, satisfy conditions* (a) *and* (c) *in*

Theorem 33. *Suppose also that a sequence* $(a_i)_{i=1}^n \in \mathbb{C}^n$ *is fixed and that for any* $(c_i)_{i=1}^n \in \mathbb{C}^n$

$$\Big\|\sum_{i=1}^n c_i f_i\Big\|_{L_2(\mu)} \leqslant MR(\{a_i\})^p\Big(\sum_{i=1}^n |c_i|^2\Big)^{1/2},$$

where $M > 0$ *and* $p \in [0, 1/2)$ *are some constants and*

$$R = R(\{a_i\}) = \Big(\sum_{i=1}^n |a_i|^2\Big)^2\Big(\sum_{i=1}^n |a_i|^4\Big)^{-1}.$$

Then there exist constants $q = q(p, M) > 0$ *and* $C_j = C_j(p, M) > 0$ $(j = 1, 2, 3)$ *such that*

$$\tau\Big\{t \in T : \Big\|\sum_{i=1}^n a_i\xi_i(t)f_i\Big\|_{m,\infty} \leqslant C_1\Big(\sum_{i=1}^n |a_i|^2 \ln P\Big)^{1/2}\Big\} \leqslant \frac{C_2}{P^q} \tag{91}$$

and

$$\mathsf{E}\Big\|\sum_{i=1}^n a_i\xi_i f_i\Big\|_{m,\infty} \geqslant C_3\Big(\sum_{i=1}^n |a_i|^2 \ln P\Big)^{1/2}, \tag{92}$$

where $P := \min\{m, R\} + 1$.

It is shown in the same paper that, under some additional restrictions on the sequence $\{\xi_i\}_{i=1}^n$ the estimate (92) is sharp if $m \leqslant n$.

Theorem 35. *Let* ξ_i $(i = 1, \dots, n)$ *be independent r.v.'s such that the distributions of polynomials in these functions satisfy the exponential estimate*

$$\tau\Big\{t \in T : \Big|\sum_{i=1}^n c_i\xi_i(t)\Big| \geqslant s\Big(\sum_{i=1}^n |c_i|^2\Big)^{1/2}\Big\} \leqslant C_4 e^{-C_5 s^2} \qquad (s > 0), \tag{93}$$

where the constants $C_4 > 0$ *and* $C_5 > 0$ *do not depend on the coefficient sequence* $\{c_i\}_{i=1}^n \in \mathbb{C}^n$. *Then there exists an absolute constant* $C_6 > 0$ *such that for any* $m \geqslant 1$ *and any* $f_i \in L_1(X)$ $(i = 1, \dots, n)$

$$\mathsf{E}\Big\|\sum_{i=1}^n \xi_i f_i\Big\|_{m,\infty} \leqslant C_6\Big\|\Big(\sum_{i=1}^n |f_i|^2\Big)^{1/2}\Big\|_{m,\infty}\sqrt{1 + \ln m}\,. \tag{94}$$

In particular, if $f_i \in L_\infty(X)$ $(i = 1, \dots, n)$, *then*

$$\mathsf{E}\Big\|\sum_{i=1}^n \xi_i f_i\Big\|_{m,\infty} \leqslant C_6\Big(\sum_{i=1}^n \|f_i\|_\infty^2\Big)^{1/2}\sqrt{1 + \ln m}\,.$$

Corollary 8. *Suppose that* $\|f_i\|_{L_\infty(X)} \leqslant M$ $(i = 1, \dots, n)$, *and that* ξ_i $(i = 1, \dots, n)$ *are independent r.v.'s satisfying the exponential estimate* (93). *Then for arbitrary* $a_i \in \mathbb{C}$

$$\mathsf{E}\Big\|\sum_{i=1}^n a_i\xi_i f_i\Big\|_{m,\infty} \leqslant MC_6\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2}\sqrt{1 + \ln m}\,.$$

So if $m \leqslant n$ *and* $m \leqslant C_7 R(\{a_i\}_{i=1}^n)$ (*see Theorem* 34), *then the inequality* (92) *in Theorem* 34 *is sharp under the assumptions of the theorem.*

In the case when $a_i = 1\,(i = 1, 2, \dots, n)$, Grigoriev [79] succeeded in extending the statement of Theorem 34 to systems $\{f_i\}$ which are in general unbounded in L_p for $p > 1$.

Theorem 36. *Let a sequence of functions $\{f_i\}_{i=1}^n$ defined on a probability space (X, Σ, μ) be such that $\|f_i\|_{L_1(\mu)} = 1$ $(i = 1, \dots, n)$ and $\big\|\sum_{i=1}^n \theta_i f_i\big\|_{L_1(\mu)} \leqslant M n^{1/2+p}$ for any collection of signs $(\theta_i)_{i=1}^n$, $\theta_i = \pm 1$, with some constants $M > 0$ and $p \in [0, 1/12)$. Moreover, suppose that a sequence $\{\xi_i\}_{i=1}^n$ of r.v.'s defined on a probability space $(T, \mathcal{T}, \tau)$ satisfies the condition (c) in Theorem 33.*

Then for any $m \leqslant n$

$$\tau\biggl\{t \in T : \biggl\|\sum_{i=1}^n \xi_i(t) f_i\biggr\|_{m,\infty} \leqslant C_1'\bigl(n(1+\ln m)\bigr)^{1/2}\biggr\} \leqslant C_2' m^{-q'},$$

and therefore

$$\mathsf{E}\biggl\|\sum_{i=1}^n \xi_i f_i\biggr\|_{m,\infty} \geqslant C_3'\bigl(n(1+\ln m)\bigr)^{1/2}$$

with some constants $q' = q'(p) > 0$ and $C_j' = C_j'(p, M) > 0$ $(j = 1, 2, 3)$.

The last theorem, the definition of the norm in a Marcinkiewicz space (see § 2), and the relation (90) immediately yield the following result.

Corollary 9. *Let $\varphi \in \Psi$. If the conditions of Theorem 36 hold, then there exists a constant $A > 0$ such that*

$$\mathsf{E}\biggl\|\sum_{i=1}^n \xi_i f_i\biggr\|_{M(\varphi)} \geqslant A\sqrt{n}\max_{m=2,\dots,n}\biggl\{\frac{\sqrt{\ln m}}{m\varphi(1/m)}\biggr\}.$$

9. Isomorphisms between symmetric spaces on $[0,1]$ and $[0,\infty)$

The general problem of the existence of isomorphisms between symmetric spaces on $[0,1]$ and on $[0,\infty)$ (other than L_p-spaces) was first posed by Mityagin in [82]. This and other closely linked questions were intensively studied in [38] (see also [17]) using an approach based on the construction of a stochastic integral with respect to a symmetrized Poisson process. Here we follow ideas from our paper [52], where a technically much simpler approach based on the use of Theorem 25 was proposed; it should be noted that similar arguments were used earlier in [34] in the special case of the Lorentz spaces $L_{p,q}$. The first part of the following theorem strengthens results in [38] on the isomorphic embedding of some symmetric space on the semi-axis into a given symmetric space X on $[0,1]$ by replacing the assumption that the lower Boyd index of X is non-trivial by the weaker assumption that the operator $\mathscr{K}$ is bounded. The second part of this theorem is proved in [83]. A similar statement was known earlier (see [38], § 8, or [17], p. 203) only in the case when the symmetric space X has non-trivial Boyd indices.

Theorem 37. (i) *If the operator $\mathscr{K}$ acts boundedly in a symmetric space X, then X contains a subspace isomorphic to the space Z_X^2.*

(ii) *If a symmetric space X is separable or has the Fatou property and the operator $\mathscr{K}$ acts boundedly in X and in $X^\times$, then the spaces X and Z_X^2 are isomorphic.*

Proof. Let $(\Omega, \mathscr{P}) := \prod_{k=1}^{\infty}([0,1],\lambda_k)$ (as above, λ_k is the Lebesgue measure). Then by Theorem 25 the linear operator

$$Qx(t,\omega_1,\omega_2,\dots) = \sum_{k=1}^{\infty} x_k(\omega_k) r_k(t), \quad \text{where } x_k(\omega_k) = x(k-1+\omega_k) \quad (0 \leqslant \omega_k \leqslant 1),$$

acts boundedly from the space Z_X^2 to the space $X(\Omega \times [0,1])$. Moreover, by Theorem 9 (see the first inequality in (26)) there exists a constant $c > 0$ such that $\|Qf\|_{X(\Omega\times[0,1])} \geqslant c\|f\|_{Z_X^2}$. So the image of the operator $\operatorname{Im}(Q)$ as a subspace of $X(\Omega\times[0,1])$ is isomorphic to the space Z_X^2. Since the spaces X and $X(\Omega\times[0,1])$ are isometric, the assertion (i) is proved.

We pass to the proof of (ii). We shall show that the image $\operatorname{Im}(Q)$ of the operator Q, which consists of all functions $g \in X(\Omega \times [0,1])$ representable in the form of a series

$$g(t,\omega_1,\omega_2,\dots) = \sum_{k=1}^{\infty} g_k(\omega_k) r_k(t)$$

convergent almost everywhere in $\Omega \times [0,1]$, is complemented in the space $X(\Omega \times [0,1])$.

We define a sequence of conditional expectations on $L_1(\Omega \times [0,1])$ by

$$\mathsf{E}(g \mid \omega_k)(\omega_k) := \biggl(\int_0^1\int_0^1 \cdots\biggr) g(t,\omega_1,\omega_2,\dots)\, dt\, d\omega_1 \cdots d\omega_{k-1}\, d\omega_{k+1} \cdots \qquad (k \geqslant 1),$$

and we define the operator

$$Pg(t,\omega_1,\omega_2,\dots) := \sum_{k=1}^{\infty} \mathsf{E}(gr_k|\omega_k) r_k(t).$$

Let

$$\langle f,g\rangle := \biggl(\int_0^1\int_0^1 \cdots\biggr) f(t,\omega_1,\omega_2,\dots) g(t,\omega_1,\omega_2,\dots)\, dt\, d\omega_1\, d\omega_2 \cdots .$$

One can easily check that for any $x, y \in L_2(0,\infty)$

$$\langle Qx, Qy\rangle = \int_0^{\infty} x(t)y(t)\, dt \tag{95}$$

and that the operator P is self-adjoint, that is, for every $f, g \in L_2(\Omega \times [0,1])$

$$\langle Pf, g\rangle = \langle f, Pg\rangle. \tag{96}$$

Let us prove that P is bounded on $X(\Omega\times[0,1])$. To this end, we first check that for every $f \in L_\infty(\Omega \times [0,1])$

$$x(t) := \sum_{k=1}^{\infty} \mathsf{E}(fr_k|\omega_k)(t+k-1)\chi_{[k-1,k)}(t) \in L_\infty \cap L_2(0,\infty). \tag{97}$$

First, it is easy to see that $|\mathsf{E}(fr_k|\omega_k)| \leqslant \|f\|_\infty$ $(k = 1, 2, \dots)$, and hence $x \in L_\infty$. Furthermore, from the obvious inequality

$$\biggl(\int_0^1 \cdots \biggl(\int_0^1 f(u, \omega_1, \dots, \omega_{k-1}, t, \omega_{k+1}, \dots) r_k(u)\, du\biggr)\, d\omega_1 \cdots d\omega_{k-1}\, d\omega_{k+1} \cdots\biggr)^2$$
$$\leqslant \int_0^1 \cdots \biggl(\int_0^1 f(u, \omega_1, \dots, \omega_{k-1}, t, \omega_{k+1}, \dots) r_k(u)\, du\biggr)^2 d\omega_1 \cdots d\omega_{k-1}\, d\omega_{k+1} \cdots,$$

which holds for arbitrary $0 \leqslant t \leqslant 1$ and $k = 1, 2, \dots$, and from the Bessel inequality it follows that

$$\begin{aligned}\|x\|^2_{L_2(0,\infty)} &= \sum_{k=1}^\infty \int_0^1 \biggl(\int_0^1 \cdots f(u, \omega_1, \dots, \omega_{k-1}, t, \omega_{k+1}, \dots) \\ &\qquad\qquad \times r_k(u)\, du\, d\omega_1 \cdots d\omega_{k-1}\, d\omega_{k+1} \cdots\biggr)^2 dt \\ &\leqslant \biggl(\int_0^1\int_0^1 \cdots\biggr) \sum_{k=1}^\infty \biggl(\int_0^1 f(u, \omega_1, \omega_2, \dots, \omega_k, \dots) r_k(u)\, du\biggr)^2 d\omega_1\, d\omega_2 \cdots d\omega_k \cdots \\ &\leqslant \biggl(\int_0^1\int_0^1 \cdots\biggr) f(u, \omega_1, \omega_2, \dots, \omega_k, \dots)^2\, du\, d\omega_1\, d\omega_2 \cdots = \|f\|_2^2 \leqslant \|f\|_\infty^2.\end{aligned}$$

Thus, (97) is proved, and hence $x \in Z_X^2$. In addition, $Pf = Qx$. Since by assumption X is separable or has the Fatou property, it is easy to check that the space Z_X^2 has the same property. Moreover, $(Z_X^2)^\times = Z_{X^\times}^2$. By the second assumption of the theorem, $\mathscr{K} : X \to X$. Then by Theorem 25,

$$\begin{aligned}\|Pf\|_{X(\Omega\times[0,1])} &= \|Qx\|_{X(\Omega\times[0,1])} \leqslant C\|x\|_{Z_X^2} \\ &= C\sup\biggl\{\biggl|\int_0^\infty x(t)y(t)\, dt\biggr| : \|y\|_{Z^2_{X^\times}} \leqslant 1\biggr\}.\end{aligned}$$

Since $\mathscr{K}$ is also bounded on $X^\times$, it follows from (95), (96), and Theorem 25 (applied to $X^\times$) that

$$\begin{aligned}\|Pf\|_{X(\Omega\times[0,1])} &\leqslant C\sup\{|\langle Qx, Qy\rangle| : \|y\|_{Z^2_{X^\times}} \leqslant 1\} \\ &= C\sup\{|\langle Pf, Qy\rangle| : \|y\|_{Z^2_{X^\times}} \leqslant 1\} = C\sup\{|\langle f, Qy\rangle| : \|y\|_{Z^2_{X^\times}} \leqslant 1\} \\ &\leqslant C\sup\{\|Qy\|_{X^\times(\Omega\times[0,1])} : \|y\|_{Z^2_{X^\times}} \leqslant 1\}\, \|f\|_{X(\Omega\times[0,1])} \leqslant CC'\|f\|_{X(\Omega\times[0,1])}.\end{aligned}$$

As a result, for all $f \in L_\infty(\Omega \times [0,1])$

$$\|Pf\|_{X(\Omega\times[0,1])} \leqslant CC'\|f\|_{X(\Omega\times[0,1])}.$$

Using the fact that X is separable or has the Fatou property and applying standard arguments, we infer that the projection P is bounded on this space. In addition, it is easy to see that its image $\mathrm{Im}(P)$ coincides with the image $\mathrm{Im}(Q)$ of Q. Hence, the subspace $\mathrm{Im}(Q)$ (which is isomorphic to Z_X^2 by the first part of the theorem) is complemented in $X(\Omega \times [0,1])$, which is isometric to X. On the other hand,

X is clearly isometric to a complemented subspace of the space Z_X^2. Since $X \oplus X$ ($Z_X^2 \oplus Z_X^2$, respectively) is isomorphic to X (Z_X^2, respectively), the Pełczyński decomposition method ([17], p. 172) yields the existence of an isomorphism between X and Z_X^2.

The following result proved in [52] shows that the Poisson stochastic integral, which is a key tool in the constructions in § 8 of [38] and § 2.f of [17], and the Kruglov operator $\mathscr{K}$ are closely related.

The stochastic integration operator with respect to a symmetrized Poisson process (we denote it by T) is defined on functions measurable on the semi-axis $(0,\infty)$ ([17], pp. 205-206). If f is a function defined on $[0,1]$, then $Tf := T'f - T''f$, where

$$T'f(u,v) := \sum_{k=1}^{\infty} \chi_{F'_k}(u) \sum_{j=1}^{k} f(\varphi'_j(v)) \qquad (u,v \in [0,1]).$$

Here $\{F'_k\}_{k\geqslant 0}$ is a partition of $[0,1]$ into disjoint subsets such that $\lambda(F'_k) = 1/(ek!)$, $k = 0,1,2,\dots$, and $\{\varphi'_j\}_{j=1}^{\infty}$ is a sequence of uniformly distributed independent r.v.'s on $[0,1]$. The function $T''f$ is defined similarly, under the assumption that the corresponding sequences $\{F''_k\}_{k\geqslant 0}$ and $\{\varphi''_j\}_{j=1}^{\infty}$ are independent with respect to the sequences $\{F'_k\}_{k\geqslant 0}$ and $\{\varphi'_j\}_{j=1}^{\infty}$, respectively. Thus, the functions $T'f$ and $T''f$ are identically distributed and independent. In the case of a function f defined on $(0,\infty)$, we consider the functions $f_n := f\chi_{(n-1,n)}$ $(n = 1,2,\dots)$. If functions b_n defined on $[0,1]$ are independent copies of f_n, then by definition, $Tf := \sum_{n=1}^{\infty} Tf_n$, where the functions Tf_n are independent copies of the functions Tb_n $(n = 1,2,\dots)$.

The following characterization of symmetric spaces X such that the operator T acts boundedly from Z_X^2 to X was obtained in [52] using the Kruglov operator. This strengthens the results in § 8 of [38] (see also [17], 2.f), where the sufficient condition $\alpha_X > 0$ was found.

Theorem 38. *If X is a symmetric space on $[0,1]$ which is separable or has the Fatou property, then the operator T is bounded from Z_X^2 to X if and only if the operator $\mathscr{K}$ is bounded on X.*

Proof. Suppose first that T is bounded from Z_X^2 to X. If f is a function defined on $[0,1]$, then the sequence $\{f(\varphi'_j)\}_{j=1}^{\infty}$ from the definition of the operator T consists of independent r.v.'s having the same distribution as f. Hence (see (33)) the r.v.'s $T'f$ and $\mathscr{K}f$ are identically distributed, and so $\|T'f\|_X = \|\mathscr{K}f\|_X$. Furthermore, due to the independence of $T'f$ and $T''f$, for any $\tau > 0$ we have

$$\begin{aligned}\lambda^2\{(u,v) : |Tf(u,v)| > \tau\} &\geqslant \lambda^2\{(u,v) : |T'f(u,v)| > \tau \text{ and } u \in F''_0\} \\ &= e^{-1}\lambda^2\{(u,v) : |T'f(u,v)| > \tau\},\end{aligned}$$

and thus $\|\mathscr{K}f\|_X = \|T'f\|_X \leqslant e\|Tf\|_X$ by Proposition 1. Hence if the operator $T\colon Z_X^2 \to X$ is bounded, then so is the operator $\mathscr{K} : X \to X$.

Conversely, suppose that $\mathscr{K} : X \to X$ is bounded. If f is defined on $[0,1]$, then as before,

$$\|Tf\|_X = \|T'f - T''f\|_X \leqslant 2\|\mathscr{K}f\|_X. \tag{98}$$

Now let $f \in Z_X^2$ be arbitrary. Since the operators T' and T'' are positive, by the definition of T ([17], p. 207) we can assume that f is non-negative. Let

$f_n := f\chi_{(n-1,n)}$ $(n = 1, 2, \dots)$. It is easy to see that there exist sequences $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ such that $f_n = g_n + h_n$, $g_n h_n = 0$ and $\int_{n-1}^n g_n(t)\,dt = \int_{n-1}^n h_n(t)\,dt$ $(n = 1, 2, \dots)$. Therefore, if $f_n' := g_n - h_n$, then $|f_n'| = f_n$ and $\int_{n-1}^n f_n'(t)\,dt = 0$, $n = 1, 2, \dots$.

Further, let the functions b_n be independent copies on $[0, 1]$ of the functions f_n' $(n = 1, 2, \dots)$. By Theorem 25 we have

$$\biggl\|\sum_{k=1}^n b_k\biggr\|_X \leqslant C\biggl\|\sum_{k=1}^n f_k'\biggr\|_{Z_X^2} = C\biggl\|\sum_{k=1}^n f_k\biggr\|_{Z_X^2}. \tag{99}$$

Moreover, by the definition of T ([17], p. 207) it follows that $\{Tf_n\}_{n=1}^\infty$ and $\{Tb_n\}_{n=1}^\infty$ are sequences of independent symmetrically distributed r.v.'s and that Tf_n and Tb_n are identically distributed for every $n = 1, 2, \dots$. Therefore, in view of (98) and (99) and the boundedness of the operator $\mathscr{K}$ on the space X, we get that

$$\begin{aligned}\biggl\|T\biggl(\sum_{k=1}^n f_k\biggr)\biggr\|_X &= \biggl\|\sum_{k=1}^n Tf_k\biggr\|_X = \biggl\|\sum_{k=1}^n Tb_k\biggr\|_X = \biggl\|T\biggl(\sum_{k=1}^n b_k\biggr)\biggr\|_X \\ &\leqslant 2\biggl\|\mathscr{K}\biggl(\sum_{k=1}^n b_k\biggr)\biggr\|_X \leqslant 2\|\mathscr{K}\|_{X\to X}\biggl\|\sum_{k=1}^n b_k\biggr\|_X \\ &\leqslant 2\|\mathscr{K}\|_{X\to X}C\biggl\|\sum_{k=1}^n f_k\biggr\|_{Z_X^2} \leqslant 2\|\mathscr{K}\|_{X\to X}C\|f\|_{Z_X^2} \quad (n = 1, 2, \dots).\end{aligned}$$

Since X is separable or has the Fatou property, we conclude by standard arguments that the operator $T\colon Z_X^2 \to X$ is bounded.

By Theorems 38 and 2.f.1(i) in [17] we obtain the following result.

Corollary 10. *If a symmetric space X is separable or has the Fatou property and $X \in \mathbb{K}$, then the image of the stochastic integration operator with respect to the symmetrized Poisson process is isomorphic to the space Z_X^2.*

10. Hilbert subspaces of symmetric spaces spanned by independent functions

The problem of describing subspaces of symmetric spaces spanned by sequences of independent functions has been considered in numerous papers (see, for example, [34], [39], [84]). We shall show that the Kruglov operator can be rather effectively used for studying this problem.

Let X be a symmetric space on $[0, 1]$. Let us consider the question as to what conditions should be imposed on X so that every sequence of independent identically distributed functions in X spans a Hilbert subspace of it, that is, for some constant $C > 0$ (possibly depending on f_1) and for every such sequence $\{f_k\}_{k=1}^\infty \subset X$ and all $(a_k)_{k=1}^\infty \in l_2$

$$C^{-1}\biggl(\sum_{k=1}^\infty a_k^2\biggr)^{1/2} \leqslant \biggl\|\sum_{k=1}^\infty a_k f_k\biggr\|_X \leqslant C\biggl(\sum_{k=1}^\infty a_k^2\biggr)^{1/2}.$$

Note that the left-hand inequality in this expression holds in an arbitrary symmetric space X ([39], Lemma 1, p. 52). Moreover, it is not difficult to show ([85], Theorem 3.4) that the right-hand inequality holds if and only if there is a constant $C>0$ independent of f_1 such that

$$\biggl\|\sum_{k=1}^{\infty} a_k f_k\biggr\|_X \leqslant C\|f_1\|_X \biggl(\sum_{k=1}^{\infty} a_k^2\biggr)^{1/2} \tag{100}$$

for every sequence of independent identically distributed functions $\{f_k\}_{k=1}^{\infty} \subset X$ and all $(a_k)_{k=1}^{\infty} \in l_2$. The following necessary conditions for (100) to hold are listed in [39], p. 71:

(a) the space X contains the space $(\exp L_2)_\circ$;

(b) $\int_0^1 f_1(x)\,dx = 0$;

(c) $f_1 \in L_2$.

The first of them is an immediate consequence of the Rodin–Semenov theorem (see the beginning of § 7 or [66]): it is sufficient to take the functions f_k to be the Rademacher functions. Since the functions f_k are identically distributed, the second condition follows from the estimate

$$Cn^{1/2} \geqslant \biggl\|\sum_{k=1}^{n} f_k\biggr\|_X \geqslant \mathsf{E}\biggl|\sum_{k=1}^{n} f_k\biggr| \geqslant \biggl|\mathsf{E}\sum_{k=1}^{n} f_k\biggr| = n|\mathsf{E}f_1| \qquad (n\in\mathbb{N}).$$

Let us check the condition (c). For an arbitrary sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of independent identically distributed functions we set $g_k := f_{2k} - f_{2k-1}$ $(k=1,2,\dots)$. Then the functions g_k are symmetrically and identically distributed and independent. Hence, (100) also holds for them. For each $a>0$ we define $g_{k,a} := g_k\chi_{\{|g_k|<a\}}$ and $h_{k,a} := g_k - 2g_{k,a}$. Since the g_k are symmetrically distributed, the functions $h_{k,a}$ and g_k are identically distributed. Hence,

$$\mathsf{E}\biggl|\sum_{k=1}^{n} g_{k,a}\biggr| = \frac{1}{2}\,\mathsf{E}\biggl|\sum_{k=1}^{\infty}(g_k - h_{k,a})\biggr| \leqslant \mathsf{E}\biggl|\sum_{k=1}^{n} g_k\biggr| \leqslant C_1 n^{1/2}.$$

Let $\sigma^2(a) := \mathsf{E}g_{1,a}^2$ and let γ be a Gaussian r.v. with $\mathsf{E}\gamma = 0$ and $\mathsf{E}\gamma^2 = 1$. Then by the central limit theorem [50], $\lim_{n\to\infty}\mathsf{E}\bigl|n^{-1/2}\sum_{k=1}^{n} g_{k,a}\bigr| = \sigma(a)\mathsf{E}|\gamma|$, whence by the preceding inequality we have $\sigma(a)\mathsf{E}|\gamma| \leqslant C_1$ for every $a>0$. Passing to the limit as $a\to\infty$, we get that $\mathsf{E}g_1^2 < \infty$, and this means that $\mathsf{E}f_1^2 < \infty$.

To state the first result about sufficient conditions guaranteeing (100), we need some notation from [39], § 3.2. For an arbitrary scalar sequence $a = (a_k)_{k=1}^{\infty}$ and for a measurable function f on $[0,1]$, we set

$$Q_a f(t) = \sum_{k=1}^{\infty} \lambda\{s\in[0,1] : |a_k f(s)| > t\}, \qquad t>0. \tag{101}$$

Let X be a symmetric space on $[0,1]$. We say that the function f has the property $A_2(X)$ $(f\in A_2(X))$ if for every $a\in\ell_2$ the space X contains all functions g such that $\lambda\{s\in[0,1] : |g(s)|>t\} \leqslant CQ_a f(t)$ $(t>0)$ for some $C>0$.

Theorem 39 ([39], Theorem 3.7). *If X is a symmetric space in $\mathbb{K}$ and $\{f_k\}_{k=1}^\infty$ is a sequence of independent identically distributed functions such that $f_1 \in A_2(X)$, $f_1 \in L_2$, and $\int_0^1 f_1(x)\,dx = 0$, then (100) holds.*

The proof of the last statement in § 3.2 of [39] is based on the theory of infinitely divisible distributions and it is quite long and not easy. At the same time, it is not difficult to see that Theorem 39 is an immediate consequence of Theorem 25 proved earlier (see details in [85]). Here we shall prove another result from [85] demonstrating the effectiveness of the operator approach to the treatment of similar problems: a simple interpolation condition guaranteeing the validity of the embedding $X \subset A_2(X)$ will be stated.

Theorem 40. *Let X be a symmetric space on $[0,1]$ which is an interpolation space with respect to the couple (L_2, L_∞), with $X \in \mathbb{K}$. Then there exists a $C > 0$ such that for any sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent identically distributed functions with $\int_0^1 f_1(x)\,dx = 0$ and for any $(a_k)_{k=1}^\infty \in \ell_2$ the inequality (100) holds.*

Proof. For an arbitrary fixed sequence $a = (a_k)_{k=1}^\infty \in \ell_2$ we define the linear operator $T_a \colon S(0,1) \to S(0,\infty)$ by

$$T_a f(t) = \sum_{k=1}^{\infty} a_k f(t-k+1)\chi_{(k-1,k]}(t).$$

Since for every function $f \in L_2$ ($f \in L_\infty$, respectively) $\|T_a f\|_2 = \|a\|_2\,\|f\|_2$ ($\|T_a f\|_\infty = \sup_k |a_k|\,\|f\|_\infty \leqslant \|a\|_2\,\|f\|_\infty$, respectively), in view of the equalities

$$Z_{L_2{}^2} = L_2(0,\infty) \quad \text{and} \quad Z^2_{L_\infty} = L_\infty \cap L_2(0,\infty)$$

we get that

$$\begin{aligned} \|T_a f\|_{Z^2_{L_2}} &= \|a\|_2\,\|f\|_2 \qquad && (f \in L_2[0,1]), \\ \|T_a f\|_{Z^2_{L_\infty}} &\leqslant 2\|a\|_2\,\|f\|_\infty \qquad && (f \in L_\infty[0,1]). \end{aligned} \tag{102}$$

In other words, T_a acts boundedly from $L_2[0,1]$ to $Z^2_{L_2}$ and from $L_\infty[0,1]$ to $Z^2_{L_\infty}$. In order to 'interpolate' the relations (102), extending them to an arbitrary symmetric space $X \in \operatorname{Int}(L_2, L_\infty)$, we need two interpolation lemmas, which we only state here (see the proof of the first one in [86], Lemma 4, and that of the second one in [85], Lemma 3.3).

Lemma 18. *For any Banach couple (X_0, X_1) and any parameter Φ of the real $\mathscr{K}$-method the following equality holds*:

$$(X_0, X_0 \cap X_1)^{\mathscr{K}}_{\Phi} = (X_0, X_1)^{\mathscr{K}}_{\Phi} \cap X_0.$$

Since the couple (L_2, L_∞) is $\mathscr{K}$-monotone [87] and $X \in \operatorname{Int}(L_2, L_\infty)$, by the Brudnyi–Krugljak theorem (see § 2 or [22], Theorem 3.3.20) there exists a parameter Φ of the real $\mathscr{K}$-method such that

$$X = (L_2, L_\infty)^{\mathscr{K}}_{\Phi}. \tag{103}$$

Lemma 19. *If the equality* (103) *holds, then up to equivalence of norms*

$$Z_X^2(0,\infty) = \big(L_2(0,\infty), L_\infty(0,\infty)\big)_\Phi^{\mathscr{K}} \cap L_2(0,\infty).$$

Let us continue the proof of Theorem 40. In view of (102) and Lemmas 18 and 19 there exists a constant $C_1 > 0$ depending only on the space X such that for arbitrary $f \in X$ and $a \in \ell_2$

$$\|T_a f\|_{Z_X^2} \leqslant C_1 \|a\|_2 \, \|f\|_X. \tag{104}$$

On the other hand, since $X \in \mathbb{K}$ by assumption, we can use Theorem 25. Thus, for an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions with $\int_0^1 f_k(x)\,dx = 0$ $(k = 1, 2, \dots)$, and for any $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ we have

$$\bigg\|\sum_{k=1}^n a_k f_k\bigg\|_X \leqslant \beta \|\mathscr{K}\|_{X\to X} \bigg\|\sum_{k=1}^n a_k \mathbf{f}_k\bigg\|_{Z_X^2}, \tag{105}$$

where $\beta > 0$ is some universal constant. Since the functions f_k $(k = 1, 2, \dots)$ are identically distributed, the functions $\sum_{k=1}^n a_k \mathbf{f}_k$ and $T_{a^n} f_1$, where $a^n = (a_k^n)$, $a_k^n = a_k$ for $k \leqslant n$, and $a_k^n = 0$ for $k > n$, have the same property. Thus, from (104) and (105) it follows that

$$\bigg\|\sum_{k=1}^n a_k f_k\bigg\|_X \leqslant C_1 \beta \|\mathscr{K}\|_{X\to X} \|f_1\|_X \, \|a\|_2, \qquad n = 1, 2, \dots .$$

The right-hand estimate in (100) can now be proved by standard arguments.

Corollary 11. *Let X be a symmetric space on $[0,1]$ such that the inequality* (100) *holds for any sequence $\{f_k\}_{k=1}^\infty \subset X$ of identically distributed independent functions with $\int_0^1 f_1(x)\,dx = 0$. Then there exists a constant $C > 0$ such that for any sequence $\{g_k\}_{k=1}^n \in X$ of identically distributed disjoint functions with $\|g_1\|_X = 1$, any $n \in \mathbb{N}$, and any $a_k \in \mathbb{R}$ $(k = 1, 2, \dots, n)$ the following inequality holds*:

$$\bigg\|\sum_{k=1}^n a_k g_k\bigg\|_X \leqslant C \bigg(\sum_{k=1}^n a_k^2\bigg)^{1/2}. \tag{106}$$

In particular,

$$\bigg\|\sum_{k=1}^n a_k \chi_{[(k-1)/n, k/n]}\bigg\|_X \leqslant C \phi_X\Big(\frac{1}{n}\Big) \bigg(\sum_{k=1}^n a_k^2\bigg)^{1/2}. \tag{107}$$

Proof. It is not difficult to see that there exist sets $\{g_k^+\}_{k=1}^n$ and $\{g_k^-\}_{k=1}^n$ of identically distributed functions such that $|g_k| = g_k^+ + g_k^-$, $g_k^+ g_k^- = 0$, and $\int_0^1 g_k^+(x)\,dx = \int_0^1 g_k^-(x)\,dx$ $(k = 1, \dots, n)$. Then the functions $g_k' := g_k^+ - g_k^-$ $(k = 1, \dots, n)$ are identically distributed and $\int_0^1 g_1(x)\,dx = 0$. Let $\{f_k\}_{k=1}^n$ be a sequence of independent copies of the functions g_k'. Then $\|f_1\|_X = \|g_1'\|_X = \|g_1\|_X = 1$, and by Theorem 40

$$\bigg\|\sum_{k=1}^n a_k f_k\bigg\|_X \leqslant C \bigg(\sum_{k=1}^n a_k^2\bigg)^{1/2} \tag{108}$$

for any $a_1, \dots, a_n \in \mathbb{R}$, where the constant $C > 0$ depends only on X. In addition, by the first part of Theorem 9 (see also [39], Lemma 5, pp. 14–15) we get that

$$\biggl\| \sum_{k=1}^{n} a_k f_k \biggr\|_X \geqslant \frac{1}{4} \biggl\| \sum_{k=1}^{n} a_k g_k' \biggr\|_X = \frac{1}{4} \biggl\| \sum_{k=1}^{n} a_k g_k \biggr\|_X . \tag{109}$$

The inequality (106) follows from (108) and (109), and (107) is a special case of (106).

Corollary 12. *If $X \overset{\neq}{\subset} L_2$ is a symmetric space and $\phi_X(u) = u^{1/2}$, then there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of identically distributed independent functions with $\int_0^1 f_1(x)\,dx = 0$ which spans a subspace of X not isomorphic to ℓ_2.*

Proof. Suppose that every sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of identically distributed independent functions with $\int_0^1 f_1(x)\,dx = 0$ spans in X a subspace isomorphic to ℓ_2. Then in view of the observation made at the beginning of this section and also by the preceding corollary, we obtain the inequality (107), which is equivalent to the inequality

$$\biggl\| \sum_{k=1}^{n} a_k \chi_{[(k-1)/n, k/n]} \biggr\|_X \leqslant C \biggl\| \sum_{k=1}^{n} a_k \chi_{[(k-1)/n, k/n]} \biggr\|_{L_2}$$

where $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$ $(k = 1, \dots, n)$ are arbitrary. Hence it follows immediately that $L_2 \subseteq X$. At the same time, by the assumption of the corollary, we have $X \subseteq L_2$, that is, $X = L_2$, which contradicts to the assumption.

Remark 12. In particular, if $X = L_{2,q}$ $(1 \leqslant q < 2)$ (see the definition in § 2), then, in view of the preceding corollary, there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of identically distributed independent functions with $\int_0^1 f_1(x)\,dx = 0$ which spans a subspace of X not isomorphic to ℓ_2. Thus, the above conditions (a)–(c) on a symmetric space X are not sufficient for (100) to hold for every sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of identically distributed independent functions with $\int_0^1 f_1(x)\,dx = 0$. This provides an answer in the negative to the question raised on p. 71 of the book [39].

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