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On Stably $\mathscr{K}$-Monotone Banach Couples

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Abstract. The $\mathscr{K}$-monotonicity of Banach couples which is stable with respect to multiplication of weight by a constant is studied. Suppose that E is a separable Banach lattice of two-sided sequences of reals such that $\|e_n\| = 1$ $(n \in \mathbb{N})$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical basis. It is shown that $\vec{E} = (E, E(2^{-k}))$ is a stably $\mathscr{K}$-monotone couple if and only if $\vec{E}$ is $\mathscr{K}$-monotone and E is shift-invariant. A non-trivial example of a shift-invariant separable Banach lattice E such that the couple $\vec{E}$ is $\mathscr{K}$-monotone is constructed. This result contrasts with the following well-known theorem of Kalton: If E is a separable symmetric sequence space such that the couple $\vec{E}$ is $\mathscr{K}$-monotone, then either $E = l_p$ $(1 \leqslant p < \infty)$ or $E = c_0$.

Key words: interpolation of operators, Peetre $\mathscr{K}$-functional, $\mathscr{K}$-monotone Banach couple, shift-invariant space.

1. Introduction. Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be Banach couples. Banach spaces X and Y such that $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and $Y_0 \cap Y_1 \subset Y \subset Y_0 + Y_1$ are called *C-interpolation* spaces with respect to the couples $\vec{X}$ and $\vec{Y}$ if every linear operator T bounded as an operator from X_0 to Y_0 and from X_1 to Y_1 is bounded as an operator from X to Y and satisfies the condition $\|T\|_{X\to Y} \leqslant C \max_{i=0,1} \|T\|_{X_i \to Y_i}$. For any Banach couple $\vec{X} = (X_0, X_1)$, $x \in X_0 + X_1$, and $t > 0$, the *Peetre $\mathscr{K}$-functional* is defined as

$$\mathscr{K}(t, x; \vec{X}) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_i \in X_i\}.$$

Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ are said to be (*uniformly*) *relatively $\mathscr{K}$-monotone with constant* $\lambda > 0$ if, whenever X and Y are C-interpolation spaces with respect to couples $\vec{X}$ and $\vec{Y}$ for some $C > 0$, $x \in X$, and $\mathscr{K}(t, y; \vec{Y}) \leqslant \mathscr{K}(t, x; \vec{X})$ for all $t > 0$ and some $y \in Y_0 + Y_1$, we have $y \in Y$ and $\|y\|_Y \leqslant \lambda C \|x\|_X$. In particular, if $\vec{X} = \vec{Y}$, then such a couple is said to be (*uniformly*) *$\mathscr{K}$-monotone*. The importance of $\mathscr{K}$-monotone couples is mainly due to the possibility of describing all interpolation spaces with respect to them. Specifically, if X is an interpolation space with respect to a $\mathscr{K}$-monotone couple $\vec{X} = (X_0, X_1)$, then we have $X = (X_0, X_1)^{\mathscr{K}}_E$, where $(X_0, X_1)^{\mathscr{K}}_E$ is the space of the *real $\mathscr{K}$-method of interpolation*, which is generated by some Banach lattice E of two-sided sequences of reals and consists of all $x \in X_0 + X_1$ such that $(\mathscr{K}(2^k, x; \vec{X}))_{k=-\infty}^{\infty} \in E$, with norm $\|x\| := \|(\mathscr{K}(2^k, x; \vec{X}))_k\|_E$ [1, Theorem 4.4.5].

In what follows, we shall mainly consider separable Banach lattices E of two-sided sequences of reals such that $\|e_n\| = 1$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical basis in sequence spaces. Every such lattice generates the Banach couple $\vec{E} = (E, E(2^{-k}))$ (for any fixed sequence $v = (v_k)_{k=-\infty}^{\infty}$ with $v_k > 0$, $E(v_k)(= E(v))$ stands for the *weighted space* consisting of all sequences $a = (a_k)_{k=-\infty}^{\infty}$ such that $a \cdot v \in E$ with norm $\|a\|_{E(v_k)} := \|a \cdot v\|_E$). The $\mathscr{K}$-monotonicity of such a couple is connected with the following subtle geometric properties of E [2].

Let $x = (x_k)_{k=-\infty}^{\infty}$ be a sequence of reals, and let $\operatorname{supp} x = \{k \in \mathbb{Z} : x_k \neq 0\}$ be its *support*. The inequality $A < B$ (where $A, B \subset \mathbb{Z}$) means that $a < b$ for any $a \in A$ and $b \in B$. A Banach lattice E of sequences is said to have the *right-shift property* if there exists a $C > 0$ such that, for arbitrary families $\{x^n\}_{n=1}^m$ and $\{y^n\}_{n=1}^m$ of finite sequences x^n and y^n of reals satisfying the conditions

$$\operatorname{supp} x^n < \operatorname{supp} y^n \quad (1 \leqslant n \leqslant m), \qquad \operatorname{supp} y^n < \operatorname{supp} x^{n+1} \quad (1 \leqslant n \leqslant m-1),$$

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and $\|y^n\|_E \leqslant \|x^n\|_E = 1$ $(1 \leqslant n \leqslant m)$, we have $\| \sum_{n=1}^m a_n y^n \|_E \leqslant C \| \sum_{n=1}^m a_n x^n \|_E$ for all $a_n \in \mathbb{R}$. Similarly, E has the *left-shift property* if there exists a $C > 0$ such that for arbitrary families $\{x^n\}_{n=1}^m$ and $\{y^n\}_{n=1}^m$ of finite sequences x^n and y^n whose supports satisfy the same conditions and for all $a_n \in \mathbb{R}$, the inequality $\| \sum_{n=1}^m a_n x^n \|_E \leqslant C \| \sum_{n=1}^m a_n y^n \|_E$ holds. It was shown by Kalton [2, Theorem 4.5] that the couple $\vec{E}$ is $\mathscr{K}$-monotone if and only if E possesses both the left-shift and right-shift properties.

One of the main results of this note is a characterization of the couples $\vec{E} = (E, E(2^{-k}))$ whose $\mathscr{K}$-monotonicity is stable when the weight is multiplied by a constant. Specifically, we show that any such couple is generated by a shift-invariant Banach lattice. It may be of interest to compare this result with the well-known theorem of Cwikel and Nilsson [3], which implies, in particular, that if E is a separable Banach lattice such that the couple $\vec{E} = (E, E(v))$ is $\mathscr{K}$-monotone for any weight v, then E coincides with either l_p $(1 \leqslant p < \infty)$ or c_0. At the same time, the class of $\mathscr{K}$-monotone couples $\vec{E}$ generated by separable shift-invariant Banach lattices is not trivial. We construct an example of a shift-invariant lattice F which has, in addition to the properties mentioned above, the right-shift and left-shift properties and satisfies the conditions $F \neq l_p$ $(1 \leqslant p < \infty)$ and $F \neq c_0$. Note that the situation changes completely if only symmetric sequence spaces are considered. As Kalton showed in [2, Theorem 4.6], for such a space, the couple $\vec{E}$ is $\mathscr{K}$-monotone if and only if either $E = l_p$ $(1 \leqslant p < \infty)$ or $E = c_0$.

2. $\mathscr{K}_u$-monotone Banach couples. If X is a Banach lattice of measurable functions defined on some σ-finite measure space and $v(t) > 0$, then the weighted space $X(v)$ is defined by analogy with weighted sequence spaces. Recall that a Banach lattice X has the Fatou property if the conditions $0 \leqslant x_n \uparrow x$ almost everywhere and $\sup_{n=1,2,\dots} \|x_n\|_X < \infty$ imply $x \in X$ and $\|x_n\|_X \uparrow \|x\|_X$. It follows from results of Sedaev [4] and Cwikel and Nilsson [3] that for a separable Banach lattice X with the Fatou property, the following conditions are equivalent:

(i) for arbitrary weights v_0, v_1, w_0, and w_1, the couples $(X(v_0), X(v_1))$ and $(X(w_0), X(w_1))$ are relatively $\mathscr{K}$-monotone with constant $\lambda > 0$ independent of the weights;

(ii) there exist a weight v and a number $p \in [1, \infty)$ such that $X = L_p(v)$ (the norms are equivalent).

We are concerned with Banach couples whose $\mathscr{K}$-monotonicity is stable with respect to a "weaker" perturbation, namely, multiplication of the weight by a constant. Specifically, suppose that (X_0, X_1) is a couple of Banach lattices of measurable functions defined on some σ-finite measure space. Are the couples $(X_0, X_1(u))$ and (X_0, X_1) relatively $\mathscr{K}$-monotone with constant $\lambda > 0$ independent of the number $u > 0$? As is well known (see, for instance, [5, Chap. 15] or [1]), the answer is positive if and only if, for some $\lambda > 0$, the conditions $x, y \in X_0 + X_1$, $\lambda' > \lambda$, $u > 0$, and $\mathscr{K}(t, y; X_0, X_1) \leqslant \mathscr{K}(ut, x; X_0, X_1)$ for all $t > 0$ imply the existence of linear operators $T \colon X_i \to X_i$ $(i = 0, 1)$ such that $\|T\|_{X_0 \to X_0} \leqslant \lambda'$, $\|T\|_{X_1 \to X_1} \leqslant \lambda' u$, and $Tx = y$. We refer to (X_0, X_1) for which such λ exists as a *$\mathscr{K}_u$-monotone* couple. Note that this notion also makes sense for general Banach couples. In particular, this property is enjoyed by pairs $(\vec{X}_{\theta_0, p_0}, \vec{X}_{\theta_1, p_1})$ of spaces of the classical $\mathscr{K}$-method of interpolation, which are generated by l_p-spaces with exponential weights (see [6] for definitions).

Proposition 1. *For an arbitrary Banach couple $\vec{X} = (X_0, X_1)$ and any θ_j and p_j $(0 < \theta_j < 1$, $1 \leqslant p_j \leqslant \infty$, $j = 0, 1)$, the couple $(\vec{X}_{\theta_0, p_0}, \vec{X}_{\theta_1, p_1})$ is $\mathscr{K}_u$-monotone.*

3. Couples generated by shift-invariant Banach lattices. Suppose that, for any $k \in \mathbb{Z}$, the shift operator $P_k((a_j)_{j=-\infty}^{\infty}) := (a_{k+j})_{j=-\infty}^{\infty}$ is bounded in a Banach lattice E of two-sided sequences of reals. We call E a *shift-invariant* lattice if $\sup_{k \in \mathbb{Z}} \|P_k\|_{E \to E} < \infty$. Let E_θ stand for the Banach lattice $E(2^{-\theta k})$ $(0 < \theta < 1)$.

Theorem 1. *For an arbitrary Banach lattice E of two-sided sequences of reals such that $\|e_n\| = 1$ $(n \in \mathbb{Z})$, the following conditions are equivalent*:

(a) *the couple $\vec{E} = (E, E(2^{-k}))$ is $\mathscr{K}_u$-monotone*;

(b) *the couple $\vec{E}$ is $\mathscr{K}$-monotone and, in addition, there exists a constant $C > 0$ such that if $\theta \in (0,1)$, $a \in E_\theta$, $b \in E + E(2^{-k})$, $u > 0$, and*

$$\mathscr{K}(t, b; \vec{E}) \leqslant \mathscr{K}(ut, a; \vec{E}) \qquad (t > 0), \tag{1}$$

then $b \in E_\theta$ and

$$\|b\|_{E_\theta} \leqslant C u^\theta \|a\|_{E_\theta}; \tag{2}$$

(c) *the couple $\vec{E}$ is $\mathscr{K}$-monotone and there exists a $C > 0$ and a $\theta_0 \in (0,1)$ such that if $a \in E_{\theta_0}$, $b \in E + E(2^{-k})$, and $u > 0$ satisfy* (1), *then $b \in E_{\theta_0}$ and inequality* (2) *holds for $\theta = \theta_0$;*

(d) *the couple $\vec{E}$ is $\mathscr{K}$-monotone, and the lattice E is shift-invariant.*

Theorem 1 allows us to construct examples of Banach couples that are $\mathscr{K}$-monotone but not $\mathscr{K}_u$-monotone. Consider an arbitrary Banach sequence lattice G with the following properties: (a) G is separable; (b) $\|e^k\|_G = 1$; (c) G has the left-shift and right-shift properties; (d) G is not shift-invariant. Then the Banach couple $(G, G(2^{-k}))$ is $\mathscr{K}$-monotone but not $\mathscr{K}_u$-monotone. Examples of lattices with properties (a)–(d) are the direct sum $l_q(\mathbb{Z}_-) \oplus l_r(\mathbb{Z}_+)$ $(1 \leqslant q \neq r < \infty)$ with the natural norm and the Tsirelson space T [7].

A natural question arises in connection with the preceding examples: Does there exist a separable shift-invariant Banach lattice with right-shift and left-shift properties not coinciding with l_p $(1 \leqslant p < \infty)$ and c_0? In view of results of [2] and Theorem 1, this question can be reformulated in the following way: Does there exist a $\mathscr{K}_u$-monotone couple of the form $(E, E(2^{-k}))$, where E is a separable Banach lattice, such that $\|e_n\|_E = 1$ $(n \in \mathbb{N})$ but $E \neq l_p$ $(1 \leqslant p < \infty)$ and $E \neq c_0$? The question is also interesting in connection with the well-known result of Kalton [2, Theorem 4.6] that if E is a symmetric Banach sequence lattice, then such a couple is $\mathscr{K}$-monotone only if $E = l_p$ $(1 \leqslant p < \infty)$ or $E = c_0$. In the last section, we give an example answering this question and, as we hope, interesting in other respects too.

4. A shift-invariant Banach lattice with right-shift and left-shift properties. The following construction is somewhat similar to that of the Tsirelson space T suggested by Figiel and Johnson (see [8] or [7]). At the same time, there are significant differences between our example and T, since we construct a shift-invariant space (the Tsirelson space does not enjoy this property).

By a *partition* we mean an ordered set $c = \{c[j]\}_{j=1}^p$ of intervals $c[j] = \{k \in \mathbb{Z} : a_j \leqslant k \leqslant a_{j+1} - 1\}$, where $a_j \in \mathbb{Z}$ and $a_{j+1} > a_j$ for all $j = 1, \dots, p$. A partition c is called *admissible* if $\operatorname{card}\{j : \operatorname{card} c[j] \leqslant k\} \leqslant 14k$ for any $k > 0$. The set of all admissible partitions will be denoted by $\mathscr{A}$.

On the space c_{00} of two-sided eventually zero real sequences we define a sequence of norms $\{\|\cdot\|_m\}_{m=0}^\infty$ as follows: for $x = \sum_n \alpha_n e_n \in c_{00}$, we set

$$\|x\|_0 = \max_n |\alpha_n|, \qquad \|x\|_{m+1} = \max\left(\|x\|_m, \frac{1}{2} \max_{c \in \mathscr{A}} \sum_{j=1}^{\operatorname{card} c} \|x\chi_{c[j]}\|_m\right), \qquad m \geqslant 0,$$

where χ_E is the indicator function of E. The set c_{00} with norm $\|\cdot\|_m$ is a normed shift-invariant lattice, and $\|x\|_{c_0} \leqslant \|x\|_m \leqslant \|x\|_{l_1}$ $(x \in c_{00})$. Since $\|x\|_{m+1} \geqslant \|x\|_m$ for any x, the limit $\|x\|_{\mathscr{A}} = \lim_{m\to\infty} \|x\|_m$ exists. Let us denote the $\|\cdot\|_{\mathscr{A}}$-completion of c_{00} by F. It is not difficult to show that

$$\|x\|_{\mathscr{A}} = \max\left(\|x\|_{c_0}, \frac{1}{2} \sup_{c \in \mathscr{A}} \sum_{j=1}^{\operatorname{card} c} \|x\chi_{c[j]}\|_{\mathscr{A}}\right) \qquad \text{for any } x \in F.$$

Theorem 2. *The Banach lattice F is separable and shift-invariant, and it has the right-shift and left-shift properties, but this lattice does not coincide with l_p $(1 \leqslant p < \infty)$ and c_0.*

Corollary 1. *The Banach couple $(F, F(2^{-k}))$ is $\mathscr{K}_u$-monotone, but it is not a uniform partial retract of any weighted L_p-couple.*

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