Rev Mat Complut (2010) 23: 113–137
DOI 10.1007/s13163-009-0003-1

On stability of $\mathcal{K}$-monotonicity of Banach couples

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Received: 3 November 2008 / Accepted: 31 March 2009 / Published online: 4 November 2009

Abstract Suppose that E is a separable Banach lattice of two-sided real sequences such that $\|e_n\| = 1$ $(n \in \mathbb{N})$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the standard basis. One of the main aims of this paper is a characterization of couples $\vec{E} = (E, E(2^{-k}))$ whose $\mathcal{K}$-monotonicity is stable when multiplying the weight by a constant. It is shown that such a property holds only for a couple $\vec{E}$ constructed upon a shift-invariant lattice. We construct also a non-trivial example of shift-invariant separable Banach lattice E such that the couple $\vec{E}$ is $\mathcal{K}$-monotone. The last result contrasts with the following well-known theorem due to Kalton: if E is a separable symmetric Banach lattice such that the couple $\vec{E}$ is $\mathcal{K}$-monotone then either $E = l_p$ $(1 \le p < \infty)$ or $E = c_0$.

Keywords Banach lattices · Interpolation of operators · Peetre $\mathcal{K}$-functional · Real method of interpolation · $\mathcal{K}$-monotone Banach couples · Shift-invariant spaces · Calderón-Lozanovskiĭ spaces

Mathematics Subject Classification (2000) 46M35 · 46E30

1 Introduction

A pair (X_0, X_1) of Banach spaces X_0 and X_1 is called a (compatible) Banach couple if there is a Hausdorff topological vector space in which each of X_0 and X_1 is continuously embedded. Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be two Banach

The first author was partly supported by the Russian Fund of Basic Research grant 07-01-96603-a.

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couples and X and Y be Banach spaces such that $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and $Y_0 \cap Y_1 \subset Y \subset Y_0 + Y_1$. The spaces X and Y are said to be C-*interpolation* spaces with respect to $\vec{X}$ and $\vec{Y}$ if every linear operator T bounded from X_0 into Y_0 and from X_1 into Y_1 is bounded from X into Y and

$$\|T\|_{X \to Y} \le C \max(\|T\|_{X_0 \to Y_0}, \|T\|_{X_1 \to Y_1}).$$

For an arbitrary Banach couple $\vec{X} = (X_0, X_1)$, any $x \in X_0 + X_1$ and $t > 0$ we define the *Peetre $\mathcal{K}$-functional* as follows:

$$\mathcal{K}(t, x; \vec{X}) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i\}.$$

Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ are called (*uniformly*) *relatively $\mathcal{K}$-monotone with constant* $\lambda > 0$ if for any $C > 0$ and arbitrary C-interpolation spaces X and Y the inequality

$$\mathcal{K}(t, y; \vec{Y}) \le \mathcal{K}(t, x; \vec{X}),$$

fulfilled for some $x \in X$, $y \in Y_0 + Y_1$ and all $t > 0$, implies: $y \in Y$ and $\|y\|_Y \le \lambda C\|x\|_X$. In particular, if $\vec{X} = \vec{Y}$ and $X = Y$ then we get the definition of a (*uniformly*) *$\mathcal{K}$-monotone couple with constant* λ, or *Calderón couple*. $\mathcal{K}$-monotone Banach couples are very important in interpolation theory because it is possible to give a complete description of all interpolation spaces for such a couple. Specifically, if X is an interpolation space with respect to a $\mathcal{K}$-monotone couple $\vec{X} = (X_0, X_1)$ then $X = (X_0, X_1)^{\mathcal{K}}_E$, where $(X_0, X_1)^{\mathcal{K}}_E$ is a space of the *real $\mathcal{K}$-method of interpolation* constructed by some Banach lattice E of two-sided real sequences and consisting of all $x \in X_0 + X_1$ such that $(\mathcal{K}(2^k, x; \vec{X}))_{k=-\infty}^{\infty} \in E$ with the norm $\|x\| := \|(\mathcal{K}(2^k, x; \vec{X}))_k\|_E$ [4, Theorem 4.4.5].

Here, mainly, we will consider separable Banach lattices E of two-sided real sequences with the Fatou property and such that $\|e_n\|_E = 1$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical vector basis. We recall that E has *the Fatou property* if $x^k \in E$, $\|x^k\|_E \le 1$ and $x^k \to x$ (coordinate-wise) imply $x \in E$ and $\|x\|_E \le 1$. Every such lattice generates the Banach couple $\vec{E} = (E, E(2^{-k}))$, where $E(v_k)$ is *the weighted space* consisting of all sequences $a = (a_k)_{k=-\infty}^{\infty}$ such that $a \cdot v \in E$, with the norm $\|a\|_{E(v_k)} := \|a \cdot v\|_E$. $\mathcal{K}$-monotonicity of such a couple is connected with some highly specific geometric properties of E which we define below.

If $x = (x_k)_{k=-\infty}^{\infty}$ is a real sequence then its *support* is defined by $\operatorname{supp} x = \{k \in \mathbb{Z} : x_k \ne 0\}$. If $A \subset \mathbb{Z}$ and $B \subset \mathbb{Z}$ then the inequality $A < B$ means that $a < b$ for arbitrary $a \in A$, $b \in B$. Let $\{x^n\}_{n=1}^m, \{y^n\}_{n=1}^m$ be two families of sequences. The pair $(x^n, y^n)_{n=1}^m$ is *interlaced* if supports of x^n, y^n are finite and

$$\operatorname{supp} x^n < \operatorname{supp} y^n \quad (1 \le n \le m),$$
$$\operatorname{supp} y^n < \operatorname{supp} x^{n+1} \quad (1 \le n \le m-1).$$

We say that a Banach sequence space E has *the right-shift property* (RSP) if there exists $C_{RS} > 0$ such that for any interlaced pair $(x^n, y^n)_{n=1}^m$ with $\|y^n\|_E \le \|x^n\|_E = 1$

$(1 \le n \le m)$ and for all $a_n \in \mathbb{R}$ we have

$$\left\| \sum_{n=1}^{m} a_n y^n \right\|_E \le C_{RS} \left\| \sum_{n=1}^{m} a_n x^n \right\|_E .$$

Analogously, E has *the left-shift property* (LSP) if for some $C_{LS} > 0$, any interlaced pair $(x^n, y^n)_{n=1}^m$, $\|x^n\|_E \le \|y^n\|_E = 1$ $(1 \le n \le m)$ and all $a_n \in \mathbb{R}$

$$\left\| \sum_{n=1}^{m} a_n x^n \right\|_E \le C_{LS} \left\| \sum_{n=1}^{m} a_n y^n \right\|_E .$$

Kalton showed [13, Theorem 4.5] that a couple $\vec{E} = (E, E(2^{-k}))$ is $\mathcal{K}$-monotone if and only if E has (RSP) and (LSP).

One of the main results of this paper is a characterization of couples $\vec{E} = (E, E(2^{-k}))$ whose $\mathcal{K}$-monotonicity is stable when multiplying the weight by a constant. Specifically, it is shown that such a property holds only for a couple $\vec{E}$ constructed upon a shift-invariant lattice. It may be interesting to compare this result with the well-known Cwikel-Nilsson theorem [8] which implies, in particular, that if E is separable and $\mathcal{K}$-monotonicity of a couple $\vec{E} = (E, E(2^{-k}))$ is stable for *any* replacement of the weight then E coincides either with l_p $(1 \le p < \infty)$ or with c_0. At the same time, we show that a class of $\mathcal{K}$-monotone couples $\vec{E}$ given by separable shift-invariant Banach lattices is not trivial. Specifically, we construct a separable shift-invariant lattice F which has the properties (RSP), (LSP) and such that $F \ne l_p$ $(1 \le p < \infty)$, $F \ne c_0$. Let us note that the situation changes completely if we consider *symmetric* sequence spaces. As Kalton showed [13, Theorem 4.6], in that case a couple $\vec{E}$ is $\mathcal{K}$-monotone if and only if either $E = l_p$ $(1 \le p < \infty)$ or $E = c_0$.

2 $\mathcal{K}_u$-monotone Banach couples

Let X be a lattice of measurable functions defined on some measure space with a σ-finite measure, $v(t) > 0$ be a weighted function, and $X(v)$ be the corresponding weighted space consisting of all functions $f(t)$ such that $fv \in X$, with the norm $\|f\|_{X(v)} = \|fv\|_X$. By the Sedaev result [17], if $X = L_p$ $(1 \le p \le \infty)$ then for any weights v_0, v_1, w_0, w_1 couples $(X(v_0), X(v_1))$ and $(X(w_0), X(w_1))$ are relatively $\mathcal{K}$-monotone with a constant depending on p only. The converse statement was proved by Cwikel and Nilsson in [8]. To be more precise, they showed that in the case when X is a separable lattice of functions with the Fatou property, the following conditions are equivalent:

(1) for any weights v_0, v_1, w_0, w_1 the couples $(X(v_0), X(v_1))$ and $(X(w_0), X(w_1))$ are relatively $\mathcal{K}$-monotone with a constant $\lambda > 0$ independent of weights;
(2) there are a weight v and $p \in [1, \infty)$ such that X coincides with $L_p(v)$ with equivalence of norms.

In this article we are concerned with Banach couples whose $\mathcal{K}$-monotonicity is stable concerning to a weaker disturbance of weight, namely, multiplying by a constant.

Specifically, let us suppose that (X_0, X_1) is a couple of Banach lattices of measurable functions defined on some space with a σ-finite measure. We will consider the question when the couples $(X_0, X_1(u))$ and (X_0, X_1) are relatively $\mathcal{K}$-monotone with a constant $\lambda > 0$ independent of a real number $u > 0$? As it is well known (see, for example, [10, Chap. 15] or [4]), the answer is positive if and only if there exists a constant $\lambda > 0$ such that the following is true: if $x, y \in X_0 + X_1$, $\lambda' > \lambda$ and $u > 0$ then the inequality

$$\mathcal{K}(t, y; X_0, X_1) \le \mathcal{K}(ut, x; X_0, X_1) \quad (t > 0) \tag{2.1}$$

implies that $Tx = y$ for some linear operator $T : X_i \to X_i$ $(i = 0, 1)$ such that $\|T\|_{X_0 \to X_0} \le \lambda'$ and $\|T\|_{X_1 \to X_1} \le \lambda' u$. If a couple (X_0, X_1) satisfies the last condition we will call it $\mathcal{K}_u$*-monotone.* Let us remark that this notion makes sense for arbitrary Banach couples as well.

Example 1 Show that the Banach couple (L_1, L_∞) of function spaces on the semiaxis $(0, \infty)$ with Lebesgue measure is $\mathcal{K}_u$-monotone. Indeed, if $f, g \in L_1 + L_\infty$ and $u > 0$ then in view of [3, Theorem 5.2.1] inequality (2.1) can be rewritten as follows:

$$\int_0^t g^*(s)\, ds \le \int_0^{ut} f^*(s)\, ds \quad (t > 0)$$

or, after change of variables,

$$\int_0^t \frac{g^*(s)}{u}\, ds \le \int_0^t \sigma_{1/u} f^*(s)\, ds \quad (t > 0),$$

where $\sigma_v f(s) := f(s/v)$. Then, by the Calderón-Mityagin theorem (see [5] or [15]), for an arbitrary small $\varepsilon > 0$ there exists a linear operator T bounded in L_1 and in L_∞ such that $\|T\|_{L_1 \to L_1} \le 1 + \varepsilon$, $\|T\|_{L_\infty \to L_\infty} \le 1 + \varepsilon$ and $g = uT(\sigma_{1/u} f)$. Since $\|\sigma_v\|_{L_1 \to L_1} = v$ and $\|\sigma_v\|_{L_\infty \to L_\infty} = 1$ $(v > 0)$, the operator $T_u h := u \cdot T(\sigma_{1/u} h)$ satisfies the conditions $\|T_u\|_{L_1 \to L_1} \le 1 + \varepsilon$, $\|T_u\|_{L_\infty \to L_\infty} \le u(1 + \varepsilon)$. Moreover, $T_u f = g$. Therefore, the couple (L_1, L_∞) is $\mathcal{K}_u$-monotone with constant 1.

The next simple statement generalizes the preceding example. We recall that a Banach couple $\vec{X} = (X_0, X_1)$ is *a partial retract* of a Banach couple $\vec{Y} = (Y_0, Y_1)$ if every $x \in X_0 + X_1$ is orbitally equivalent to some $y \in Y_0 + Y_1$. This means that there are linear operators $U : \vec{X} \to \vec{Y}$ and $V : \vec{Y} \to \vec{X}$ such that $Ux = y$ and $Vy = x$. If there is a constant $C > 0$ independent of $x \in X_0 + X_1$ such that for any $C' > C$ we can choose operators U and V which satisfy additionally

$$\|U\|_{\vec{X} \to \vec{Y}} = \max_{i=0,1} \{\|U\|_{X_i \to Y_i}\} \le C',$$

$$\|V\|_{\vec{Y} \to \vec{X}} = \max_{i=0,1} \{\|V\|_{Y_i \to X_i}\} \le C',$$

then $\vec{X}$ is called *a uniform partial retract* of $\vec{Y}$ (with constant C).

Proposition 2.1 *Let a Banach couple $\vec{X} = (X_0, X_1)$ be a uniform partial retract of a weighted L_p-couple. Then $\vec{X}$ is $\mathcal{K}_u$-monotone.*

Proof By hypothesis, $\vec{X}$ is a uniform partial retract of a couple $\vec{L} = (L_{p_0}(w_0), L_{p_1}(w_1))$, where $1 \le p_0, p_1 \le \infty$, $w_0 > 0$, $w_1 > 0$.

Suppose that $x, y \in X_0 + X_1$ and $u > 0$ satisfy (2.1). Then for some $C > 0$ and any $\varepsilon > 0$ there are $f, g \in L_{p_0}(w_0) + L_{p_1}(w_1)$ and linear operators $U_i : \vec{X} \to \vec{L}$ and $V_i : \vec{L} \to \vec{X}$ such that $\|U_i\|_{\vec{X}\to\vec{L}} \le C + \varepsilon$, $\|V_i\|_{\vec{L}\to\vec{X}} \le C + \varepsilon$ $(i = 1, 2)$, $U_1 x = f$, $U_2 y = g$, $V_1 f = x$, $V_2 g = y$. From the definition of the Peetre $\mathcal{K}$-functional we get

$$(C+\varepsilon)^{-1}\mathcal{K}(t, x; \vec{X}) \le \mathcal{K}(t, f; \vec{L}) \le (C+\varepsilon)\mathcal{K}(t, x; \vec{X}) \quad (t > 0)$$

and

$$(C+\varepsilon)^{-1}\mathcal{K}(t, y; \vec{X}) \le \mathcal{K}(t, g; \vec{L}) \le (C+\varepsilon)\mathcal{K}(t, y; \vec{X}) \quad (t > 0).$$

So, (2.1) implies

$$\mathcal{K}(t, g; \vec{L}) \le (C+\varepsilon)^2\mathcal{K}(tu, f; \vec{L}) = (C+\varepsilon)^2\mathcal{K}(t, f; \vec{L_u}) \quad (t > 0),$$

where $\vec{L_u} := (L_{p_0}(w_0), L_{p_1}(uw_1))$. Since $\vec{L_u}$ and $\vec{L}$ are weighted L_p-couples, then there exists $\lambda > 0$ independent of u and a linear operator $W : \vec{L_u} \to \vec{L}$ such that $\|W\|_{L_{p_0}(w_0)\to L_{p_0}(w_0)} \le \lambda(C+\varepsilon)^2$, $\|W\|_{L_{p_1}(uw_1)\to L_{p_1}(w_1)} \le \lambda(C+\varepsilon)^2$ and $g = Wf$ (see [18] and [9]). Consequently, $\|W\|_{L_{p_0}(w_0)\to L_{p_0}(w_0)} \le \lambda(C+\varepsilon)^2$, $\|W\|_{L_{p_1}(w_1)\to L_{p_1}(w_1)} \le \lambda(C+\varepsilon)^2 u$ and the operator $T := V_2 W U_1$ satisfies all conditions from the definition of $\mathcal{K}_u$-monotonicity with constant $(C+\varepsilon)^4\lambda$. □

It is well known (see [11] or [7]) that the conditions of the last proposition are fulfilled for couples of the form $(\vec{X}_{\theta_0,p_0}, \vec{X}_{\theta_1,p_1})$ $(0 < \theta_0, \theta_1 < 1,\ 1 \le p_0, p_1 \le \infty)$, where $(X_0, X_1)_{\theta,p} := (X_0, X_1)^{\mathcal{K}}_{l_p(2^{-k\theta})}$, i.e., the space with the norm

$$\|x\|_{(X_0,X_1)_{\theta,p}} = \left(\sum_{k=-\infty}^{\infty} (\mathcal{K}(2^k, x; X_0, X_1)2^{-k\theta})^p\right)^{1/p}$$

(with appropriate modification when $p = \infty$) [3]. Thus, we come to

Corollary 2.2 *For an arbitrary Banach couple $\vec{X} = (X_0, X_1)$ and any $0 < \theta_0, \theta_1 < 1$ and $1 \le p_0, p_1 \le \infty$ the Banach couple $(\vec{X}_{\theta_0,p_0}, \vec{X}_{\theta_1,p_1})$ is $\mathcal{K}_u$-monotone.*

3 Couples given by shift-invariant Banach lattices

Suppose that the shift operator $P_k(a_j) := (a_{k+j})_{j=-\infty}^{\infty}$ is bounded in a Banach lattice of two-sided real sequences E for any $k \in \mathbb{Z}$. We call E a shift-invariant lattice if $\sup_{k\in\mathbb{Z}} \|P_k\|_{E\to E} < \infty$. Later on by $\{e_n\}_{n\in\mathbb{Z}}$ we denote the canonical basis in sequence spaces, i.e., $e_n = (\dots, 0, \underbrace{1}_{n}, 0, \dots)$, and by E_θ a Banach lattice $E(2^{-\theta k})$ $(\theta \in (0, 1))$.

Theorem 3.1 *For an arbitrary Banach lattice E of two-sided sequences such that $\|e_n\|_E = 1$ ($n \in \mathbb{Z}$) the following conditions are equivalent*:

(a) *the couple $\vec{E} = (E, E(2^{-k}))$ is $\mathcal{K}_u$-monotone*;
(b) *the couple $\vec{E}$ is $\mathcal{K}$-monotone and, in addition, there exists $C > 0$ such that for any $\theta \in (0,1)$ the inequality*

$$\mathcal{K}(t, b; \vec{E}) \le \mathcal{K}(ut, a; \vec{E}) \quad (t > 0), \tag{3.1}$$

where $a \in E_\theta$, $b \in E + E(2^{-k})$ and $u > 0$, implies that $b \in E_\theta$ and

$$\|b\|_{E_\theta} \le C u^\theta \|a\|_{E_\theta}; \tag{3.2}$$

(c) *the couple $\vec{E}$ is $\mathcal{K}$-monotone and, in addition, there exist $C > 0$ and $\theta \in (0,1)$ such that inequality* (3.1), *fulfilled for some $a \in E_\theta$, $b \in E + E(2^{-k})$ and $u > 0$, implies $b \in E_\theta$ and* (3.2);
(d) *the couple $\vec{E}$ is $\mathcal{K}$-monotone and the space E is shift-invariant.*

Proof (a) $\Rightarrow$ (b). First of all, a direct examination shows that E_θ isometrically coincides with the Calderón-Lozanovskiĭ space $E^{1-\theta}E(2^{-k})^\theta$. Recall that it consists of all
$a = (a_k)_{k=-\infty}^{\infty}$ such that

$$|a_k| \le C|a_k^0|^{1-\theta}|a_k^1|^\theta \quad (k \in \mathbb{Z}), \tag{3.3}$$

for some $C > 0$, $a^0 = (a_k^0) \in E$, $a^1 = (a_k^1) \in E(2^{-k})$, $\|a^0\|_E \le 1$, $\|a^1\|_{E(2^{-k})} \le 1$. Moreover, $\|a\|_{E^{1-\theta}E(2^{-k})^\theta} = \inf C$, where the infimum is taken over all $C > 0$ satisfying (3.3).

Since E is separable then from [14, Theorem 4.1.14 and remarks after it] it follows that the spaces $E_\theta^u := E^{1-\theta}(uE(2^{-k}))^\theta$ and E_θ are 2-interpolation with respect to the couples $\vec{E}^u := (E, uE(2^{-k}))$ and $\vec{E} := (E, E(2^{-k}))$, for all $\theta \in (0,1)$ and $u > 0$ (after the standard complexification these spaces isometrically coincide with corresponding 1-interpolation spaces of the complex method of interpolation). Therefore, inequality (3.1) implies

$$\|b\|_{E_\theta} \le 2\lambda \|a\|_{E_\theta^u}, \tag{3.4}$$

for some $\lambda > 0$. We remark that (3.3) is equivalent to the following inequality:

$$|a_k| \le C u^\theta |a_k^0|^{1-\theta} \left|\frac{a_k^1}{u}\right|^\theta \quad (k \in \mathbb{Z}),$$

and also $\|a^1/u\|_{uE(2^{-k})} = \|a^1\|_{E(2^{-k})}$. Thus, $\|a\|_{E_\theta^u} = u^\theta \|a\|_{E_\theta}$ and (3.4) leads to (3.2).

Implication (b) $\Rightarrow$ (c) is evident, and so we turn to proving that (c) implies (d).

First, since the couple $\vec{E} = (E, E(2^{-k}))$ is (uniformly) $\mathcal{K}$-monotone, Theorem 4.3 from [13] implies that E has (RSP), (LSP), and then, by [2, Lemma 3], $\|P_n\|_{E \to E} \le$

$C'(n+1)$ $(n \geq 0)$ and $\|P_n\|_{E\to E} \leq C'(1-n)$ $(n<0)$. Hence, the Calderón operator defined by

$$Q(\xi_k)_n := \sum_{k=-\infty}^{\infty} \min(1, 2^{n-k})\xi_k$$

is bounded in the Banach lattice E_θ for every $\theta \in (0,1)$, which implies that any linear operator bounded from l_1 into l_∞ and from $l_1(2^{-k})$ into $l_\infty(2^{-k})$ is bounded in E_θ. Moreover, since $\|e_n\| = 1$ $(n \in \mathbb{Z})$, we have

$$l_1 \subset E \subset l_\infty. \tag{3.5}$$

Thus,

$$(l_1, l_1(2^{-k}))^{\mathcal{K}}_{E_\theta} = (E, E(2^{-k}))^{\mathcal{K}}_{E_\theta} = (l_\infty, l_\infty(2^{-k}))^{\mathcal{K}}_{E_\theta} = E_\theta \tag{3.6}$$

(more general results in this direction can be found in [1]).

Suppose that E is not shift-invariant. Then, there exist a sequence of integers $\{n_k\}_{k=1}^{\infty}$, converging either to $+\infty$ or to $-\infty$, and a sequence $a^k = (a^k_j)_{j=-\infty}^{\infty} \in E$ such that $\|a^k\|_E = 1$ $(k = 1, 2, \ldots)$ and

$$\lim_{k\to\infty} \|P_{n_k} a^k\|_E = \infty. \tag{3.7}$$

If we set $b^k := (a^k_j 2^{j\theta})_{j\in\mathbb{Z}}$, then $b^k \in E_\theta$, $\|b^k\|_{E_\theta} = 1$ $(k = 1, 2, \ldots)$ and

$$P_{n_k} b^k = (a^k_{j-n_k} 2^{(j-n_k)\theta})_{j\in\mathbb{Z}} = 2^{-n_k\theta}((P_{n_k}a^k)_j 2^{j\theta})_{j\in\mathbb{Z}}.$$

Consequently, $\|P_{n_k} b^k\|_{E_\theta} = 2^{-n_k\theta}\|P_{n_k} a^k\|_E$ and in view of (3.7)

$$\lim_{k\to\infty} 2^{n_k\theta}\|P_{n_k} b^k\|_{E_\theta} = \infty. \tag{3.8}$$

Let us denote $c^k_j := \mathcal{K}(2^j, b^k; \vec{E})$ and $c^k = (c^k_j)_{j\in\mathbb{Z}}$. Taking into consideration (3.5), we get:

$$\begin{aligned} |b^k_j| &\leq \sup_{i\in\mathbb{Z}} \min(1, 2^{j-i})|b^k_i| = \mathcal{K}(2^j, b^k; l_\infty, l_\infty(2^{-i})) \\ &\leq \mathcal{K}(2^j, b^k; \vec{E}) = c^k_j \quad (j \in \mathbb{Z},\ k = 1, 2, \ldots). \end{aligned}$$

By that, if $d^k := P_{n_k} c^k$, then $d^k \geq P_{n_k}(|b^k|)$, hence, having in mind (3.8),

$$\lim_{k\to\infty} 2^{n_k\theta}\|d^k\|_{E_\theta} = \infty. \tag{3.9}$$

Let us define a sequence of functions

$$f_k(t) := \mathcal{K}(2^{-n_k}t, b^k; \vec{E}) \quad (t > 0),\ k = 1, 2, \ldots.$$

Since E is separable, the intersection $E \cap E(2^{-k})$ is dense in the sum $E + E(2^{-k})$. Therefore, f_k belongs to Φ_0 ($k = 1, 2, \dots$), where Φ_0 is the set of all increasing concave functions φ defined on the semiaxis $(0, \infty)$ and such that

$$\lim_{t\to 0+} \varphi(t) = \lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$$

(see, for example, [16, p. 386]). □

Next, we need an auxiliary statement which, in fact, is already known [16, p. 398]. Nevertheless, we give its proof for the sake of completeness as well as to emphasize universality of constants appearing.

Lemma 3.2 *For any function* $\varphi \in \Phi_0$ *there exists a sequence* $\xi = (\xi_i)_{i\in\mathbb{Z}} \in l_1 + l_1(2^{-k})$ *such that*

$$\frac{1}{2}\varphi(t) \le \mathcal{K}(t, \xi; l_1, l_1(2^{-k})) \le \varphi(t) \quad (t > 0).$$

Proof As it is well known (see, for example, [3, p. 152]), a function $\varphi \in \Phi_0$ has the following integral representation:

$$\varphi(t) = -\int_0^\infty \min(t, s)\, d(\varphi'(s)) \quad (t > 0), \tag{3.10}$$

with the usual Riemann-Stieltjes integral on the right (since φ is concave, derivative φ' decreases). Recall also the following well-known formula:

$$\mathcal{K}(t, (c_i); l_1, l_1(2^{-k})) = \sum_{i=-\infty}^{\infty} \min(1, t2^{-i})|c_i| \quad (t > 0).$$

So, if $\xi_i := 2^i \cdot (\varphi'(2^i) - \varphi'(2^{i+1}))$ and $\xi = (\xi_i)$, then in view of (3.10)

$$\begin{aligned}\varphi(t) &= \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \min(t, s)\, d(-\varphi'(s)) \\ &\ge \sum_{i=-\infty}^{\infty} \min(t, 2^i)(\varphi'(2^i) - \varphi'(2^{i+1})) \\ &= \mathcal{K}(t, \xi; l_1, l_1(2^{-k})) \quad (t > 0),\end{aligned}$$

whence, in particular, $\xi \in l_1 + l_1(2^{-k})$. At the same time,

$$\begin{aligned}\varphi(t) &\le \sum_{i=-\infty}^{\infty} \min(t, 2^{i+1})(\varphi'(2^i) - \varphi'(2^{i+1})) \\ &\le 2\mathcal{K}(t, \xi; l_1, l_1(2^{-k})) \quad (t > 0),\end{aligned}$$

and the proof is completed. □

Continuation of the proof of implication (c) $\Rightarrow$ (d). Since $f_k \in \Phi_0$ then, by Lemma 3.2, for each $k = 1, 2, \dots$ there exists a sequence $\xi^k \in l_1 + l_1(2^{-k})$ such that

$$\frac{1}{2} f_k(t) \le \mathcal{K}(t, \xi^k; l_1, l_1(2^{-k})) \le f_k(t) \quad (t > 0). \tag{3.11}$$

From (3.5) it follows that $\xi^k \in E + E(2^{-k})$. Having in mind the definition of f_k we get

$$\mathcal{K}(t, \xi^k; \vec{E}) \le \mathcal{K}(t, \xi^k; l_1, l_1(2^{-k})) \le \mathcal{K}(2^{-n_k} t, b^k; \vec{E}) \quad (t > 0).$$

Therefore, the sequences ξ^k, b^k and $u = 2^{-n_k}$ satisfy condition (3.1), so it follows that $\xi^k \in E_\theta$ and

$$\|\xi^k\|_{E_\theta} \le C 2^{-n_k \theta} \|b^k\|_{E_\theta} = C 2^{-n_k \theta} \quad (k = 1, 2, \dots).$$

But then the definition of d^k and relations (3.11) and (3.6) imply that

$$\begin{aligned} \|d^k\|_{E_\theta} &= \|(\mathcal{K}(2^{j-n_k}, b^k; \vec{E}))_j\|_{E_\theta} \le 2\|(K(2^j, \xi^k; l_1, l_1(2^{-k})))_j\|_{E_\theta} \\ &\le 2C_1 \|\xi^k\|_{E_\theta} \le 2C_1 C 2^{-n_k \theta} \quad (k = 1, 2, \dots), \end{aligned}$$

which contradicts (3.9). Thus, our assumption is wrong and E is shift-invariant.

(d) $\Rightarrow$ (a). We may assume that $\|P_n\|_{E \to E} = 1$, $n \in \mathbb{Z}$ (this can be easily achieved by replacing the norm in E with the following equivalent one $\|x\|_E := \sup_n \|P_n x\|_E$). Suppose that inequality (3.1) is true, for some $a, b \in E + E(2^{-k})$ and $u > 0$. We have to find a linear operator $T : \vec{E} \to \vec{E}$ such that $\|T\|_{E \to E} \le \lambda$, $\|T\|_{E(2^{-k}) \to E(2^{-k})} \le \lambda u$ and $Tx = y$, where a constant λ is independent of a, b and u. Note, that it is enough to prove the last statement only in the case when $u = 2^k$ $(k \in \mathbb{Z})$.

In accordance with terminology of [13], the couple $\vec{E}$ is exponentially separated and $\rho(n) := \|e_n\|_E / \|e_n\|_{E(2^{-k})} = 2^n$ $(n \in \mathbb{Z})$. Therefore, in view of [13, Lemma 4.1] there exists $C_2 > 0$ such that for any $x \in E + E(2^{-k})$

$$\mathcal{K}(t, x; \vec{E}) \le \|x \cdot (-\infty, n]\|_E + t\|x \cdot (n, \infty)\|_{E(2^{-k})} \le C_2 \mathcal{K}(t, x; \vec{E}), \tag{3.12}$$

if $2^n \le t \le 2^{n+1}$ (for any $x = (x_j)_{j=-\infty}^{\infty}$ and $V \subset \mathbb{Z}$ we put $x \cdot V = (x \cdot V)_j$, where $(x \cdot V)_j = x_j$ if $j \in V$ and $(x \cdot V)_j = 0$ if $j \notin V$).

Let $u = 2^k$, $k \in \mathbb{Z}$. If $2^n \le t \le 2^{n+1}$, then $2^{n+k} \le ut \le 2^{n+k+1}$, and, by concavity of $\mathcal{K}$-functional and relations (3.12) and (3.1), we get

$$\mathcal{K}(t, b; \vec{E}) \le 2(\|a \cdot (-\infty, n+k]\|_E + 2^{n+k}\|a \cdot (n+k, \infty)\|_{E(2^{-k})}). \tag{3.13}$$

Since E is a shift-invariant lattice, then

$$\|a \cdot (-\infty, n+k]\|_E = \left\| \sum_{i=-\infty}^{n+k} a_i e_i \right\|_E = \left\| \sum_{i=-\infty}^{n} a_{i+k} e_i \right\|_E = \|(P_{-k} a) \cdot (-\infty, n]\|_E,$$

and analogously

$$2^k\|a\cdot(n+k,\infty)\|_{E(2^{-k})}=2^k\left\|\sum_{i=n+k+1}^{\infty}a_i2^{-i}e_i\right\|_E$$
$$=\left\|\sum_{i=n+1}^{\infty}a_{i+k}2^{-i}e_i\right\|_E=\|(P_{-k}a)\cdot(n,\infty)\|_{E(2^{-i})}.$$

Thus, (3.13) can be rewritten in the following way:

$$\mathcal{K}(t,b;\vec{E})\le 2(\|(P_{-k}a)\cdot(-\infty,n]\|_E+2^n\|(P_{-k}a)\cdot(n,\infty)\|_{E(2^{-i})}).$$

Since the last inequality holds for all $2^n\le t\le 2^{n+1}$, $n\in\mathbb{Z}$, then applying (3.12) once again we get

$$\mathcal{K}(t,b;\vec{E})\le 2C_2\mathcal{K}(t,P_{-k}a;\vec{E})\quad (t>0). \tag{3.14}$$

By hypotheses, the Banach couple $\vec{E}$ is (uniformly) $\mathcal{K}$-monotone. So, there exists a linear operator $T':\vec{E}\to\vec{E}$ satisfying $\|T'\|_{E\to E}\le C_3$, $\|T'\|_{E(2^{-k})\to E(2^{-k})}\le C_3$ and $T'(P_{-k}a)=b$. Now we put $T:=T'P_{-k}$. Since E is shift-invariant, then

$$\|T\|_{E\to E}\le\|T'\|_{E\to E}\cdot\|P_{-k}\|_{E\to E}\le C_3$$

and

$$\|T\|_{E(2^{-k})\to E(2^{-k})}\le\|T'\|_{E(2^{-k})\to E(2^{-k})}\cdot\|P_{-k}\|_{E(2^{-k})\to E(2^{-k})}\le C_3 2^k.$$

Moreover, $Ta=b$. Hence, $\vec{E}$ is $\mathcal{K}_u$-monotone and the proof is completed. □

Now we can give examples of $\mathcal{K}$-monotone but not $\mathcal{K}_u$-monotone Banach couples.

Example 2 We consider an arbitrary Banach lattice of sequences G having the following properties: (a) G is separable; (b) $\|e^k\|_G=1$; (c) G has (LSP) and (RSP); (d) G is not shift-invariant. Note that such lattices exist. The first example is the direct sum $l_q(\mathbb{Z}_-)\oplus l_r(\mathbb{Z}_+)$ $(1\le q\ne r<\infty)$ with the norm

$$\|(x_k)\|=\left(\sum_{k=-1}^{-\infty}|x_k|^q\right)^{1/q}+\left(\sum_{k=0}^{\infty}|x_k|^r\right)^{1/r}.$$

The second one is the well-known Tsirelson space T and its versions [6]. Right- and left-shift properties for the former of these spaces are checked directly, for the latter see [6, Proposition II.4].

In view of (c) and [13, Theorem 4.5] every such Banach couple $(G,G(2^{-k}))$ is $\mathcal{K}$-monotone. At the same time, since G is not shift-invariant, by Theorem 3.1, the last couple is not $\mathcal{K}_u$-monotone. In particular, according to Proposition 2.1 $(G,G(2^{-k}))$ is not a uniformly partial retract of any weighted L_p-couple.

A natural question arises in connection with the preceding examples, namely, does there exist a separable shift-invariant Banach lattice with (RSP), (LSP) which does not coincide with l_p $(1 \le p < \infty)$ and c_0? In view of [13] and Theorem 3.1 the question can be reformulated this way: does there exist a $\mathcal{K}_u$-monotone couple of the form $(E, E(2^{-k}))$, where E is a separable Banach lattice, $\|e^k\|_E = 1$, $E \ne l_p$ $(1 \le p < \infty)$, $E \ne c_0$? This question is also interesting in connection with the well-known Kalton result [13, Theorem 4.6] saying that for a *symmetric* sequence space E such a couple is $\mathcal{K}$-monotone if and only if $E = l_p$ $(1 \le p < \infty)$ or $E = c_0$. In the next part of the article such an example, interesting as we hope in other respects as well, is constructed.

4 A shift-invariant Banach lattice with (RSP) and (LSP)

Our construction of a shift-invariant Banach lattice F possessing the properties (RSP) and (LSP) is somewhat similar to the one of the dual T^* for Tsirelson's space proposed by Figiel and Johnson [12]. The choice of the recursive norm in F is caused by results stated in [6] where, in particular, it is proved that T^* has (RSP) and (LSP). However, the proof of these properties in the case of F significantly differs from the one in the case of T^*. In this connection it may be interesting to compare our Lemma 4.7 with Lemma I.11 from [6].

Definition 4.1 If $a, b \in \mathbb{Z}$, $a < b$, then $[a; b] := \{a, a+1, \dots, b\}$. By *a partition* we will mean a finite ordered set of intervals $\{[a_j; a_{j+1} - 1]\}_{j=1}^{p}$, where $a_j \in \mathbb{Z}$ and $a_{j+1} > a_j$ for all $j = 1, 2, \dots, p$.

We denote by $\mathcal{C}$ a set of all partitions. Let $c \in \mathcal{C}$. We use next the following notation:
card c is number of intervals in the partition c;
$c[j]$—j-th interval in c (numeration begins with 1);
$\operatorname{card} c[j] = \max c[j] - \min c[j] + 1$.

Definition 4.2 A partition c is called *admissible* if

$$\operatorname{card}\{j : \operatorname{card} c[j] \le k\} \le 14k$$

for any $k > 0$. The set of all admissible partitions will be denoted by $\mathcal{A}$.

At first, we define the sequence of the norms $\{\|\cdot\|_m\}_{m=0}^{\infty}$ on the space c_{00} of all real sequences which are eventually 0. If $x = \sum_n \alpha_n e_n \in c_{00}$ then

$$\begin{cases} \|x\|_0 = \max_n |\alpha_n|; \\ \|x\|_{m+1} = \max(\|x\|_m, \frac{1}{2}\max_{c\in\mathcal{A}} \sum_{j=1}^{\operatorname{card} c} \|c[j]x\|_m), \quad m \ge 0. \end{cases}$$

It is easy to see that the space c_{00} with the norm $\|\cdot\|_m$ is a shift-invariant lattice, and for all $x \in c_{00}$

$$\|x\|_{c_0} \le \|x\|_m \le \|x\|_{l_1}.$$

Since $\|x\|_{m+1} \geq \|x\|_m$ for any x, there exists a limit $\|x\|_{\mathcal{A}} = \lim_{m\to\infty} \|x\|_m$. Let F be the $\|\cdot\|_{\mathcal{A}}$-completion of c_{00}. The simplest properties of F are listed in the following proposition.

Proposition 4.1

(i) *F is a separable shift-invariant Banach lattice with the Fatou property.*
(ii) *For any $x \in F$*

$$\|x\|_{\mathcal{A}} = \max\left(\|x\|_{c_0}, \frac{1}{2}\sup_{c\in\mathcal{A}} \sum_{j=1}^{\operatorname{card} c} \|c[j]x\|_{\mathcal{A}}\right). \tag{4.1}$$

(iii) *For any sequence $(y^j)_{j=1}^{\infty}$ of vectors $y^j \in c_{00}$ such that*

$$\operatorname{supp} y^j \subset [b_j; b_{j+1}-1], \quad b_{j+1} = b_j + \left\lceil \frac{j}{14} \right\rceil, \quad j = 1, 2, \ldots$$

we have

$$\left\| \sum_{j=1}^{\infty} y^j \right\|_{\mathcal{A}} \geq \frac{1}{2} \sum_{j=1}^{\infty} \|y^j\|_{\mathcal{A}}.$$

Proposition 4.2 *F does not coincide with l_p $(1 \leq p < \infty)$ and c_0.*

Proof According to property (iii) from Proposition 4.1, F contains a basic sequence equivalent to the canonical basis in l_1. Thus, it is enough to show that $F \neq l_1$. Suppose the contrary. Let

$$K := \sup_{x\in F} \frac{\|x\|_{l_1}}{\|x\|_{\mathcal{A}}} < \infty.$$

We choose vector $y^1 = \sum_i \alpha_i e_i \in c_{00}$ such that

$$\|y^1\|_{\mathcal{A}} < \frac{8}{7K} \|y^1\|_{l_1},$$

and define

$$y^n = \sum_{i=-\infty}^{\infty} \alpha_i e_{i+(n-1)d}, \quad d = \operatorname{diam} \operatorname{supp} y^1 + 1, \quad n > 1,$$

where here and next $\operatorname{diam} U := \max U - \min U$ $(U \subset \mathbb{Z})$. Now, using (4.1), let us estimate norm of the vector $y = \sum_{n=1}^{N} y^n$, where $N = 28(2d-1)+4$. If $\|y\|_{\mathcal{A}} = \|y\|_{c_0}$, then we immediately come to contradiction. Therefore,

$$\|y\|_{\mathcal{A}} = \frac{1}{2} \sum_{j=1}^{\operatorname{card} c} \|c[j]y\|_{\mathcal{A}}$$

for some partition c. Let

$$H = \{n : \operatorname{card}\{j : c[j]y^n \neq 0\} > 1\}.$$

Since the interval $I = [\min \operatorname{supp} y^{\min H}; \max \operatorname{supp} y^{\max H}]$ contains at least $\operatorname{card} H - 1$ intervals from c, we get

$$\begin{aligned}
Nd \geq \operatorname{card} I &\geq \sum_{c[j]\subset I} \operatorname{card} c[j] = \sum_{k=1}^{\infty} \operatorname{card}\{c[j] \subset I : \operatorname{card} c[j] \geq k\} \\
&\geq \sum_{k=1}^{\infty} \max(0, \operatorname{card} H - 1 - 14(k-1)) \geq \sum_{k=1}^{2d} (\operatorname{card} H - 1 - 14(k-1)) \\
&= 2d(\operatorname{card} H - 1) - 7(1 + (2d-1))(2d-1),
\end{aligned}$$

whence, it follows

$$\operatorname{card} H \leq \frac{N}{2} + 7(2d-1) + 1 = \frac{3N}{4}.$$

If $n \in G := [1; N] \setminus H$, then

$$\sum_{j=1}^{\operatorname{card} c} \|c[j]y^n\|_{\mathcal{A}} \leq \|y^n\|_{\mathcal{A}}.$$

Thus,

$$\begin{aligned}
\|y\|_{\mathcal{A}} &= \frac{1}{2} \sum_{j=1}^{\operatorname{card} c} \left\| c[j] \sum_{n=1}^{N} y^n \right\|_{\mathcal{A}} \\
&\leq \frac{1}{2} \sum_{j=1}^{\operatorname{card} c} \left\| c[j] \sum_{n\in H} y^n \right\|_{\mathcal{A}} + \frac{1}{2} \sum_{j=1}^{\operatorname{card} c} \left\| c[j] \sum_{n\in G} y^n \right\|_{\mathcal{A}} \\
&\leq \sum_{n\in H} \|y^n\|_{\mathcal{A}} + \frac{1}{2} \sum_{n\in G} \|y^n\|_{\mathcal{A}} \leq \frac{7}{8} N \|y^1\|_{\mathcal{A}} < \frac{1}{K} \|y\|_{l_1}.
\end{aligned}$$

We came to contradiction which proves the statement. $\square$

Now we formulate the main result of this part of the article.

Theorem 4.3 *F possesses the properties (RSP) and (LSP).*

Corollary 4.4 *The Banach couple $(F, F(2^{-k}))$ is $\mathcal{K}_u$-monotone and it is not a uniform partial retract of any weighted L_p-couple.*

Proof $\mathcal{K}_u$-monotonicity of this couple is an immediate result of applying Theorems 3.1, 4.3, Proposition 4.1 and [13, Theorem 4.5]. The second part follows from Proposition 4.2 and the proof of Theorem 1 from [2]. $\square$

We split the proof of Theorem 4.3 in two stages. First, we show that F satisfies (RSP) and (LSP) for interlaced pairs of families consisting of vectors e_n (the weak shift properties (WRSP), (WLSP)). Then, by comparing norms of arbitrary linear combinations of finite vectors and corresponding linear combinations of e_n we obtain the desired result.

Definition 4.3 We say that a Banach lattice E has *the weak right-shift property* (WRSP), if there is $C_{WRS} > 0$ such that for any interlaced pair $(e_{n_i}, e_{m_i})_{i=1}^k$ (i.e., $n_i < m_i < n_{i+1}$) and arbitrary $\alpha_i \in \mathbb{R}$

$$\left\| \sum_{i=1}^k \alpha_i e_{m_i} \right\|_E \le C_{WRS} \left\| \sum_{i=1}^k \alpha_i e_{n_i} \right\|_E .$$

Analogously, if

$$\left\| \sum_{i=1}^k \alpha_i e_{n_i} \right\|_E \le C_{WLS} \left\| \sum_{i=1}^k \alpha_i e_{m_i} \right\|_E ,$$

for any $\alpha_i \in \mathbb{R}$, then E has *the weak left-shift property* (WLSP).

Proposition 4.5 *F has (WRSP), (WLSP) and $C_{WRS} = C_{WLS} \le 2$.*

First, we introduce a number of definitions and prove some statements which will form a basis for further discussion.

Definition 4.4 Let $c = \{[a_j; a_{j+1} - 1]\}_{j=1}^p$ be a partition. We call a partition c' *enlargement* of c if $c' = \{[a_{j_l}; a_{j_{l+1}} - 1]\}_{l=1}^r$ for some $1 = j_1 < j_2 < \cdots < j_{r+1} = p + 1$. *The constant of the enlargement c'* is the value

$$\mathrm{En}(c', c) = \max_{l=1,\dots,r} (j_{l+1} - j_l).$$

In other words, $\mathrm{En}(c', c)$ is the maximal number of intervals from c which are joined together in one interval of c'.

Definition 4.5 An ordered set

$$S = \{c_1; c_2; \dots; c_r\}, \quad c_l \in \mathcal{C}\ (l = 1, \dots, r),$$

is called *a system of partitions*, if

$$\max c_l[\operatorname{card} c_l] = \min c_{l+1}[1] - 1 \quad (l = 1, \dots, r - 1).$$

Definition 4.6 We say that a system of partitions $S' = \{c_1'; c_2'; \dots; c_r'\}$ is *an enlargement* of a system $S = \{c_1; c_2; \dots; c_r\}$, if c_l' is an enlargement of c_l for all $1 \le l \le r$. We also define *the constant of the enlargement S'* as follows:

$$\mathrm{En}(S', S) = \max_{l=1,\dots,r} \mathrm{En}(c_l', c_l).$$

Definition 4.7 We will say that a partition c is *given by a system* of partitions $S = \{c_1; c_2; \ldots; c_r\}$, if

$$c = \bigcup_{l=1}^{r} c_l.$$

Now, we are ready to state the first auxiliary statement.

Lemma 4.6 *Let $c'_1, c'_2, \ldots, c'_h$ be admissible partitions ($h \leq 14$), $S = \{c_1; c_2; \ldots; c_h\}$ be a system of partitions (not necessarily admissible) such that $\operatorname{card} c_l = \operatorname{card} c'_l$ for any $1 \leq l \leq h$ and*

$$\sum_{j=j_1}^{j_2} \operatorname{card} c_l[j] \geq \frac{1}{3} \sum_{j=j_1}^{j_2-2} \operatorname{card} c'_l[j]$$

if

$$2 \leq j_1 < j_2 \leq \operatorname{card} c_l - 1, \quad j_2 \geq j_1 + 2.$$

Then there exists a system S^e being an enlargement of S with constant $\operatorname{En}(S^e, S) \leq 14$ such that the partition c^e given by the system S^e is admissible.

Proof It is enough to show that for any $1 \leq l \leq h$ we can construct an enlargement c_l^e of the partition c_l with a constant satisfying $\operatorname{En}(c_l^e, c_l) \leq 14$ and such that

$$\operatorname{card}\{r < \operatorname{card} c_l^e : \operatorname{card} c_l^e[r] \leq k\} < k \tag{4.2}$$

for all $k > 0$. We will successively join intervals from c_l, i.e., we set

$$R_l = \left\lceil \frac{\operatorname{card} c_l}{14} \right\rceil - 1;$$

$$c_l^e[r] = \bigcup_{j=14(r-1)+1}^{14r} c_l[j], \quad r = 1, \ldots, R_l;$$

$$c_l^e[R_l + 1] = \bigcup_{j=14R_l+1}^{\operatorname{card} c_l} c_l[j].$$

Now let us assume that there is a $k > 0$ which fails (4.2), i.e.,

$$\operatorname{card} c_l^e[r_v] \leq k, \ 1 \leq r_v \leq R_l, \quad v = 1, \ldots, k.$$

Let

$$j_1^v = 14(r_v - 1) + 2 \geq 2;$$

$$j_2^v = 14 r_v \leq \operatorname{card} c_l - 1.$$

According to the restrictions on c_l,

$$\sum_{j=j_1^v}^{j_2^v-2} \operatorname{card} c_l'[j] \le 3 \sum_{j=j_1^v}^{j_2^v} \operatorname{card} c_l[j] < 3 \operatorname{card} c_l^e[r_v] \le 3k.$$

Summing by v, we get

$$\sum_{v=1}^{k} \sum_{j=j_1^v}^{j_2^v-2} \operatorname{card} c_l'[j] < 3k^2. \tag{4.3}$$

Let $M = \{j : j \in [j_1^v; j_2^v - 2]; v \in [1; k]\}$. Then $\operatorname{card} M = 11k$. As $c_l' \in \mathcal{A}$,

$$\begin{aligned}
\sum_{j\in M} \operatorname{card} c_l'[j] &= \sum_{k_1=1}^{\infty} \operatorname{card}\{j \in M : \operatorname{card} c_l'[j] \ge k_1\} \\
&\ge \sum_{k_1=1}^{\infty} \max(0, 11k - 14(k_1 - 1)) = \sum_{k_1=1}^{\lfloor \frac{11k}{14} \rfloor +1} (11k - 14(k_1 - 1)) \\
&= 11k\left(\left\lfloor \frac{11k}{14} \right\rfloor + 1\right) - 14\frac{1}{2}\left(\left\lfloor \frac{11k}{14} \right\rfloor + 1\right)\left\lfloor \frac{11k}{14} \right\rfloor \\
&\ge \frac{11k}{14}\left(11k - 7\left\lfloor \frac{11k}{14} \right\rfloor\right) \ge \frac{121k^2}{28} > 3k^2.
\end{aligned}$$

The obtained estimate contradicts (4.3). Consequently, our assumption is wrong, and the lemma is proved. □

Definition 4.8 A partition c is called *a standard partition* of a vector $x \in c_{00}$, if (1) $c[j]x \neq 0$ for all $1 \le j \le \operatorname{card} c$ and (2) $\operatorname{supp} x \subset \bigcup_{j=1}^{\operatorname{card} c} c[j]$.

Definition 4.9 Let

$$\begin{aligned}
x &= \sum_{i=1}^{k} a_i e_{n_i}, \quad n_i < n_{i+1} \quad (i = 1, \dots, k-1), \\
y &= \sum_{i=1}^{k} a_i e_{m_i}, \quad m_i < m_{i+1} \quad (i = 1, \dots, k-1)
\end{aligned} \tag{4.4}$$

be two finite vectors and c_1, c_2 be two standard partitions of x and y, respectively. We say that c_1 and c_2 are *equivalent with respect to* x and y, if

$$n_i \in c_1[j] \iff m_i \in c_2[j]$$

for all possible i, j. Certainly, equivalent partitions have the same number of intervals.

Let us introduce a binary relation $\mathcal{P}$ on the set of finite vectors. For two vectors x and y representable in the form (4.4), the pair $< x, y >$ belongs $\mathcal{P}$ if and only if for all $1 \le i_1 \le i_2 \le k$ and for any standard admissible partition c' of the vector $\sum_{i=i_1}^{i_2} \alpha_i e_{m_i}$ there exists an equivalent partition c of the vector $\sum_{i=i_1}^{i_2} \alpha_i e_{n_i}$ such that

$$\sum_{j=j_1}^{j_2} \operatorname{card} c[j] \ge \frac{1}{3} \sum_{j=j_1}^{j_2-2} \operatorname{card} c'[j] \tag{4.5}$$

if

$$2 \le j_1 < j_2 \le \operatorname{card} c - 1, \quad j_2 \ge j_1 + 2. \tag{4.6}$$

Lemma 4.7 *Let* $< x, y > \in \mathcal{P}$. *Then*

$$\|x\|_{m+1} \ge \frac{1}{2} \sum_{l=1}^{\operatorname{card} c'} \|c'[l]y\|_m,$$

for any partition $c' \in \mathcal{C}$ *such that* $\operatorname{card} c' \le 14$ *and for arbitrary* $m \ge 0$.

Proof We prove this statement by induction. We may (and will) assume that c' is a standard partition of y. Let c be an equivalent partition of x. Since $\operatorname{card} c \le 14$, then c is admissible and, consequently, the statement is true for $m = 0$. Now we fix $m > 0$. We define the sets M_0 and M_{m-1} as follows:

$$M_0 = \{l : \|c'[l]y\|_m = \|c'[l]y\|_0\};$$
$$M_{m-1} = [1; \operatorname{card} c'] \setminus M_0.$$

It is easy to see that if $l \in M_{m-1}$ then

$$\|c'[l]y\|_m = \frac{1}{2} \sum_{j=1}^{\operatorname{card} c'_l} \|c'_l[j]c'[l]y\|_{m-1},$$

for some standard admissible partition c'_l of the vector $c'[l]y$. Next, we define the following system of partitions $S = \{c_l\}_{l=1}^{\operatorname{card} c}$:

(i) if $l \in M_0$ then $c_l = \{c[l]\}$;
(ii) if $l \in M_{m-1}$ then c_l is a partition of $c[l]x$, equivalent to the partition c'_l of the vector $c'[l]y$ such that c_l and c'_l satisfy (4.5) and

$$\begin{aligned} \min c_l[1] &= \min c[l]; \\ \max c_l[\operatorname{card} c_l] &= \max c[l]. \end{aligned} \tag{4.7}$$

Note that equalities (4.7) can be achieved by means of a suitable choice of $\operatorname{card} c_l[1]$ and $\operatorname{card} c_l[\operatorname{card} c_l]$. Since lengths of the first and the last intervals are missing in formula (4.5) and $< x, y > \in \mathcal{P}$, then a needed partition c_l exists.

According to Lemma 4.6, there is an enlargement S^e of the system S with constant at most 14 such that the partition c^e given by the system S^e is admissible. Consequently,

$$\|x\|_{m+1} \geq \frac{1}{2} \sum_{r=1}^{\operatorname{card} c^e} \|c^e[r]x\|_m.$$

Let

$$R_{m-1} = \left\{ r : \exists l \in M_{m-1}, c^e[r] \subset \bigcup_{j=1}^{\operatorname{card} c_l} c_l[j]c[l] \right\}.$$

It is obvious that

$$\bigcup_{r \in R_{m-1}} c^e[r] = \bigcup_{l \in M_{m-1}} \bigcup_{j=1}^{\operatorname{card} c_l} c_l[j]c[l].$$

Taking into account that each of the intervals $c^e[r]$ is an union of at most 14 intervals $c_l[j]c[l]$, we can use the inductive hypothesis. Therefore,

$$\sum_{r \in R_{m-1}} \|c^e[r]x\|_m \geq \frac{1}{2} \sum_{l \in M_{m-1}} \sum_{j=1}^{\operatorname{card} c'_l} \|c'_l[j]c'[l]y\|_{m-1} = \sum_{l \in M_{m-1}} \|c'[l]y\|_m.$$

If $r \notin R_{m-1}$ then $c^e[r] = c[l]$ for some $l \in M_0$ and

$$\|c^e[r]x\|_m \geq \|c^e[r]x\|_0 = \|c'[l]y\|_0 = \|c'[l]y\|_m.$$

Finally, we get

$$\|x\|_{m+1} \geq \frac{1}{2} \sum_{l=1}^{\operatorname{card} c'} \|c'[l]y\|_m,$$

and the proof is completed. □

Now we are ready for proving that F has (WRSP) and (WLSP).

Proof of Proposition 4.5 We prove only the property (WRSP). Let $(e_{n_i}; e_{m_i})_{i=1}^k$ be an interlaced pair,

$$x = \sum_{i=1}^k \alpha_i e_{n_i} \quad \text{and} \quad y = \sum_{i=1}^k \alpha_i e_{m_i}.$$

Having in mind Lemma 4.7, we have to establish only that the pair $< x, y >$ belongs to $\mathcal{P}$. We consider an arbitrary standard partition c' of the vector y and construct an equivalent partition c of the vector x in the following way:

$c[1] = [\min \operatorname{supp} x; \max\{n_i : m_i \in c'[1]\}]$;

$$c[j]=[\max\{n_i : m_i \in c'[j-1]\}+1; \max\{n_i : m_i \in c'[j]\}], \quad 2 \le j \le \operatorname{card} c.$$

Let j_1, j_2 satisfy inequalities (4.6). Then

$$\sum_{j=j_1}^{j_2} \operatorname{card} c[j] = \max\{n_i : m_i \in c'[j_2]\} - \max\{n_i : m_i \in c'[j_1-1]\}$$

$$= n_{i_M} - n_{i_m},$$

where $i_M = \max\{i : m_i \in c'[j_2]\}$, $i_m = \max\{i : m_i \in c'[j_1-1]\}$. Then, as $m_i - m_{i-1} < n_{i+1} - n_{i-1}$ $(2 \le i \le k-1)$,

$$\sum_{j=j_1}^{j_2-2} \operatorname{card} c'[j] < m_{i_M-1} - m_{i_m} = \sum_{i=i_m+1}^{i_M-1} (m_i - m_{i-1})$$

$$< \sum_{i=i_m+1}^{i_M-1} (n_{i+1} - n_{i-1}) < 2(n_{i_M} - n_{i_m}).$$

From the last formula we conclude

$$\sum_{j=j_1}^{j_2} \operatorname{card} c[j] > \frac{1}{2} \sum_{j=j_1}^{j_2-2} \operatorname{card} c'[j].$$

□

Now we are going to show that the "expansion" operator is bounded in F.

Proposition 4.8 *For any finite vector $\sum_i \alpha_i e_{n_i}$ $(n_i < n_{i+1})$*

$$\left\| \sum_i \alpha_i e_{2n_i} \right\|_{\mathcal{A}} \le 2 \left\| \sum_i \alpha_i e_{n_i} \right\|_{\mathcal{A}}.$$

Proof Let $x = \sum_{i=1}^k \alpha_i e_{n_i}$, $y = \sum_{i=1}^k \alpha_i e_{2n_i}$. Then we need to prove that $\langle x, y\rangle \in \mathcal{P}$. Consider an arbitrary standard partition $c' = \{[a'_j; a'_{j+1} - 1]\}_{j=1}^p$ of y. If

$$c = \{[a_j; a_{j+1} - 1]\}_{j=1}^p, \quad \text{where } a_j = \left\lceil \frac{a'_j}{2} \right\rceil,$$

then $\lceil \frac{a'_{j+1}}{2} \rceil > \lceil \frac{a'_j}{2} \rceil$. Indeed, otherwise, $a'_{j+1} = a'_j + 1$, and $a'_j \equiv 1 \pmod 2$. It would mean that the interval $[a'_j; a'_{j+1} - 1]$ does not contain even numbers and therefore the partition c' is not standard for y. Thus, $a_{j+1} > a_j$ $(j = 1, \dots, p-1)$, and c is defined correctly. It is easy to see that $2n_i \in [a'_j; a'_{j+1} - 1]$ implies $n_i \in [a_j; a_{j+1} - 1]$, that is, c is equivalent to c' with respect to the vectors x and y. Let us show that for any j

$$\left\lceil \frac{a'_{j+1}}{2} \right\rceil - \left\lceil \frac{a'_j}{2} \right\rceil \ge \frac{1}{3}(a'_{j+1} - a'_j).$$

If $a'_{j+1} \equiv 1 \pmod 2$ or both numbers a'_{j+1}, a'_j are even then the inequality is evident. We consider the case when $a'_{j+1} \equiv 0 \pmod 2$, $a'_j \equiv 1 \pmod 2$. Since there is an even number in the interval $[a'_j; a'_{j+1} - 1]$, then $a'_{j+1} \geq a'_j + 3$. Consequently,

$$\left\lceil \frac{a'_{j+1}}{2} \right\rceil - \left\lceil \frac{a'_j}{2} \right\rceil = \frac{a'_{j+1}}{2} - \frac{a'_j + 1}{2} \geq \frac{a'_{j+1} - a'_j}{2} - \frac{1}{2} \frac{a'_{j+1} - a'_j}{3} = \frac{a'_{j+1} - a'_j}{3}.$$

Thus, c and c' satisfy (4.5), and the statement is proved. □

Now we turn to comparing norms of suitable linear combinations of arbitrary finite vectors and vectors of the standard basis.

Lemma 4.9 *Let* $x = \sum_{n=1}^N u^n$, $u^n = \sum_i u_i^n e_i \in c_{00}$, $\operatorname{supp} u^n < \operatorname{supp} u^{n+1}$, $n = 1, \dots, N-1$, *and for all* $n \in [1; N-1]$

$$\max \operatorname{supp} u^n + \operatorname{diam} \operatorname{supp} u^n \leq \min \operatorname{supp} u^{n+1}.$$

Then

$$\left\| \sum_{n=1}^N \|u^n\|_{\mathcal{A}} e_{\min \operatorname{supp} u^n} \right\|_{\mathcal{A}} \leq 2 \left\| \sum_{n=1}^N u^n \right\|_{\mathcal{A}}.$$

Proof Let us denote

$$y = \sum_{n=1}^N \|u^n\|_{\mathcal{A}} e_{\min \operatorname{supp} u^n}; \qquad u^{n,2} = \sum_{i=-\infty}^{\infty} u_i^n e_{2i}; \qquad x_2 = \sum_{n=1}^N u^{n,2}.$$

Taking into account Proposition 4.8, it is enough to prove that

$$\|y\|_{\mathcal{A}} \leq \|x_2\|_{\mathcal{A}}.$$

Show that for any standard admissible partition c' of the vector y there exists a standard admissible partition c of the vector x_2 such that $\operatorname{card} c = \operatorname{card} c'$ and

$$\min \operatorname{supp} u^n \in c'[j] \iff \operatorname{supp} u^{n,2} \subset c[j]. \tag{4.8}$$

We define c in the following way:

$$\begin{aligned} &c = \{[a_j; a_{j+1} - 1]\}_{j=1}^{\operatorname{card} c'}; \\ &a_j = \min \operatorname{supp} u^{n_j} + a'_j, \quad j = 1, \dots, \operatorname{card} c', \quad a_{\operatorname{card} c + 1} = 2a'_{\operatorname{card} c' + 1}; \\ &n_j = \min\{n : \min \operatorname{supp} u^n \in c'[j]\}, \quad j = 1, \dots, \operatorname{card} c'. \end{aligned}$$

First of all, $a_{j+1} - a_j > a'_{j+1} - a'_j$, $j = 1, \dots, \operatorname{card} c$, i.e., the partition c is admissible. Let $a'_j \leq \min \operatorname{supp} u^n$ for some j, n. Then

$$n_j = \min\{h : \min \operatorname{supp} u^h \geq a'_j\} \leq n$$

and

$$a_j \le \min \operatorname{supp} u^{n_j} + \min \operatorname{supp} u^n \le 2 \min \operatorname{supp} u^n = \min \operatorname{supp} u^{n,2}.$$

Suppose, on the contrary, $a'_j > \min \operatorname{supp} u^n$. Then $n_j \ge n+1$, and, by hypotheses,

$$\min \operatorname{supp} u^{n_j} \ge \max \operatorname{supp} u^n + \operatorname{diam} \operatorname{supp} u^n,$$

whence it follows that

$$a_j \ge 2 \max \operatorname{supp} u^n - \min \operatorname{supp} u^n + a^j > \max \operatorname{supp} u^{n,2}.$$

Thus, (4.8) holds for c and c'. Then we use recursive formula (4.1) and induction by N, finishing the proof. □

Proposition 4.10 *Let* $x = \sum_{n=1}^N u^n$, $\operatorname{supp} u^n < \operatorname{supp} u^{n+1}$, $n = 1, \ldots, N-1$. *Then there exist* $i_n \in \operatorname{supp} u^n$ $(n = 1, \ldots, N)$ *such that*

$$\left\| \sum_{n=1}^N \|u^n\|_{\mathcal{A}} e_{i_n} \right\|_{\mathcal{A}} \le 8 \left\| \sum_{n=1}^N u^n \right\|_{\mathcal{A}}.$$

Proof We represent each vector u^n as a sum

$$u^n = u^{1,n} + u^{2,n},$$

where $u^{1,n}$ and $u^{2,n}$ are defined as follows:

(a) if $\operatorname{diam} \operatorname{supp} u^n = 0$ then $u^{1,n} = u^n$, $u^{2,n} = 0$;
(b) if $\operatorname{diam} \operatorname{supp} u^n > 0$ then $\operatorname{supp} u^{1,n} < \operatorname{supp} u^{2,n}$, $\operatorname{diam} \operatorname{supp} u^{1,n} = \lfloor \frac{\operatorname{diam} \operatorname{supp} u^n}{2} \rfloor$.

Then

$$\begin{aligned} \min \operatorname{supp} u^{1,n+1} &> \max \operatorname{supp} u^n \ge 2 \operatorname{diam} \operatorname{supp} u^{1,n} + \min \operatorname{supp} u^{1,n} \\ &= \max \operatorname{supp} u^{1,n} + \operatorname{diam} \operatorname{supp} u^{1,n}. \end{aligned}$$

Applying Lemma 4.9 to vectors $u^{1,n}$, we get

$$\left\| \sum_{n=1}^N \|u^{1,n}\|_{\mathcal{A}} e_{\min \operatorname{supp} u^n} \right\|_{\mathcal{A}} \le 2 \left\| \sum_{n=1}^N u^{1,n} \right\|_{\mathcal{A}}.$$

Analogous estimate holds also for vectors $-u^{2,n}$. Combining it with the equality $\min \operatorname{supp}(-u^{2,n}) = -\max \operatorname{supp} u^{2,n}$, we obtain that

$$\left\| \sum_{n=1}^N \|u^{2,n}\|_{\mathcal{A}} e_{\max \operatorname{supp} u^n} \right\|_{\mathcal{A}} \le 2 \left\| \sum_{n=1}^N u^{2,n} \right\|_{\mathcal{A}}.$$

As a result,

$$\begin{aligned}
8\left\|\sum_{n=1}^{N}u^n\right\|_{\mathcal{A}} &\ge 4\left(\left\|\sum_{n=1}^{N}u^{1,n}\right\|_{\mathcal{A}}+\left\|\sum_{n=1}^{N}u^{2,n}\right\|_{\mathcal{A}}\right)\\
&\ge 2\left\|\sum_{n=1}^{N}\|u^{1,n}\|_{\mathcal{A}}e_{i_{1,n}}\right\|_{\mathcal{A}}+2\left\|\sum_{n=1}^{N}\|u^{2,n}\|_{\mathcal{A}}e_{i_{2,n}}\right\|_{\mathcal{A}}\\
&\ge 2\left\|\sum_{n=1}^{N}\max(\|u^{1,n}\|_{\mathcal{A}},\|u^{2,n}\|_{\mathcal{A}})e_{i_n}\right\|_{\mathcal{A}}\ge\left\|\sum_{n=1}^{N}\|u^n\|_{\mathcal{A}}e_{i_n}\right\|_{\mathcal{A}},
\end{aligned}$$

where $i_{1,n}=\min\operatorname{supp}u^n$, $i_{2,n}=\max\operatorname{supp}u^n$, $i_n\in\{i_{1,n};i_{2,n}\}$. □

Later on we will use the notation: if $a,b\in\mathbb{Z}$, $a\le b$ and, as usual, $[a;b]=\{x\in\mathbb{Z}:\ a\le x\le b\}$, then

$$I[a;b]:=(a,b)=\{x\in\mathbb{Z}:\ a<x<b\}\quad\text{and}\quad J[a;b]:=\{a;b\}.$$

Proposition 4.11 *For an arbitrary sequence of finite vectors* $\{u^n\}_{n=1}^N$, $\operatorname{supp}u^n<\operatorname{supp}u^{n+1}$ $(n=1,\dots,N-1)$, *and for any* $m\ge 0$ *there exist* $i_n\in\operatorname{supp}u^n$ $(n=1,\dots,N)$ *such that*

$$\left\|\sum_{n=1}^{N}u^n\right\|_m\le\left\|4\sum_{n\in I[1;N]}\|u^n\|_m e_{i_n}+2\sum_{n\in J[1;N]}\|u^n\|_m e_{i_n}\right\|_m. \tag{4.9}$$

Proof We carry on the proof by induction. If $m=0$ the statement is evident. Let $m>0$, $x=\sum_{n=1}^N u^n$ and

$$\|x\|_m=\frac{1}{2}\sum_{j=1}^{\operatorname{card}c}\|c[j]x\|_{m-1},$$

for some standard admissible partition c.

We divide the interval $[1;N]$ into four sets (some of them may be empty):

$$H_1:=\{n:\ \{n_1:c[j]u^{n_1}\ne 0\}=\{n\}\text{ for some }j\}$$

(that is, there exists an interval $c[j]$ which intersecs only with support of u^n),

$$H_2:=\{n\notin H_1:\ c[j]u^n\ne 0,c[j+1]u^n\ne 0\text{ for some }j\}$$

(corresponding vectors u^n do not belong to the first set and their supports intersect with two intervals $c[j]$ and $c[j+1]$),

$$H_3:=\{n\in I[1;N]\setminus H_1:\ \operatorname{supp}u^n\subset c[j]\text{ for some }j\}$$

(corresponding vectors u^n again do not belong to the first set and their supports are contained within one of the intervals, excluding u^1 and u^N),

$$H_4 := J[1; N] \setminus H_1.$$

It follows easily that the sets of indices H_l, $1 \le l \le 4$, are mutually disjoint and their union coincides with $[1; N]$. Moreover, we introduce the sets G_1, G_2:

$$G_1 = \{j \in [1; \operatorname{card} c] : \operatorname{card}\{n : c[j]u^n \neq 0\} = 1\};$$
$$G_2 = \{j \in [1; \operatorname{card} c] : \operatorname{card}\{n : c[j]u^n \neq 0\} > 1\}.$$

At first,

$$\sum_{j=1}^{\operatorname{card} c} \left\| c[j] \sum_{n=1}^{N} u^n \right\|_{m-1} \le \sum_{j=1}^{\operatorname{card} c} \left\| c[j] \sum_{n\in H_1} u^n \right\|_{m-1} + \sum_{j\in G_2} \left\| c[j] \sum_{n\in H_2\cup H_3\cup H_4} u^n \right\|_{m-1}$$
$$= S_1 + S_2.$$

According to the definition of the norm $\|\cdot\|_m$, we have

$$S_1 \le \sum_{n\in H_1} \sum_{j=1}^{\operatorname{card} c} \|c[j]u^n\|_{m-1} \le 2 \sum_{n\in H_1} \|u^n\|_m = \sum_{j\in G_1} \left\| c[j] \sum_{n\in H_1} 2\|u^n\|_m e_{i_n} \right\|_{m-1},$$

where $i_n \in \bigcup_{j\in G_1} \operatorname{supp} c[j]u^n$, $n \in H_1$.

Now let us estimate the sum S_2. If $n \in H_2$ and $c[j]u^n \neq 0$ for some j, then

$$n \in J[\min\{n_1 : c[j]u^{n_1} \neq 0\}; \max\{n_1 : c[j]u^{n_1} \neq 0\}].$$

Analogously, in the case when $n \in H_4$ we have

$$n \in J[\min\{n_1 : c[1]u^{n_1} \neq 0\}; \max\{n_1 : c[1]u^{n_1} \neq 0\}] \quad \text{or}$$
$$n \in J[\min\{n_1 : c[\operatorname{card} c]u^{n_1} \neq 0\}; \max\{n_1 : c[\operatorname{card} c]u^{n_1} \neq 0\}].$$

Then, by the inductive hypothesis,

$$S_2 \le \sum_{j\in G_2} \left\| c[j] \left(\sum_{n\in H_2} 2\|u^n\|_{m-1}(e_{i_{1,n}} + e_{i_{2,n}}) \right.\right.$$
$$\left.\left. + \sum_{n\in H_3} 4\|u^n\|_{m-1} e_{i_n} + \sum_{n\in H_4} 2\|u^n\|_{m-1} e_{i_n} \right) \right\|_{m-1},$$

where $i_n \in \operatorname{supp} c[j]u^n$, $i_{1,n} \in \operatorname{supp} c[j]u^n$, $i_{2,n} \in \operatorname{supp} c[j+1]u^n$ for corresponding j. Let us note that for $\|\cdot\|_{m-1}$, as well as any other norm, the inequality

$$\sum_i \|z_1^i + z_2^i + z_3^i\|_{m-1} \le \max\left(\sum_i \|2z_1^i + z_3^i\|_{m-1}, \sum_i \|2z_2^i + z_3^i\|_{m-1} \right)$$

holds for arbitrary z_1^i, z_2^i, z_3^i. Consequently, by replacing $i_{1,n}$ or $i_{2,n}$ with i_n, we get

$$S_2 \le \sum_{j\in G_2} \left\| c[j]\left(\sum_{n\in H_2} 4\|u^n\|_m e_{i_n} + \sum_{n\in H_3} 4\|u^n\|_m e_{i_n} + \sum_{n\in H_4} 2\|u^n\|_m e_{i_n} \right) \right\|_{m-1},$$

where $i_n \in \operatorname{supp} u^n$. As a result we have

$$\begin{aligned}
2\|x\|_m &\le S_1 + S_2 \\
&\le \sum_{j=1}^{\operatorname{card} c} \left\| c[j]\left(\sum_{n\in H_1} 2\|u^n\|_m e_{i_n} + \sum_{n\in H_2} 4\|u^n\|_m e_{i_n} \right.\right. \\
&\quad \left.\left. + \sum_{n\in H_3} 4\|u^n\|_m e_{i_n} + \sum_{n\in H_4} 2\|u^n\|_m e_{i_n} \right) \right\|_{m-1} \\
&\le 2\left\| \sum_{n\in H_1} 2\|u^n\|_m e_{i_n} + \sum_{n\in H_2} 4\|u^n\|_m e_{i_n} + \sum_{n\in H_3} 4\|u^n\|_m e_{i_n} \right. \\
&\quad \left. + \sum_{n\in H_4} 2\|u^n\|_m e_{i_n} \right\|_m .
\end{aligned}$$

Taking into account the embedding $\{1; N\} \subset H_1 \cup H_4$, we get inequality (4.9). □

Proof of Theorem 4.3 We apply Propositions 4.5, 4.10 and 4.11. Note that the constant $C_{RSP} = C_{LSP} \le 2 \cdot 8 \cdot 4 = 64$. □

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