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VECTOR-VALUED SUMS OF INDEPENDENT FUNCTIONS IN REARRANGEMENT INVARIANT SPACES

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Abstract: We obtain some relations that describe the behavior of vector-valued Gaussian sums in rearrangement invariant spaces on the square whose character depends on whether the lower Boyd index of the space is trivial or not. Similar results are proven for the general systems of independent identically and symmetrically distributed random variables.

Keywords: Rademacher functions, Gaussian random variable, independent random variables, rearrangement invariant space, Orlicz space, Boyd indices

Introduction

Let $r_k(t) = \operatorname{sgn} \sin 2^k \pi t$ $(k = 1, 2, \dots)$ be the Rademacher functions on the segment $[0, 1]$. In accord with the classical Khintchine's inequality [1], for every $p > 0$, there exists a constant $C_p > 0$ such that, for every $n \in \mathbb{N}$ and arbitrary reals a_k $(k = 1, 2, \dots, n)$, we have

$$C_p^{-1}\left(\sum_{k=1}^n a_k^2\right)^{1/2} \le \left\|\sum_{k=1}^n a_k r_k\right\|_{L_p} \le C_p\left(\sum_{k=1}^n a_k^2\right)^{1/2}.$$

This relation caused a number of investigations and generalizations as well as numerous applications in various sections of analysis (for instance, see [2]). In particular, V. A. Rodin and E. M. Semenov proved in 1975 that, for a rearrangement invariant space X on $[0, 1]$, the inequalities

$$C^{-1}\left(\sum_{k=1}^n a_k^2\right)^{1/2} \le \left\|\sum_{k=1}^n a_k r_k\right\|_X \le C\left(\sum_{k=1}^n a_k^2\right)^{1/2} \tag{1}$$

hold if and only if X contains G, where G is the closure of L_∞ in the exponential Orlicz space $\operatorname{Exp} L^2$ constructed by the function $N_2(u) = e^{u^2} - 1$ [3]. The condition $X \supset G$ means that X in a certain sense "lies far" from L_∞. A similar but more restrictive condition allows us to find a natural description of the behavior of the vector-valued Rademacher sums $\sum_{k=1}^n x_k(s) r_k(t)$ in rearrangement invariant spaces on the square $I \times I$, where $I = [0, 1]$. Namely, if X is a rearrangement invariant space on $[0, 1]$ then

$$C^{-1}\left\|\left(\sum_{k=1}^n x_k^2\right)^{1/2}\right\|_X \le \left\|\sum_{k=1}^n x_k(s) r_k(t)\right\|_{X(I\times I)} \le C\left\|\left(\sum_{k=1}^n x_k^2\right)^{1/2}\right\|_X \tag{2}$$

with a constant $C > 0$ independent of $n \in \mathbb{N}$ and $\{x_k\}_{k=1}^n \subset X$ if and only if the Boyd index α_X of X is positive [4, Proposition 2.d.1; 5].

The Rademacher system is a classical example of a sequence of uniformly bounded and identically and symmetrically distributed random variables (r.v.'s). In the present article we carry over relations of the form (2) to similar systems of unbounded r.v.'s. A particular attention is paid to a system of r.v.'s with the Gaussian distribution. Moreover, we demonstrate that, under certain conditions on a system of r.v.'s, the behavior of vector-valued sums can be described as in (2) for the rearrangement invariant spaces X "close" to L_∞, i.e., in case $\alpha_X = 0$.

We confine exposition to considering r.v.'s whose domain is $I = [0, 1]$ (or the square $I \times I$) with the Lebesgue measure. However, the results remain valid for an arbitrary nonatomic probability space.

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1. Preliminaries

A Banach space X of Lebesgue measurable real functions on $I = [0,1]$ is called *rearrangement invariant* (or *symmetric*) if

1) $y \in X$ and $|x(t)| \le |y(t)|$ for a.e. $t \in I$ imply that $x \in X$ and $\|x\|_X \le \|y\|_X$;

2) $y \in X$ and

$$m\{t \in I : |x(t)| > \tau\} = m\{t \in I : |y(t)| > \tau\} \quad (\tau > 0)$$

(here and in what follows m stands for the Lebesgue measure) yield $x \in X$ and $\|x\|_X = \|y\|_X$.

If X is a rearrangement invariant space then the *dual* (or *associated*) space X' consists of measurable functions y such that $\|y\|_{X'} := \sup\left\{\int_0^1 |x(t)y(t)|\,dt : \ x \in X,\ \|x\|_X \le 1\right\} < \infty$. The space X is naturally embedded into the second dual X'' and $\|x\|_{X''} \le \|x\|_X$ $(x \in X)$. The space X'' possesses the following *maximality* property: if $\{x_n\}_{n=1}^\infty \subseteq X''$, $x_n \to x$ a.e. on $[0,1]$, and $\sup_n \|x_n\|_{X''} < \infty$ then $x \in X''$ and $\|x\|_{X''} \le \liminf_{n\to\infty} \|x_n\|_{X''}$.

If X is a rearrangement invariant space on $[0,1]$ then we have the continuous embeddings $L_\infty \subseteq X \subseteq L_1$. The *dilation operator*

$$\sigma_\tau x(t) := x(t/\tau) \cdot \chi_{[0,\min(1,\tau)]}(t) \quad (\tau > 0)$$

is continuous in every rearrangement invariant space X and $\|\sigma_\tau\|_{X\to X} \le \max(1,\tau)$ [6, Theorem 2.4.5]. Using this operator, we can define the *Boyd indices* of X as follows:

$$\alpha_X = \lim_{\tau\to 0+} \frac{\log \|\sigma_\tau\|_{X\to X}}{\log \tau} \quad \text{and} \quad \beta_X = \lim_{\tau\to +\infty} \frac{\log \|\sigma_\tau\|_{X\to X}}{\log \tau}.$$

Clearly, $0 \le \alpha_X \le \beta_X \le 1$.

An Orlicz space, a natural generalization of the L_p-spaces, serves as an important example of a rearrangement invariant space. If $N(x)$ is an increasing convex function on $[0,\infty)$ and $N(0) = 0$ then the Orlicz space L_N contains the measurable functions $x = x(t)$ on $[0,1]$ such that

$$\|x\|_{L_N} = \inf\left\{u > 0 : \int_0^1 N\left(\frac{|x(t)|}{u}\right) dt \le 1\right\} < \infty.$$

In particular, if an increasing convex function $N_q(x)$ is equivalent to the function e^{x^q} $(q > 0)$ then we arrive at the exponential Orlicz space (or Zygmund space) $\operatorname{Exp} L^q$. As it is known [7, 8],

$$\|x\|_{\operatorname{Exp} L^q} \asymp \sup_{0<u\le 1} (x^*(u) \log^{-1/q}(e/u)). \tag{3}$$

Here and in what follows $x^*(u)$ is the nonincreasing left continuous rearrangement of a function $|x(t)|$, i.e.,

$$x^*(u) = \inf\{\tau > 0 : m\{t \in [0,1] : |x(t)| > \tau\} < u\} \quad (0 < u \le 1).$$

The spaces $\operatorname{Exp} L^q$ are "close" to L_∞ in the sense that $\alpha_{\operatorname{Exp} L^q} = \beta_{\operatorname{Exp} L^q} = 0$ for every $q > 0$.

If X is a rearrangement invariant space on $I = [0,1]$ then the corresponding rearrangement invariant space on the square, $X(I \times I)$, consists of all measurable functions $x(s,t)$ on $I \times I$ such that $x^* \in X$. We have $\|x\|_{X(I\times I)} := \|x^*\|_X$.

Denote by $[a]$ the integer part of a real a and by $\|x\|_p$, the norm of a measurable function $x = x(t)$ in $L_p[0,1]$. At last, the equality of two Banach spaces is understood to be the coincidence of the underlying sets and the equivalence of their norms. The expression $f \asymp g$ means that $cf \le g \le Cf$ for some $c > 0$ and $C > 0$ and these constants are independent of all or part of arguments of the functions (quasinorms) f and g.

2. Vector-Valued Gaussian Sums in Rearrangement Invariant Spaces

Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of independent standard Gaussian r.v.'s on $I = [0,1]$. The following result is similar to the corresponding theorem for Rademacher's functions which was mentioned in the Introduction.

Theorem 1. *If X is a rearrangement invariant space on I then the following are equivalent:*
1) $\alpha_X > 0$;
2) *there exists a constant $A > 0$ such that, for every $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in X$, we have*

$$A^{-1}\left\|\left(\sum_{k=1}^{n} x_k^2\right)^{1/2}\right\|_X \le \left\|\sum_{k=1}^{n} x_k(s)g_k(t)\right\|_{X(I\times I)} \le A\left\|\left(\sum_{k=1}^{n} x_k^2\right)^{1/2}\right\|_X. \tag{4}$$

We employ some auxiliary propositions in the proof. They are stated as lemmas.

The following exponential estimate for the Rademacher sums is well known [2, Section 2, p. 11]:

$$m\left\{t \in [0,1] : \left|\sum_{k=1}^{n} a_k r_k(t)\right| > u\right\} \le D\exp\left(-\frac{u^2}{2\sum\limits_{k=1}^{n} a_k^2}\right) \quad (u > 0), \tag{5}$$

where $D = e\sqrt{2}$ (a similar result is valid also for the general sequences of mean zero uniformly bounded independent functions [9, Theorem 2.5]).

Note that the proof of (5) in [2, Section 2, p. 11] relies on the following estimate for the moments of the Rademacher sums:

$$\left\|\sum_{k=1}^{n} a_k r_k\right\|^{2m} \le B_{2m}\left(\sum_{i=1}^{n} a_i^2\right)^m,$$

where

$$B_{2m} := \frac{(2m)!}{2^m m!} \quad (m = 1, 2, \dots). \tag{6}$$

At the same time, it is a well-known fact that the moment of order $2m$ of a standard Gaussian variable g is defined by the equality $\|g\|_{2m}^{2m} = B_{2m}$. Therefore, repeating the arguments of [2, Section 2, p. 11], we obtain the same exponential estimate in the Gaussian case.

Lemma 1. *If $D = e\sqrt{2}$ then, for all $n \in \mathbb{N}$ and arbitrary reals $a_1, \dots, a_n$,*

$$m\left\{t \in [0,1] : \left|\sum_{k=1}^{n} a_k g_k(t)\right| > u\right\} \le D\exp\left(-\frac{u^2}{2\sum\limits_{k=1}^{n} a_k^2}\right) \quad (u > 0).$$

Given a measurable function $\varphi(t)$ (on $I = [0,1]$), define the tensor product operator as follows: $T_\varphi x(s,t) = x(s)\varphi(t)$. Assume that $\varphi(t)$ is unbounded on $I = [0,1]$ and T_φ is bounded as an operator from a rearrangement invariant space $X(I)$ into $X(I \times I)$. In this case $\alpha_X > 0$ [5, Lemma 1] (also see [10, Lemma 17]).

Show that the converse proposition is valid as well.

Lemma 2. *Let a rearrangement invariant space X be such that $\alpha_X > 0$. Then if $\varphi \in L_p(I)$ for every $p < \infty$ then T_φ is bounded as an operator from $X(I)$ into $X(I \times I)$.*

Proof. By hypothesis and the definition of Boyd indices, there exist $\delta \in (0, 1/2)$ and $C > 0$ such that $\|\sigma_\tau\|_{X\to X} \le C\tau^{2\delta}$ for all $\tau \in (0,1]$. Let $x \in X(I)$. Applying the Hölder inequality and the last relation, we infer

$$
\begin{aligned}
\|T_\varphi x\|_{X(I\times I)} &\le \Big\| x(s)\sum_{k=1}^\infty \varphi^*(2^{-k})\chi_{(2^{-k},2^{-k+1}]}(t)\Big\|_{X(I\times I)}\\
&\le \sum_{k=1}^\infty \varphi^*(2^{-k})\|\sigma_{2^{-k}}x\|_X \le C\sum_{k=1}^\infty \varphi^*(2^{-k})2^{-2k\delta}\|x\|_X\\
&\le C\Big(\sum_{k=1}^\infty 2^{-k\delta/(1-\delta)}\Big)^{1-\delta}\Big(\sum_{k=1}^\infty \varphi^*(2^{-k})^{1/\delta}2^{-k}\Big)^{\delta}\|x\|_X\\
&= C(2^{\delta/(1-\delta)}-1)^{\delta-1}\Big\|\sum_{k=1}^\infty \varphi^*(2^{-k})\chi_{(2^{-k},2^{-k+1}]}(t)\Big\|_{L_{1/\delta}}\|x\|_X \le C_\delta\|\varphi\|_{L_{1/\delta}}\|x\|_X.
\end{aligned}
$$

Thereby T_φ is bounded as an operator from $X(I)$ into $X(I\times I)$ and

$$\|T_\varphi\|_{X(I)\to X(I\times I)} \le C_\delta\|\varphi\|_{L_{1/\delta}}. \qquad \square$$

The last result deals with comparison between Rademacher and Gaussian sums.

Lemma 3. *For every $n\in\mathbb{N}$ and all reals $a_1,\dots,a_n$,*

$$m\Big\{t\in I:\Big|\sum_{k=1}^n a_k r_k(t)\Big|>\tau\Big\} \le \frac{2}{\alpha}m\Big\{t\in I:\Big|\sum_{k=1}^n a_k g_k(t)\Big|>\alpha\tau\Big\} \quad (\tau>0),$$

where $\alpha := \frac{2}{\sqrt{2\pi}}\int_1^\infty e^{-t^2/2}\,dt$.

Proof. Note that $\alpha = m\{t\in[0,1]:|g(t)|>1\}$, where g is a standard Gaussian r.v. Hence, for arbitrary $\tau>0$ and $k=1,2,\dots$, we have

$$m\{t\in[0,1]:|r_k(t)|>\tau\} \le \frac{1}{\alpha}m\{t\in[0,1]:|g_k(t)|>\tau\}.$$

Since r.v.'s r_k (respectively, g_k) are symmetrically distributed and independent, the Kwapień–Rychlik inequality [11, Theorem 5.4.4] yields the claim.

Proof of Theorem 1. We assume first that $\alpha_X>0$ and

$$f := \Big(\sum_{i=1}^n x_i^2\Big)^{1/2} \in X.$$

By Lemma 1, for a.e. $s\in I$,

$$m\Big\{t\in I:\Big|\sum_{i=1}^n x_i(s)g_i(t)\Big|>\tau\Big\} \le D\exp\Big(-\frac{\tau^2}{2f(s)^2}\Big).$$

It is immediate that

$$\exp\Big(-\frac{\tau^2}{2f(s)^2}\Big) \le m\{t\in I: 2f(s)\log^{1/2}e/t>\tau\} \quad (0<s\le 1).$$

Thereby, integrating the previous inequality over I, we obtain

$$m\Big\{(s,t)\in I\times I:\Big|\sum_{i=1}^n x_i(s)g_i(t)\Big|>\tau\Big\} \le Dm\{(s,t)\in I\times I: 2f(s)\log^{1/2}e/t>\tau\}$$

and so

$$\Big\|\sum_{i=1}^n x_i(s)g_i(t)\Big\|_{X(I\times I)} \le 2D\|T_\varphi f\|_{X(I\times I)},$$

where $\varphi(t)=\log^{1/2}(e/t)$. Lemma 2 implies that

$$\Big\|\sum_{i=1}^n x_i(s)g_i(t)\Big\|_{X(I\times I)} \le M\|f\|_X,$$

and the right-hand side inequality in (4) is proven.

The left-hand side inequality in (4) is fulfilled in every rearrangement invariant space. Indeed, if $x_1,\dots,x_n\in X$ then Lemma 3 implies that

$$m\Big\{t\in I:\Big|\sum_{k=1}^n x_k(s)r_k(t)\Big|>\tau\Big\}\le\frac{2}{\alpha}m\Big\{t\in I:\Big|\sum_{k=1}^n x_k(s)g_k(t)\Big|>\alpha\tau\Big\}$$

for a.e. $s\in I$ and every $\tau>0$. The Fubini theorem yields the relation

$$m\Big\{(s,t)\in I\times I:\Big|\sum_{k=1}^n x_k(s)r_k(t)\Big|>\tau\Big\}\le\frac{2}{\alpha}m\Big\{(s,t)\in I\times I:\Big|\sum_{k=1}^n x_k(s)g_k(t)\Big|>\alpha\tau\Big\},$$

which in view of [6, Corollary 2.4.2] validates the inequality

$$\Big\|\sum_{k=1}^n x_k r_k\Big\|_{X(I\times I)}\le\frac{2}{\alpha^2}\Big\|\sum_{k=1}^n x_k g_k\Big\|_{X(I\times I)}.$$

On the other hand, for every rearrangement invariant space X and arbitrary $x_1,\dots,x_n\in X$ in view of [4, Proposition 2.d.1], we have the inequality

$$\Big\|\sum_{k=1}^n x_k r_k\Big\|_{X(I\times I)}\ge c\Big\|\Big(\sum_{k=1}^\infty x_k^2\Big)^{1/2}\Big\|_X.$$

From this and the previous inequality, we obtain

$$\Big\|\Big(\sum_{k=1}^n x_k^2\Big)^{1/2}\Big\|_X\le\frac{2}{\alpha^2 c}\Big\|\sum_{k=1}^n x_k g_k\Big\|_{X(I\times I)}.$$

Assume now that (4) is fulfilled in a given rearrangement invariant space X for every $n\in\mathbb{N}$ and all $x_1,\dots,x_n\in X$. As it is easily seen, the operator T_g, with a standard Gaussian r.v. g is bounded in X. Since g is an unbounded function on $[0,1]$, we conclude that $\alpha_X>0$ [5, Lemma 1] and the theorem is proven. $\square$

Corollary 1. *If X is a rearrangement invariant space on I then the following are equivalent:*
1) $\alpha_X>0$*;*
2) *there exists a constant $B>0$ such that*

$$B^{-1}\Big\|\sum_{k=1}^n x_k(s)r_k(t)\Big\|_{X(I\times I)}\le\Big\|\sum_{k=1}^n x_k(s)g_k(t)\Big\|_{X(I\times I)}\le B\Big\|\sum_{k=1}^n x_k(s)r_k(t)\Big\|_{X(I\times I)}$$

for all $n\in\mathbb{N}$ and $x_1,\dots,x_n\in X$.

Remark 1. Let the operator T_φ, with $\varphi(t)=\log^{1/2}(e/t)$, be bounded as an operator from a rearrangement invariant space $X(I)$ into a rearrangement invariant space $Y(I\times I)$. In this case the proof of Theorem 1 shows that

$$\Big\|\sum_{k=1}^n x_k(s)g_k(t)\Big\|_{Y(I\times I)}\le C\Big\|\Big(\sum_{k=1}^n x_k^2\Big)^{1/2}\Big\|_{X(I)},$$

where the constant $C > 0$ is independent of $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$.

Proceed with examining the vector-valued Gaussian sums $\sum_{k=1}^n x_k(s) g_k(t)$ in rearrangement invariant spaces X "close" to L_∞, i.e., in case $\alpha_X = 0$. Let X be a rearrangement invariant space such that $\beta_X < 1$, $\alpha > 0$. Denote by $X(\log^{-\alpha})$ the set of all measurable functions $x(t)$ on $[0,1]$ such that the following quasinorm is finite:

$$\|x\|_{X(\log^{-\alpha})} := \|x^*(t) \log^{-\alpha} e/t\|_X.$$

By [12], we have

$$\|x\|_{X(\log^{-\alpha})} \asymp \|x^{**}(t) \log^{-\alpha} e/t\|_X, \quad \text{where } x^{**}(t) = \frac{1}{t} \int_0^t x^*(s)\, ds$$

and thereby $X(\log^{-\alpha})$ is a rearrangement invariant space.

Theorem 2. *Assume that X is a rearrangement invariant space and $\beta_X < 1$. Then there exists a constant $A > 0$ such that*

$$\Big\| \sum_{k=1}^n x_k(s) g_k(t) \Big\|_{X(\log^{-1/2})(I\times I)} \le A \Big\| \Big(\sum_{k=1}^n x_k^2 \Big)^{1/2} \Big\|_X \tag{7}$$

for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$.

To prove Theorem 2, we need the following technical lemma.

Lemma 4. *If nonnegative functions $x(t)$ and $y(t)$ decrease on $[0,1]$ then*

$$(x(s) \cdot y(t))^*(2u) \le x(u) \cdot y(u), \quad 0 < u \le 1/2.$$

Proof. Since $x(s) \le x(u)$ for $s \in [u,1]$ and $y(t) \le y(u)$ for $t \in [u,1]$, and $m\{[u,1]\times[u,1]\} = (1-u)^2$, we have

$$m\{(s,t) \in I \times I : x(s)y(t) > x(u)y(u)\} \le 1 - (1-u)^2 < 2u$$

for every $0 < u \le 1/2$. The claim follows now from the definition of the rearrangement of a measurable function. $\square$

Proof of Theorem 2. In view of Remark 1, it suffices to establish that T_φ, where $\varphi(t) = \log^{1/2}(e/t)$, is bounded from $X(I)$ into $X(\log^{-1/2})(I \times I)$. In fact, if $x \in X(I)$ then Lemma 4 yields

$$\begin{aligned} \|T_\varphi x\|_{X(\log^{-1/2})(I\times I)} &\le \|x^*(u/2) \log^{1/2}(2e/u) \log^{-1/2}(e/u)\|_{X(I)} \\ &\le \sqrt{2} \|\sigma_2 x\|_{X(I)} \le 2\sqrt{2} \|x\|_{X(I)}. \qquad \square \end{aligned}$$

Remark 2. By Lemma 3, a similar assertion is valid also for vector-valued Rademacher series.

Let us show now that Theorem 2 is sharp for the exponential Orlicz spaces $\operatorname{Exp} L^q$ $(q > 0)$.

Theorem 3. *Let $q > 0$ be arbitrary. Assume that*

$$\Big\| \sum_{k=1}^n x_k(s) g_k(t) \Big\|_{Y(I\times I)} \le A \Big\| \Big(\sum_{k=1}^n x_k^2 \Big)^{1/2} \Big\|_{\operatorname{Exp} L^q} \tag{8}$$

for some rearrangement invariant space Y and some constant $A > 0$ independent of $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \operatorname{Exp} L^q$. Then $Y \supset \operatorname{Exp} L^q(\log^{-1/2})$.

Moreover, $\operatorname{Exp} L^q(\log^{-1/2}) = \operatorname{Exp} L^r$, with $r = \frac{2q}{2+q}$.

To prove Theorem 3, we need the following two lemmas.

Lemma 5. *For all $\alpha > 0$ and $\beta > 0$, we have*

$$(\log^{\alpha}(e/s) \cdot \log^{\beta}(e/t))^*(u) \ge 2^{-\alpha-\beta} \log^{\alpha+\beta}(e/u) \quad (0 < u \le 1).$$

Proof. First of all, note that

$$m\{u \in I : 2^{-\alpha-\beta} \log^{\alpha+\beta}(e/u) > \tau\} \le m\{u \in I : 2^{-\alpha-\beta} \log^{\alpha+\beta}(e^2/2u) > \tau\} = \frac{1}{2} \exp(2 - 2\tau^{\frac{1}{\alpha+\beta}}).$$

In view of [6, Corollary 2.4.2], it suffices to justify the relation

$$L_{\alpha,\beta}(\tau) := m\{(s,t) \in I \times I : \log^{\alpha}(e/s) \cdot \log^{\beta}(e/t) > \tau\} \ge \frac{1}{2} \exp(2 - 2\tau^{\frac{1}{\alpha+\beta}})$$

for all $\tau \ge 1$. Indeed, changing variables, we infer

$$L_{\alpha,\beta}(\tau) = \int_0^1 m\Big\{t \in I : \log^{\beta}(e/t) > \frac{\tau}{\log^{\alpha}(e/s)}\Big\}\, ds$$

$$= e^2 \int_1^{\infty} \operatorname{Exp}\Big(-u - \frac{\tau^{1/\beta}}{u^{\alpha/\beta}}\Big)\, du \ge e^2 \int_{\tau^{1/(\alpha+\beta)}}^{\infty} e^{-2u}\, du = \frac{1}{2} \exp(2 - 2\tau^{\frac{1}{\alpha+\beta}}). \qquad \square$$

Lemma 6. *For $q > 0$ and $p > 0$, we have*

$$\operatorname{Exp} L^q(\log^{-1/p}) = \operatorname{Exp} L^r, \quad \textit{where } r = \frac{pq}{p+q}.$$

Proof. Using the definition of the rearrangement of a measurable function and relation (3), we infer

$$\|x\|_{\operatorname{Exp} L^q(\log^{-1/p})} \asymp \sup_{0<u\le 1} (x^*(t) \log^{-1/p}(e/t))^*(u) \log^{-1/q}(e/u)$$

$$\le \sup_{0<u\le 1} (x^*(u) \log^{-1/q-1/p}(e/u)) \asymp \|x\|_{\operatorname{Exp} L^r}.$$

On the other hand, if $x \in \operatorname{Exp} L^q(\log^{-1/p})$ then

$$(x^*(t) \log^{-1/p}(e/t))^*(u) \le C \log^{1/q}(e/u) \quad (0 < u \le 1).$$

Applying [6, Theorem 2.2.1], we can find a measure-preserving mapping $w : [0,1] \to [0,1]$ such that

$$x^*(t) \log^{-1/p}(e/t) \le 2C \log^{1/q}(e/w(t)) \quad (0 < t \le 1)$$

and thereby

$$x^*(u) \le 2C(\log^{1/p}(e/t) \log^{1/q}(e/w(t)))^*(u) \quad (0 < u \le 1).$$

Hence, by [6, p. 93],

$$x^*(u) \le 2C \log^{1/q}(2e/u) \log^{1/p}(2e/u) \le C_1 \log^{1/q+1/p}(e/u) \quad (0 < u \le 1).$$

Taking (3) into account once more, we see that $x \in \operatorname{Exp} L^r$ and the lemma is proven. $\square$

Proof of Theorem 3. By hypothesis, $T_g x(s,t) = g(s)x(t)$, where g is a standard Gaussian r.v., is bounded from $X := \operatorname{Exp} L^q$ into $Y(I \times I)$. Since

$$m\{s \in I : |g(s)| > \tau\} = \frac{2}{\sqrt{2\pi}} \int_{\tau}^{\infty} e^{-t^2/2}\, dt \ge \frac{2}{\sqrt{2\pi}} \int_{\tau}^{2\tau} e^{-t^2/2}\, dt \ge \frac{2}{\sqrt{2\pi}} e^{-2t^2}$$

for every $\tau \ge 1$ and Y is a rearrangement invariant space, the operator $T_\varphi x(s,t) = \varphi(s)x(t)$, where $\varphi(s) = \log^{1/2}(e/s)$, is bounded from X into $Y(I \times I)$ as well. Thereby from (3) it follows that $\log^{1/q}(e/s) \cdot \log^{1/2}(e/t) \in Y(I \times I)$ and so Lemma 5 yields $\log^{1/q+1/2}(e/u) \in Y$. Appealing again to (3), we conclude that $Y \supset \operatorname{Exp} L^r$, where $r = \frac{2q}{2+q}$.

The second claim of the theorem results from Lemma 6. $\square$

A similar statement is valid also for the Rademacher sums.

Theorem 4. *Let $q > 0$ be arbitrary. Assume that for some rearrangement invariant space Y*

$$\Big\|\sum_{k=1}^{n} x_k(s) r_k(t)\Big\|_{Y(I\times I)} \le A\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_{\operatorname{Exp} L^q} \tag{9}$$

where the constant $A > 0$ is independent of $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \operatorname{Exp} L^q$. Then $Y'' \supset \operatorname{Exp} L^r(\log^{-1/2})$, where $r = \frac{2q}{2+q}$.

Proof. As it is seen from the proof of the previous theorem, it suffices to show that the operator $T_g x(s,t) = g(s)x(t)$, where g is a standard Gaussian r.v., is bounded from $X := \operatorname{Exp} L^q$ into $Y''(I \times I)$.

Set $x_k(t) = x(t)/\sqrt{n}$, $k = 1, 2, \dots, n$, where $n \in \mathbb{N}$ and $x(t) \in X$ are arbitrary, and

$$v_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} r_k(t), \quad n = 1, 2, \dots.$$

The condition (9) implies that

$$\|x(s)v_n^*(t)\|_{Y(I\times I)} \le A\|x\|_X \quad (n \in \mathbb{N}). \tag{10}$$

Since $v_n(t)$ are symmetrically distributed, the central limit theorem [13, Theorem 7.2] yields the equality

$$\lim_{n\to\infty} m\{t \in [0,1] : |v_n(t)| > \tau\} = \frac{2}{\sqrt{2\pi}} \int_{\tau}^{\infty} e^{-t^2/2}\, dt \quad (\tau > 0).$$

By [14, Lemma 3.1], we have

$$\lim_{n\to\infty} v_n^*(t) = g^*(t) \quad (0 < t \le 1).$$

Hence, passing to the limit in (10) as $n \to \infty$ and using the maximality of the second dual Y'', we infer that

$$\|T_g x(s,t)\|_{Y''(I\times I)} = \|g^*(t)x(s)\|_{Y''(I\times I)} \le A\|x\|_X (x \in X). \qquad \square$$

3. Vector-Valued Sums of Independent Functions in the General Case

We examine a more general situation. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of independent identically and symmetrically distributed r.v.'s on $[0,1]$ such that $f := f_1^* \in L_p[0,1]$ for every $p < \infty$. If $m \in \mathbb{N}$ and $k_1 + k_2 + \dots + k_n = m$ then

$$\int_0^1 f(t)^{2k_1}\, dt \cdot \ldots \cdot \int_0^1 f(t)^{2k_n}\, dt \le \|f\|_{2m}^{2k_1} \cdot \|f\|_{2m}^{2k_n} = \|f\|_{2m}^{2m}.$$

Therefore, applying the standard arguments (see [1, 2]) and taking the properties of r.v.'s f_k into account, for arbitrary $m \in \mathbb{N}$ and $a_k \in \mathbb{R}$, we obtain

$$\begin{aligned}
\int_0^1 \Big(\sum_{k=1}^{n} a_k f_k(t)\Big)^{2m} dt &= \sum_{k_1+\dots+k_n=m, k_i\ge 0} \frac{(2m)!}{(2k_1)!\dots(2k_n)!} a_1^{2k_1}\dots a_n^{2k_n} \int_0^1 f_{k_1}(t)^{2k_1}\, dt \cdot \ldots \cdot \int_0^1 f_{k_n}(t)^{2k_n}\, dt \\
&= \sum_{k_1+\dots+k_n=m, k_i\ge 0} \frac{(2m)!}{(2k_1)!\dots(2k_n)!} a_1^{2k_1}\dots a_n^{2k_n} \int_0^1 f(t)^{2k_1}\, dt \cdot \ldots \cdot \int_0^1 f(t)^{2k_n}\, dt \\
&\le \|f\|_{2m}^{2m} \cdot \frac{(2m)!}{2^m m!} \sum_{k_1+\dots+k_n=m, k_i\ge 0} \frac{m!}{k_1!\dots k_n!} a_1^{2k_1}\dots a_n^{2k_n} = \|f\|_{2m}^{2m} \cdot B_{2m}\Big(\sum_{i=1}^{n} a_i^2\Big)^m,
\end{aligned}$$

where B_{2m} is defined by (6). By the Stirling formula

$$n! = \sqrt{2\pi n} n^n e^{-n} e^{R_n}, \quad \text{where } \frac{1}{12n+1} < R_n < \frac{1}{12n}.$$

Hence,

$$B_{2m} \le \sqrt{2} \cdot 2^m e^{-m} m^m.$$

Thus,

$$\int_0^1 \left(\sum_{k=1}^n a_k f_k(t)\right)^{2m} dt \le \chi(m)^m \left(\sum_{i=1}^n a_i^2\right)^m, \tag{11}$$

where $\chi(t) = t\|f\|_{2t}^2$ $(t \ge 1)$. Clearly, χ is an increasing continuous function; we may assume that $\chi(1) = \|f\|_2^2 = 1$.

Given $n \in \mathbb{N}$, $u > 0$, and $a_k \in \mathbb{R}$ $(k = 1, 2, \dots, n)$, we put

$$m = \left[\chi^{-1}\left(\frac{u^2}{e\sum\limits_{k=1}^n a_k^2}\right)\right],$$

where χ^{-1} is the inverse of χ. Applying (11), we infer

$$m\left\{t \in [0,1] : \left|\sum_{k=1}^n a_k f_k(t)\right| > u\right\} \le u^{-2m} \int_0^1 \left|\sum_{k=1}^n a_k f_k(t)\right|^{2m} dt$$

$$\le u^{-2m}\chi(m)^m \left(\sum_{k=1}^n a_k^2\right)^m \le e^{-m} \le \operatorname{Exp}\left(-\chi^{-1}\left(\frac{u^2}{e\sum\limits_{k=1}^n a_k^2}\right) + 1\right).$$

Therefore,

$$m\left\{t \in I : \left|\sum_{i=1}^n x_i(s) f_i(t)\right| > \tau\right\} \le \operatorname{Exp}\left(-\chi^{-1}\left(\frac{u^2}{ey(s)^2}\right) + 1\right)$$

for a.e. $s \in I$, where $y := \left(\sum_{i=1}^n x_i^2\right)^{1/2} \in X$. Since

$$\exp\left(-\chi^{-1}\left(\frac{u^2}{ey(s)^2}\right) + 1\right) \le m\{t \in I : \sqrt{e}y(s)\sqrt{\chi(\log e/t)} > \tau\} \quad (0 < s \le 1),$$

integrating over I, we obtain

$$m\left\{(s,t) \in I \times I : \left|\sum_{i=1}^n x_i(s) f_i(t)\right| > \tau\right\} \le m\{(s,t) \in I \times I : \sqrt{e}y(s)\sqrt{\chi(\log e/t)} > \tau\}.$$

Thus, if $\xi(t) := \sqrt{\chi(\log e/t)}$ then

$$\left\|\sum_{i=1}^n x_i(s) f_i(t)\right\|_{X(I\times I)} \le \sqrt{e}\|T_\xi y\|_{X(I\times I)}. \tag{12}$$

Observe that $\xi \in L_p[0,1]$ for every $p < \infty$ if and only if $\|f\|_{2\log e/t} \in L_p[0,1]$ for every $p < \infty$. Hence, the following statement is an immediate consequence of (12) and the arguments of the proof of Theorem 1.

Theorem 5. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of independent identically and symmetrically distributed r.v.'s on $[0,1]$ such that $f := f_1^* \in L_p[0,1]$ for every $p < \infty$ and $f := f_1^* \neq 0$. Assume furthermore that*

$$\eta(t) := \|f\|_{2\log e/t} \in L_p[0,1] \quad \text{for every } p < \infty. \tag{13}$$

Then, for an arbitrary rearrangement invariant space X on I, the following are equivalent:
1) $\alpha_X > 0$;
2) *there exists a constant $A > 0$ such that*

$$A^{-1}\bigg\|\bigg(\sum_{k=1}^{n} x_k^2\bigg)^{1/2}\bigg\|_X \le \bigg\|\sum_{k=1}^{n} x_k(s) f_k(t)\bigg\|_{X(I\times I)} \le A\bigg\|\bigg(\sum_{k=1}^{n} x_k^2\bigg)^{1/2}\bigg\|_X$$

for every $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in X$.

Taking [5, Lemma 1] into account, we conclude that

Corollary 2. *Assume that a sequence of r.v.'s $\{f_k\}_{k=1}^{\infty}$ meets the conditions of Theorem 5 and $\lim_{t\to 0+} f^*(t) = \infty$. Then, for an arbitrary rearrangement invariant space X on I, the following are equivalent:*
1) $\alpha_X > 0$;
2) *there exists a constant $B > 0$ such that*

$$B^{-1}\bigg\|\sum_{k=1}^{n} x_k(s) r_k(t)\bigg\|_{X(I\times I)} \le \bigg\|\sum_{k=1}^{n} x_k(s) f_k(t)\bigg\|_{X(I\times I)} \le B\bigg\|\sum_{k=1}^{n} x_k(s) r_k(t)\bigg\|_{X(I\times I)}$$

for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$.

Assuming a slightly stronger condition than (13) we can obtain the next result for rearrangement invariant spaces "close" to L_∞.

Theorem 6. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of independent identically and symmetrically distributed r.v.'s on $[0,1]$ such that $f := f_1^* \in L_p[0,1]$ for every $p < \infty$. Assume furthermore that*

$$\|f\|_p \le Cp^{\gamma} \quad (1 \le p < \infty) \tag{14}$$

for some $\gamma > 0$. Then if X is a rearrangement invariant space and $\beta_X < 1$ then there exists a constant $A > 0$ such that

$$\bigg\|\sum_{k=1}^{n} x_k(s) f_k(t)\bigg\|_{X(\log^{-1/2-\gamma})(I\times I)} \le A\bigg\|\bigg(\sum_{k=1}^{n} x_k^2\bigg)^{1/2}\bigg\|_X$$

for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$.

Similar to the case of Theorem 2, the proof is based on Lemma 4 (also see (12)) and we omit it.

In conclusion, we exhibit some examples.

Example 1. We say that a positive decreasing function h on $[0,1]$ *satisfies the Δ^2-condition* ($h \in \Delta^2$) whenever there exists a constant $K > 0$ such that

$$h(t^2) \le Kh(t) \quad (0 < t \le 1).$$

If $f \le h$, where $h \in \Delta^2$, then f satisfies (14) for $\gamma = \log_2 K$. Indeed, by [15, Lemma 2], we have

$$\|f\|_p \le \|h\|_p \le Ch(2^{-p}) \le C_1 K^{\log_2 p} = C_1 p^{\log_2 K} \quad (p \ge 1).$$

In particular, $f(t) = \log^q(e/t) \in \Delta^2$ for every $q > 0$.

Example 2. Consider the function $f_\theta(t) = e^{\log^\theta(1/t)}$ $(0 < t \le 1)$. It is immediate that $f_\theta \in L_p[0,1]$ for every $\theta \in (0,1)$ if $p < \infty$ while $f_\theta \notin \Delta^2$. Using the asymptotic Laplace method, we can show that $\|f_\theta\|_p \asymp e^{\gamma_\theta p^{\frac{\theta}{1-\theta}}}$ [15, Example 1]. Hence, f_θ satisfies (13) if and only if $\theta \in (0, 1/2)$.

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