

Rademacher series and isomorphisms of rearrangement invariant spaces on the finite interval and on the semi-axis

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Received 10 March 2010; accepted 20 August 2010

Communicated by N. Kalton

Abstract

Let X be a rearrangement invariant function space on $[0, 1]$. We consider the subspace $\text{Radi } X$ of X which consists of all functions of the form $f = \sum_{k=1}^{\infty} x_k r_k$, where x_k are arbitrary independent functions from X and r_k are usual Rademacher functions independent of $\{x_k\}$. We prove that $\text{Radi } X$ is complemented in X if and only if both X and its Köthe dual space X' possess the so-called Kruglov property. As a consequence we show that the last conditions guarantee that X is isomorphic to some rearrangement invariant function space on $[0, \infty)$. This strengthens earlier results derived in different approach in [W.B. Johnson, B. Maurey, G. Schechtman, L. Tzafriri, Symmetric structures in Banach spaces, Mem. Amer. Math. Soc. 1 (217) (1979)].

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Keywords: Rademacher functions; Rearrangement invariant spaces; Isomorphism of Banach spaces; Kruglov property

0. Introduction

Let $r_k(t) = \text{sign} \sin 2^k \pi t$ ($k = 1, 2, \dots$) be the Rademacher functions on the interval $[0, 1]$. By classical Khintchine inequality [13], for each $p > 0$ there exist positive constants A_p and B_p

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such that for arbitrary real a_k ($k = 1, 2, \dots$) we have

$$A_p \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{L_p[0,1]} \leq B_p \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2}.$$

This result caused a great number of investigations and generalizations and found many applications in various areas of analysis (see, for instance, [19,3]). In particular, in 1975, Rodin and Semenov [20] proved that the inequality

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \leq C \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2}$$

holds in a rearrangement invariant (r.i.) space X on $[0, 1]$ if and only if X contains the space G , the closure of L_∞ in the Orlicz space $\exp L_2$ generated by the function $\exp(u^2) - 1$ ($u > 0$). It was proved also that the closed linear span $[r_n]$ is complemented in X if and only if $G \subset X \subset G'$, where G' is the Köthe dual space to G [21], [17, Theorem 2.b.4(ii)]. Moreover, Lindenstrauss and Tzafriri [17, Proposition 2.d.1] showed that the analogous relation for series with arbitrary coefficients x_k from an r.i. space X

$$\left\| \sum_{k=1}^{\infty} x_k(s) r_k(t) \right\|_{X([0,1] \times [0,1])} \leq C \left\| \left(\sum_{k=1}^{\infty} x_k^2 \right)^{1/2} \right\|_X \quad (0.1)$$

holds provided that the lower Boyd index α_X of X is positive. They proved as well that in the case when the Boyd indices of X are non-trivial, i.e., $0 < \alpha_X \leq \beta_X < 1$, the subspace $\text{Rad } X$ of all functions defined on the square $[0, 1] \times [0, 1]$ and representable in the form

$$f(s, t) = \sum_{k=1}^{\infty} x_k(s) r_k(t), \quad x_k \in X \quad (\text{the series converges a.e. on } [0, 1] \times [0, 1]) \quad (0.2)$$

is complemented in $X([0, 1] \times [0, 1])$ [17, Proposition 2.d.2]. Later, Astashkin and Braverman [4] have proved the opposite statements: if (0.1) is fulfilled for arbitrary $x_k \in X$, with some $C > 0$, then $\alpha_X > 0$, and if $\text{Rad } X$ is a complemented subspace of $X([0, 1] \times [0, 1])$, then $0 < \alpha_X \leq \beta_X < 1$.

In [1] (see also [2]), the analogous problem was considered in the case of Rademacher series with independent coefficients. More precisely, there it was proved that inequality (0.1) holds for a some constant $C > 0$ and for every sequence of *independent* functions $\{f_k\}_{k=1}^{\infty} \subset X$ if and only if the r.i. space X has the so-called Kruglov property (which is less restrictive than the condition $\alpha_X > 0$; see the next section). In this paper we solve the problem of complementability of the subspace $\text{Radi } X$ consisting of all functions $f(s, t)$ which can be represented in the form (0.2), where x_k are arbitrary independent functions from an r.i. space X . We prove that $\text{Radi } X$ is a complemented subspace in X if and only if both X and its Köthe dual X' possess the Kruglov property (Theorem 2.1).

The last result can be applied to construct isomorphisms between r.i. spaces on the finite interval and on the semi-axis. The general problem of existence of such isomorphisms in the case

of r.i. spaces other than L_p -spaces was first posed by Mityagin in [18]. This and other related problems were extensively studied in [11] (see also [17]) via the approach using a stochastic integral with respect to a symmetrized Poisson process. We will follow here an other approach established on applying the Kruglov property of an r.i. space [7]. It is technically rather simpler; a somewhat similar approach was appeared earlier in a special case of r.i. (Lorentz) spaces $L_{p,q}$ [10]. We show that an r.i. space X on $[0, 1]$ is isomorphic to some r.i. space on the semi-axis provided that both X and its Köthe dual space X' possess the Kruglov property (Theorem 2.4). This strengthens the analogous results of [11, §8] (see also [17, p. 203]) proved under a stronger condition when an r.i. space X has non-trivial Boyd indices.

1. Preliminaries

1.1. Rearrangement invariant spaces

Detailed exposition of theory of rearrangement invariant spaces see in [15,17,8].

A Banach space X of real-valued Lebesgue-measurable functions on the interval $[0, \alpha)$, where $0 < \alpha \leq \infty$, is called *rearrangement invariant (r.i.)* if from the conditions $y \in X$ and $x^*(t) \leq y^*(t)$ a.e. on $[0, \alpha)$ it follows that $x \in X$ and $\|x\|_X \leq \|y\|_X$. Here and throughout, λ is the Lebesgue measure and $x^*(t)$ is the right-continuous non-increasing *rearrangement* of $|x(s)|$, i.e.,

$$x^*(t) := \inf\{\tau \geq 0: \lambda\{s \in [0, \alpha): |x(s)| > \tau\} < t\} \quad (t > 0).$$

If X is an r.i. space on the interval $[0, \alpha)$, then *the Köthe dual space* X' consists of all measurable functions y such that

$$\|y\|_{X'} = \sup\left\{\int_0^\alpha x(t)y(t) dt: \|x\|_X \leq 1\right\} < \infty.$$

The space X' is r.i. as well; it is embedded into the dual space X^* of X isometrically, and $X' = X^*$ if and only if X is separable. An r.i. space X is said *to have the Fatou property* if the conditions $x_n \in X$ ($n = 1, 2, \dots$), $\sup_{n=1,2,\dots} \|x_n\|_X < \infty$, and $x_n \rightarrow x$ a.e. imply that $x \in X$ and $\|x\|_X \leq \liminf_{n \rightarrow \infty} \|x_n\|_X$. X has the Fatou property if and only if the natural embedding of X into its second Köthe dual X'' is an isometric surjection. In what follows we suppose that every r.i. space X has the Fatou property or separable.

For each r.i. space X on $[0, 1]$ we have the continuous embeddings $L_\infty \subset X \subset L_1$. By $X_\circ$ we will denote *the separable part* of X , i.e., the closure of L_∞ in X ; $X_\circ$ is an r.i. space which is separable provided that $X \neq L_\infty$.

If $\tau > 0$ then *the dilation operator* $\sigma_\tau x(t) := x(t/\tau) \cdot \chi_{[0, \min(1, \tau)]}(t)$ is bounded in every r.i. space X and $\|\sigma_\tau\|_{X \rightarrow X} \leq \max(1, \tau)$. The numbers

$$\alpha_X = \lim_{\tau \rightarrow 0+} \frac{\ln \|\sigma_\tau\|_{X \rightarrow X}}{\ln \tau} \quad \text{and} \quad \beta_X = \lim_{\tau \rightarrow +\infty} \frac{\ln \|\sigma_\tau\|_{X \rightarrow X}}{\ln \tau}$$

are called the *Boyd indices* of X ; we always have $0 \leq \alpha_X \leq \beta_X \leq 1$.

Important examples of r.i. spaces are L_p -spaces ($1 \leq p \leq \infty$) and their generalization, the Orlicz spaces [14]. Let $M(u)$ be an Orlicz function on $[0, \infty)$, that is, a continuous convex increasing function on $[0, \infty)$ such that $M(0) = 0$. Then the Orlicz space $L_M = L_M[0, 1]$ consists of all measurable functions $x(t)$ on $[0, 1]$ such that $\int_0^1 M(|x(t)|/\lambda) dt \leq 1$, for some $\lambda > 0$. The Luxemburg norm $\|x\|_{L_M} := \inf \lambda$, where the infimum is taken over all λ satisfying the last inequality. An Orlicz space L_M always has the Fatou property and it is separable if and only if the function M satisfies the Δ_2 -condition at infinity (i.e., there exist $u_0 > 0$ and $C > 0$ such that $M(2u) \leq CM(u)$ for all $u > u_0$). A special interest for us will be classical exponential Orlicz spaces. The space $\exp L^q$, $q > 0$, is generated by an Orlicz function equivalent to the function $e^{t^q} - 1$ for sufficiently small $t > 0$.

Following [11] (see also [17, 2.f]), for each r.i. space X on $[0, 1]$ we define Z_X^2 as the set of all measurable on $(0, \infty)$ functions f such that

$$\|f\|_{Z_X^2} := \|f^* \chi_{[0,1]}\|_X + \|f^* \chi_{[1,\infty)}\|_{L_2[1,\infty)} < \infty.$$

It can easily be shown that the quasinorm $\|\cdot\|_{Z_X^2}$ is equivalent to an r.i. norm, so that Z_X^2 is an r.i. space on $[0, \infty)$.

1.2. The Kruglov property

Let f be a measurable function on $[0, 1]$ (equivalently, a random variable). Denote by $\pi(f)$ the random variable $\sum_{i=1}^N f_i$, where f_i are independent copies of f (that is, independent random variables equidistributed with f) and N is a random variable independent of the sequence $\{f_i\}$ and having the Poisson distribution with parameter 1. In other words, $\pi(f)$ is equidistributed with the sum

$$\sum_{k=1}^{\infty} \sum_{i=1}^k f_i(s) \chi_{E_k}(t) \quad (0 \leq s, t \leq 1),$$

where E_k are disjoint subsets of $[0, 1]$, $\lambda(E_k) = 1/(ek!)$ ($k = 1, 2, \dots$). It is not hard to check that the characteristic function $\theta_{\pi(f)}(t)$ of $\pi(f)$ is equal to the function $\exp(\theta_f(t) - 1)$ for all $t \in \mathbb{R}$, where θ_f is the characteristic function of the random variable f .

The following property has its origin in Kruglov's paper [16] and was actively studied and used by Braverman [9].

Definition 1.1. We say that an r.i. space X on $[0, 1]$ has the *Kruglov property* ($X \in \mathbb{K}$) if the relation $f \in X$ implies that $\pi(f) \in X$.

Roughly speaking, an r.i. space X has the Kruglov property if it is situated sufficiently “far” from the space L_∞ . In particular, if X contains some L_p with $p < \infty$ or, all the more, if its lower Boyd index $\alpha_X > 0$, then $X \in \mathbb{K}$. However, this is not necessary; for instance, the exponential Orlicz spaces $\exp L^q$ do not contain L_p with any $p < \infty$ but $\exp L^q \in \mathbb{K}$ if and only if $0 < q \leq 1$ [9, §2.4], [5].

The Kruglov property is closely related to the well-known Rosenthal inequality [22] and to the problem of the comparison of sums of independent functions and their disjoint copies in r.i.

spaces [9,12]. Using an operator approach developed in [5,6], Astashkin and Sukochev proved that in every r.i. space X with the Kruglov property the inequality

$$\left\| \sum_{k=1}^{\infty} f_k \right\|_X \leq C \left\| \sum_{k=1}^{\infty} \mathbf{f}_k \right\|_{Z_X^2} \quad (1.1)$$

holds for some constant $C > 0$ and for each sequence of independent functions $\{f_k\}_{k=1}^{\infty} \subset X$ such that $\int_0^1 f_k(t) dt = 0$ ($k = 1, 2, \dots$) [7]. Here, $\mathbf{f}_k$ are disjoint copies of f_k defined on the semi-axis $[0, \infty)$ (for instance, we may take $\mathbf{f}_k(t) = f_k(t - k + 1)\chi_{[k-1, k)}(t)$, $k = 1, 2, \dots$). Earlier inequality (1.1) was proved in [12] under a stronger assumption that an r.i. space X contains some L_p with $p < \infty$.

2. Results and proofs

Let X be an r.i. space on $[0, 1]$. Equivalently, the set $\text{Radi } X$ (see Introduction) can be defined in the following way. Denote $\Omega = [0, 1]^{\infty}$ (respectively, $\bar{\Omega} = \Omega \times [0, 1]$) with the probability measure $\prod_{k=1}^{\infty} \lambda_k$ (respectively, $\prod_{k=0}^{\infty} \lambda_k$), where λ_k is the usual Lebesgue measure on $[0, 1]$. If X is an r.i. space on $[0, 1]$ then the space $X(\bar{\Omega})$ consists of all functions $f(t_0, t_1, \dots)$ measurable on $\bar{\Omega}$ such that the norm $\|f\|_{X(\bar{\Omega})} := \|f^*\|_X < \infty$. Then $\text{Radi } X$ is the set of all functions from $X(\bar{\Omega})$ which are representable in the form

$$f(t_0, t_1, \dots) = \sum_{k=1}^{\infty} x_k(t_k) r_k(t_0), \quad x_k \in X \text{ (the series converges a.e. on } \bar{\Omega}), \quad (2.1)$$

where, as above, r_k are usual Rademacher functions. It is easily to check that the functions $x_k(t_k) r_k(t_0)$, where $x_k \in X$ ($k = 1, 2, \dots$), form in $X(\bar{\Omega})$ an unconditional basic sequence with constant 1 [9, Prop. 1.14]. This implies immediately that $\text{Radi } X$ is a closed linear subspace of the r.i. space $X(\bar{\Omega})$. Since there exists a measure-preserving mapping from $([0, 1], \lambda)$ onto $(\bar{\Omega}, \prod_{k=0}^{\infty} \lambda_k)$, the spaces X and $X(\bar{\Omega})$ are isometric and we may (and will) identify them.

The following theorem is the main result of this paper.

Theorem 2.1. *Let X be an r.i. space on $[0, 1]$. Then $\text{Radi } X$ is a complemented subspace of X if and only if $X \in \mathbb{K}$ and $X' \in \mathbb{K}$.*

This theorem is an immediate consequence of the following two propositions.

Denoting

$$\Omega_k := \{(t_1, \dots, t_{k-1}, t_{k+1}, \dots) : 0 \leq t_i \leq 1, i = 1, \dots, k-1, k+1, \dots\} \quad (k \in \mathbb{N}),$$

we define on $L_1(\bar{\Omega})$ the sequence of following conditional expectations:

$$\mathbb{E}(f|t_k)(t_k) := \int_0^1 \int_{\Omega_k} f(t_0, t_1, \dots, t_{k-1}, t_k, t_{k+1}, \dots) dt_0 dt_1 \dots dt_{k-1} dt_{k+1} \dots \quad (k \geq 1).$$

Moreover, let $fr_k := f(t_0, t_1, \dots)r_k(t_0)$ ($k = 1, 2, \dots$) and

$$Pf(t_0, t_1, t_2, \dots) := \sum_{k=1}^{\infty} \mathbb{E}(fr_k|t_k)r_k(t_0).$$

Proposition 2.2. *The operator P is bounded in an r.i. space X if and only if $X \in \mathbb{K}$ and $X' \in \mathbb{K}$.*

Proof. Firstly, suppose that $X \in \mathbb{K}$ and $X' \in \mathbb{K}$ and prove that P is bounded in $X(\bar{\Omega})$. If $f \in L_{\infty}(\bar{\Omega})$, then the function

$$x(t) := \sum_{k=1}^{\infty} \mathbb{E}(fr_k|t_k)(t - k + 1)\chi_{[k-1, k)}(t) \quad \text{belongs to } L_{\infty} \cap L_2(0, \infty). \quad (2.2)$$

In fact, it is clear that $|\mathbb{E}(fr_k|t_k)| \leq \|f\|_{\infty}$ ($k = 1, 2, \dots$), whence $x \in L_{\infty}$. Moreover, from Minkowski and Bessel inequalities it follows

$$\begin{aligned} \|x\|_{L_2(0, \infty)}^2 &= \sum_{k=1}^{\infty} \int_0^1 \left(\int_{\Omega_k} \int_0^1 f(u, t_1, \dots, t_{k-1}, t, t_{k+1}, \dots)r_k(u) du dt_1 \dots dt_{k-1} dt_{k+1} \dots \right)^2 dt \\ &\leq \int_{\Omega} \sum_{k=1}^{\infty} \left(\int_0^1 f(u, t_1, t_2, \dots, t_k, \dots)r_k(u) du \right)^2 dt_1 dt_2 \dots dt_k \dots \\ &\leq \int_{\Omega} \int_0^1 f(u, t_1, t_2, \dots)^2 du dt_1 dt_2 \dots = \|f\|_{L_2(\bar{\Omega})}^2 \leq \|f\|_{L_{\infty}(\bar{\Omega})}^2. \end{aligned}$$

Thus, (2.2) is proved. Since $L_{\infty}[0, 1] \subset X$, then from (2.2) it follows that $x \in Z_X^2$. Furthermore, it is not hard to check that Z_X^2 has the Fatou property or separable just as X itself and that $(Z_X^2)' = Z_{X'}^2$. Note that $Pf = Qx$, where the linear operator Q is defined as follows

$$Qx(t_0, t_1, t_2, \dots) := \sum_{k=1}^{\infty} x_k(t_k)r_k(t_0),$$

where

$$x_k(t_k) = x(k - 1 + t_k) \quad (0 \leq t_k \leq 1). \quad (2.3)$$

Since $X \in \mathbb{K}$, then, by [7], inequality (1.1) holds and therefore Q is bounded from the space Z_X^2 into $X(\bar{\Omega})$. Thus,

$$\begin{aligned}
\|Pf\|_{X(\bar{\Omega})} &= \|Qx\|_{X(\bar{\Omega})} \leq C\|x\|_{Z_X^2} \\
&= C \sup \left\{ \left| \int_0^\infty x(t)y(t) dt \right| : \|y\|_{Z_{X'}^2} \leq 1 \right\}.
\end{aligned} \tag{2.4}$$

For any $f \in X(\bar{\Omega})$ and $g \in X'(\bar{\Omega})$ denote

$$\langle f, g \rangle := \int_{\bar{\Omega}} \int_0^1 f(t_0, t_1, \dots) g(t_0, t_1, \dots) dt_0 dt_1 \dots$$

It can be easily checked that for any $x \in Z_X^2$ and $y \in Z_{X'}^2$ we have

$$\langle Qx, Qy \rangle = \int_0^\infty x(t)y(t) dt \tag{2.5}$$

and that the projection P is self-conjugate, i.e., if $\langle f, Pg \rangle$ is well defined then

$$\langle Pf, g \rangle = \langle f, Pg \rangle. \tag{2.6}$$

Thus, using (2.4), (2.5) and (2.6), the condition $X' \in \mathbb{K}$ and again (1.1), we obtain that

$$\begin{aligned}
\|Pf\|_{X(\bar{\Omega})} &\leq C \sup \{ |\langle Qx, Qy \rangle| : \|y\|_{Z_{X'}^2} \leq 1 \} \\
&= C \sup \{ |\langle Pf, Qy \rangle| : \|y\|_{Z_{X'}^2} \leq 1 \} \\
&= C \sup \{ |\langle f, Qy \rangle| : \|y\|_{Z_{X'}^2} \leq 1 \} \\
&\leq C \sup \{ \|Qy\|_{X'} : \|y\|_{Z_{X'}^2} \leq 1 \} \cdot \|f\|_{X(\bar{\Omega})} \\
&\leq CC' \|f\|_{X(\bar{\Omega})}.
\end{aligned}$$

Since X is separable or has the Fatou property, then applying standard arguments (see also the proof of Proposition 2.3 below for this argument in the Fatou case) we are able to extend the last inequality to the whole X . Thus, P is bounded in $X(\bar{\Omega}) \approx X$.

Conversely, suppose that P is bounded in an r.i. space X . Taking into account (2.6), we see that it is bounded in X' as well, with the same norm. Denote $A := \|P\|_{X \rightarrow X} = \|P\|_{X' \rightarrow X'}$.

Assume that a function $f(s_0, s_1, \dots) = \sum_{k=1}^\infty x_k(s_k)r_k(s_0) \in \text{Radi } X$. Since $Pf = f$, then again by (2.6)

$$\begin{aligned}
\|f\|_{X(\bar{\Omega})} &= \sup \{ \langle f, g \rangle : \|g\|_{X'} \leq 1 \} \\
&= \sup \{ \langle f, Pg \rangle : \|g\|_{X'} \leq 1 \} \\
&\leq \sup \{ \langle f, g \rangle : g \in \text{Radi } X, \|g\|_{X'} \leq A \}.
\end{aligned} \tag{2.7}$$

If $g(s_0, s_1, \dots) = \sum_{k=1}^{\infty} y_k(s_k) r_k(s_0)$ ($y_k \in X'$), then it is easy to see that

$$\langle f, g \rangle = \int_{\Omega} \sum_{k=1}^{\infty} y_k(s_k) x_k(s_k) ds_1 ds_2 \dots$$

Therefore,

$$\begin{aligned} |\langle f, g \rangle| &\leq \int_{\Omega} \left(\sum_{k=1}^{\infty} x_k(s_k)^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} y_k(s_k)^2 \right)^{1/2} ds_1 ds_2 \dots \\ &\leq \left\| \left(\sum_{k=1}^{\infty} x_k(s_k)^2 \right)^{1/2} \right\|_{X(\Omega)} \left\| \left(\sum_{k=1}^{\infty} y_k(s_k)^2 \right)^{1/2} \right\|_{X'(\Omega)}. \end{aligned} \quad (2.8)$$

It is an easy consequence of Khintchine inequality (see also [17, Proposition 2.d.1]) that for an arbitrary r.i. space Y there is a constant $M(Y)$ such that

$$\left\| \left(\sum_{k=1}^{\infty} y_k^2 \right)^{1/2} \right\|_Y \leq M(Y) \left\| \sum_{k=1}^{\infty} y_k(t) r_k(s) \right\|_{Y([0,1] \times [0,1])}. \quad (2.9)$$

Therefore, from (2.7) and (2.8) it follows that

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} x_k(s_k) r_k(s_0) \right\|_{X(\tilde{\Omega})} &\leq \sup \left\{ \langle f, g \rangle : \left\| \left(\sum_{k=1}^{\infty} y_k(s_k)^2 \right)^{1/2} \right\|_{X'(\Omega)} \leq M(X') A \right\} \\ &\leq M(X') A \left\| \left(\sum_{k=1}^{\infty} x_k(s_k)^2 \right)^{1/2} \right\|_{X(\Omega)}. \end{aligned}$$

Thus, inequality (0.1) holds in X for arbitrary independent x_k ($k = 1, 2, \dots$). Since X has the Fatou property or separable, then from [1, Theorem 2] we infer that $X \in \mathbb{K}$.

The second condition $X' \in \mathbb{K}$ may be proved in the same way only inequality (2.9) should be applied to the space X . $\square$

Proposition 2.3. *If $\text{Radi } X$ is a complemented subspace of an r.i. space X , then the projection P is bounded in X .*

Proof. Let $u = \sum_{i=1}^{\infty} \alpha_i 2^{-i}$ and $v = \sum_{i=1}^{\infty} \beta_i 2^{-i}$ ($\alpha_i, \beta_i = 0, 1$) be dyadic expansions of numbers $u, v \in [0, 1]$. For a dyadic rational number we choose the nonterminating binary expansion (note that the set of all dyadic rational numbers from $[0, 1]$ has the Lebesgue measure zero and, therefore, it is not essential in the question). Following [21] (see also [4]), we set

$$u \oplus v := \sum_{i=1}^{\infty} 2^{-i} [(\alpha_i + \beta_i) \bmod 2],$$

where by $(\alpha + \beta) \bmod 2$ ($\alpha, \beta = 0, 1$) we define, as usual, the addition modulo 2, i.e.,

$$(\alpha + \beta) \bmod 2 = \begin{cases} 0, & \text{if } \alpha + \beta = 0 \text{ or } \alpha + \beta = 2, \\ 1, & \text{if } \alpha + \beta = 1. \end{cases}$$

Moreover, the set $\bar{\Omega}$ is a compact abelian group with respect to the operation $t \oplus s := \{t_i \oplus s_i\}_{i=0}^{\infty}$, where $t = \{t_i\}_{i=0}^{\infty} \in \bar{\Omega}$ and $s = \{s_i\}_{i=0}^{\infty} \in \bar{\Omega}$. It is clear that $w_p(t) := t \oplus p$, where $p = \{p_i\}_{i=0}^{\infty} \in \bar{\Omega}$, is a measure-preserving mapping from the probability space $(\bar{\Omega}, \prod_{k=0}^{\infty} \lambda_k)$ onto itself and therefore the operators

$$T_p f(t_0, t_1, \dots) := f \circ w_p(t_0, t_1, \dots) = f(t \oplus p) \quad (p \in \bar{\Omega})$$

act isometrically in arbitrary r.i. space $X(\bar{\Omega})$. Note that the subspace $\text{Radi } X$ is invariant related to them. In fact, if

$$\gamma_i = \left\{ u \in [0, 1]: u = \sum_{j=1}^{\infty} \alpha_j 2^{-j}, \alpha_i = 0 \right\} \quad \text{and} \quad \bar{\gamma}_i = [0, 1] \setminus \gamma_i \quad (i = 1, 2, \dots),$$

then

$$r_i(t \oplus u) = \begin{cases} r_i(t), & u \in \gamma_i, \\ -r_i(t), & u \in \bar{\gamma}_i. \end{cases} \quad (2.10)$$

Furthermore, the mapping $f \mapsto f(\cdot \oplus u)$ is an isometry in an r.i. space X for every $u \in [0, 1]$. Therefore, by Rudin's theorem [23] (or [24, Theorem 5.18]), there is a bounded projection $S : X(\bar{\Omega}) \rightarrow \text{Radi } X$ commuting with all T_p ($p \in \bar{\Omega}$). We will show that the operators S and P coincide on the separable part X_o of X .

First of all, since the Rademacher functions form a basic sequence in every r.i. space then the operator S is representable in the form:

$$Sf(t_0, t_1, \dots) = \sum_{i=1}^{\infty} S_i f(t_i) r_i(t_0), \quad (2.11)$$

where S_i are bounded linear operators from $X(\bar{\Omega})$ into X . Since the operators T_p and S commute, then $S(f \circ w_p) = S(T_p f) = T_p S f$. Hence, using (2.11) and (2.10), we obtain that

$$S_i(f \circ w_p) = \begin{cases} S_i f, & p_0 \in \gamma_i, \quad p_i = 0, \\ -S_i f, & p_0 \in \bar{\gamma}_i, \quad p_i = 0. \end{cases}$$

Moreover, taking into account that $\lambda(\gamma_i) = \lambda(\bar{\gamma}_i) = 1/2$, we have that

$$\int_{\Omega_i} \int_{\gamma_i} S_i(f \circ w_p) dp_0 \dots dp_{i-1} dp_{i+1} \dots = \frac{1}{2} S_i f \quad (2.12)$$

and

$$\int_{\Omega_i} \int_{\overline{\gamma_i}} S_i(f \circ w_p) dp_0 \dots dp_{i-1} dp_{i+1} \dots = -\frac{1}{2} S_i f, \quad (2.13)$$

where it is assumed that $p_i = 0$.

It is easy to see that the closed linear span $[r_n]$ is a complemented subspace of the space $\text{Radi } X$. Since, by hypothesis, $\text{Radi } X$, in turn, is a complemented subspace of X , then $[r_n]$ is complemented in X . Then, by [21], $X \neq L_\infty$, and $X_\circ$ is separable.

Now, suppose $f \in (X(\tilde{\Omega}))_\circ$. Using the separability of $(X(\tilde{\Omega}))_\circ$ and the fact that the operators T_p act isometrically in this space, it is not hard to check that the mapping $p \mapsto f \circ w_p$ is an $(X(\tilde{\Omega}))_\circ$ -valued Bochner-integrable function on $\tilde{\Omega}$. Thus, since S_i is a linear bounded operator from $X(\tilde{\Omega})$ to X , it can be taken out of the integral in equalities (2.12) and (2.13). Therefore, we obtain that

$$S_i f = S_i \left(\int_{\Omega_i} \left(\int_{\gamma_i} (f \circ w_p) dp_0 - \int_{\overline{\gamma_i}} (f \circ w_p) dp_0 \right) dp_1 \dots dp_{i-1} dp_{i+1} \dots \right). \quad (2.14)$$

On the other hand, for any $t \in [0, 1]$

$$\{v \in [0, 1]: v = t \oplus u, u \in \gamma_i\} = \begin{cases} \gamma_i, & t \in \gamma_i, \\ \overline{\gamma_i}, & t \in \overline{\gamma_i} \end{cases}$$

and

$$\{v \in [0, 1]: v = t \oplus u, u \in \overline{\gamma_i}\} = \begin{cases} \overline{\gamma_i}, & t \in \gamma_i, \\ \gamma_i, & t \in \overline{\gamma_i}. \end{cases}$$

Thus,

$$\begin{aligned} & \int_{\Omega_i} \int_{\gamma_i} (f \circ w_p) dp_0 dp_1 \dots dp_{i-1} dp_{i+1} \dots \\ &= \int_{\Omega_i} \int_{\gamma_i} f(v, t_1, t_2, \dots) dv dt_1 \dots dt_{i-1} dt_{i+1} \dots \cdot \chi_{\gamma_i}(t_0) \\ &+ \int_{\Omega_i} \int_{\overline{\gamma_i}} f(v, t_1, t_2, \dots) dv dt_1 \dots dt_{i-1} dt_{i+1} \dots \cdot \chi_{\overline{\gamma_i}}(t_0) \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega_i} \int_{\overline{\gamma_i}} (f \circ w_p) dp_0 dp_1 \dots dp_{i-1} dp_{i+1} \dots \\ &= \int_{\Omega_i} \int_{\overline{\gamma_i}} f(v, t_1, t_2, \dots) dv dt_1 \dots dt_{i-1} dt_{i+1} \dots \cdot \chi_{\gamma_i}(t_0) \\ &+ \int_{\Omega_i} \int_{\gamma_i} f(v, t_1, t_2, \dots) dv dt_1 \dots dt_{i-1} dt_{i+1} \dots \cdot \chi_{\overline{\gamma_i}}(t_0). \end{aligned}$$

Since $r_i = \chi_{\gamma_i} - \chi_{\bar{\gamma}_i}$ ($i = 1, 2, \dots$), then we infer that

$$\begin{aligned} & \int_{\Omega_i} \left(\int_{\gamma_i} (f \circ w_p) dp_0 - \int_{\bar{\gamma}_i} (f \circ w_p) dp_0 \right) dp_1 \dots dp_{i-1} dp_{i+1} \dots \\ &= \int_{\Omega_i} \left(\int_{\gamma_i} f(v, t_1, t_2, \dots) dv - \int_{\bar{\gamma}_i} f(v, t_1, t_2, \dots) dv \right) dt_1 \dots dt_{i-1} dt_{i+1} \dots \cdot r_i(t_0) \\ &= \int_{\Omega_i} \int_0^1 f(v, t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots) r_i(v) dv dt_1 \dots dt_{i-1} dt_{i+1} \dots \cdot r_i(t_0) \\ &= \mathbb{E}(f r_i | t_i) \cdot r_i(t_0). \end{aligned}$$

Moreover, we have that

$$S_i(x(t_k)r_k(t_0)) = \begin{cases} x(t_k), & i = k, \\ 0, & i \neq k, \end{cases}$$

and from (2.14) it follows that $S_i f = \mathbb{E}(f r_i | t_i)$ ($i = 1, 2, \dots$). Taking into account the definition of the projection P and (2.11), we obtain that $Sf = Pf$, and the coincidence of the operators S and P on X_0 is proved. If X is separable, then $S = P$ on the whole X , and the proof is completed.

Now, let X have the Fatou property. Let $f \in X(\bar{\Omega}) \approx X$ be arbitrary. There exists a sequence $\{f_n\} \subset L_\infty$ such that $|f_n| \leq |f|$ and $f_n \rightarrow f$ a.e. It is obvious that $\mathbb{E}(f_n r_k | t_k) \rightarrow \mathbb{E}(f r_k | t_k)$ a.e. as $n \rightarrow \infty$ ($k \in \mathbb{N}$). At the same time, as it is proved,

$$\|Pf_n\|_X = \|Sf_n\|_X \leq \|S\| \|f_n\|_X \leq \|S\| \|f\|_X \quad (n \in \mathbb{N}).$$

Since $\{y_k(t_k)r_k(t_0)\}_{k=1}^\infty$ is a basic sequence with constant 1 in X for any $y_k \in X$, then the previous inequality yields

$$\left\| \sum_{k=1}^m \mathbb{E}(f_n r_k | t_k) r_k(t_0) \right\|_X \leq \|S\| \|f\|_X$$

for all $m = 1, 2, \dots$ and $n = 1, 2, \dots$. Since X has the Fatou property, then passing to the limit firstly as $n \rightarrow \infty$ and then as $m \rightarrow \infty$ we obtain that

$$\|Pf\|_X = \left\| \sum_{k=1}^\infty \mathbb{E}(f r_k | t_k) r_k(t_0) \right\|_X \leq \|S\| \|f\|_X,$$

which implies that P is bounded in X . $\square$

As a consequence of Theorem 2.1 we will show that an r.i. space X on $[0, 1]$ such that $X \in \mathbb{K}$ and $X' \in \mathbb{K}$ is isomorphic to some r.i. space on the semi-axis. This strengthens the analogous results of [11, §8] (see also [17, p. 203]) proved using a different approach under a stronger condition when an r.i. space X has non-trivial Boyd indices.

Theorem 2.4. *If X is an r.i. space on $[0, 1]$ such that $X \in \mathbb{K}$ and $X' \in \mathbb{K}$, then the spaces X and Z_X^2 are isomorphic.*

Proof. By Theorem 3.1 from [7], for every r.i. space X satisfying the Kruglov property there exists a constant $C > 0$ such that for each sequence of mean zero independent functions $\{f_k\}_{k=1}^\infty \subset X$ inequality (1.1) holds. In particular, this implies that the operator Q (see (2.3)) is bounded from the space Z_X^2 into $X(\bar{\Omega})$. Moreover, by [12, Theorem 1] (see inequality (3)), there is a constant $c > 0$ such that $\|Qf\|_{X(\bar{\Omega})} \geq c\|f\|_{Z_X^2}$ ($f \in Z_X^2$). Hence, the image of the operator Q , which coincides with $\text{Radi } X$, is isomorphic to the space Z_X^2 . Therefore, by Theorem 2.1, Z_X^2 is a complemented subspace of $X(\bar{\Omega}) \approx X$. On the other hand, it is obvious that X is isomorphic to a complemented subspace of the space Z_X^2 . Since $X \oplus X$ ($Z_X^2 \oplus Z_X^2$, respectively) is isomorphic to X (to Z_X^2 , respectively), then by the decomposition method due to Pelczynski [17, p. 172], we conclude that the spaces X and Z_X^2 are isomorphic. $\square$

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