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On the Comparison of Distribution Functions of Random Variables

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Abstract—We prove a rather general comparison principle for the distribution functions of random variables. As a consequence, we obtain a criterion for the equivalence, in distribution in the vector sense, of an arbitrary sequence of random variables to the Rademacher system; we study the applications of this principle to special cases.

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1. INTRODUCTION

If f is a random variable on a probability space $(\Omega, \Sigma, \mathsf{P})$, then, as usual,

$$\|f\|_p := \left(\int_\Omega |f(\omega)|^p \, d\mathsf{P}\right)^{1/p}, \quad 1 \le p < \infty, \qquad \mathsf{E}f := \int_\Omega f(\omega)\, d\mathsf{P}.$$

In what follows, we assume that all the random variables belong to the space $L_p = L_p(\Omega, \Sigma, \mathsf{P})$ for all $1 \le p < \infty$.

In [1] (see also [2]) the following comparison principle for the distribution functions of random variables was proved.

Theorem 1. *Suppose that $f \ge 0$ and $g \ge 0$ are two random variables given on, in general, different probability spaces and $\{f_i\}_{i=1}^\infty$, $\{g_i\}_{i=1}^\infty$ are independent "copies" of f and g, respectively. Suppose that*

$$\Big\|\max_{i=1,\dots,n} f_i\Big\|_2 \le C_1 \Big\|\max_{i=1,\dots,n} f_i\Big\|_1, \tag{1}$$

$$\Big\|\max_{i=1,\dots,n} g_i\Big\|_1 \le C_2 \Big\|\max_{i=1,\dots,n} f_i\Big\|_1, \tag{2}$$

where $C_1 > 0$ and $C_2 > 0$ are independent of $n \in \mathbb{N}$.

Then there exists a $C > 0$ depending only on C_1 and C_2 such that, for all $z > 0$,

$$\mathsf{P}\{g > z\} \le C\mathsf{P}\left\{f > \frac{z}{C}\right\}.$$

In this paper, we obtain a comparison principle for the distribution functions of random variables with conditions expressed in terms of the moments of these variables, without invoking any additional constructions. Moreover, we show that these conditions are weaker than conditions (1) and (2) of Theorem 1. Using this principle as well as the main result of [3] on the distribution of vector-valued Rademacher series, we prove a criterion for the equivalence in distribution of an arbitrary sequence of random variables to the Rademacher system. In the concluding part of the paper, some applications

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of the results obtained are presented. We shall show that any Sidon system of characters on a compact Abelian group, as well as any sequence of the form $\{e^{i\lambda_k t}\}_{k=1}^{\infty}$, where $(\lambda_k)_{k=1}^{\infty} \subset \mathbb{R}$ is a topological Sidon set, is equivalent in distribution in the vector sense to the Rademacher system. In both cases, we use the earlier proved statements on the equivalence of the corresponding moments (see [4] and [5], respectively).

Everywhere in what follows, μ is the Lebesgue measure on the line and the expression $F \asymp G$, $t \in T$, means that, for some constant $C > 0$ and all $t \in T$, the following inequalities hold:

$$C^{-1}F(t) \leq G(t) \leq CF(t).$$

2. COMPARISON PRINCIPLE FOR THE DISTRIBUTION FUNCTIONS OF RANDOM VARIABLES

For an arbitrary random variable $f \geq 0$ given on a probability space (Ω, P), the quantity $(\mathsf{E}f^p)^{1/p}$ is continuous and monotone increasing for $p \in [1, \infty)$. Therefore,

$$\alpha_f := \mathsf{E}f \leq (\mathsf{E}f^p)^{1/p} \leq \beta_f := \lim_{p\to\infty} (\mathsf{E}f^p)^{1/p}, \qquad 1 \leq p < \infty.$$

The following main result of this paper shows that, under certain conditions, inequalities for the moments of a random variable imply an inequality for their distributions.

Theorem 2. *Suppose that $f \geq 0$ and $g \geq 0$ are two random variables given on possibly different probability spaces. Suppose that there exist constants $A > 0$ and $B > 0$ such that, for all $p \geq 1$,*

$$(\mathsf{E}f^{2p})^{1/(2p)} \leq A(\mathsf{E}f^p)^{1/p}, \tag{3}$$

$$(\mathsf{E}g^p)^{1/p} \leq B(\mathsf{E}f^p)^{1/p}. \tag{4}$$

Then, for some constant $C > 0$ depending only on A and B and all $z > 0$,

$$\mathsf{P}'\{g > z\} \leq C\mathsf{P}\left\{f > \frac{z}{C}\right\}. \tag{5}$$

Proof. First, for any $p \geq 1$, using the Paley–Zygmund inequality [6, Sec. 1.6] and conditions (3), we obtain

$$\begin{aligned}\mathsf{P}\left\{f > \frac{1}{3}(\mathsf{E}f^p)^{1/p}\right\} &\geq \mathsf{P}\left\{f \geq \frac{1}{2}(\mathsf{E}f^p)^{1/p}\right\} = \mathsf{P}\{f^p \geq 2^{-p}\mathsf{E}f^p\} \\ &\geq (1-2^{-p})^2\frac{(\mathsf{E}f^p)^2}{\mathsf{E}f^{2p}} \geq (1-2^{-p})^2A^{-2p} \geq (2A)^{-2p},\end{aligned} \tag{6}$$

where the last inequality follows from the fact that

$$(1-2^{-p})^2 \geq 2^{-2p} \qquad \text{if} \quad p \geq 1.$$

Thus, in view of the definition α_f, we can write

$$\mathsf{P}\{f > z\} \geq (2A)^{-2}, \qquad \text{if} \quad 0 < z \leq \frac{\alpha_f}{3}. \tag{7}$$

On the other hand, in view of Chebyshev's inequality and inequality (4), we obtain

$$\mathsf{P}'\{g \geq \lambda(\mathsf{E}f^p)^{1/p}\} \leq \lambda^{-p}\frac{\mathsf{E}g^p}{\mathsf{E}f^p} \leq \left(\frac{B}{\lambda}\right)^p$$

for arbitrary $p \geq 1$ and $\lambda > 0$. In particular, if $\lambda = Be$, then

$$\mathsf{P}'\{g \geq Be(\mathsf{E}f^p)^{1/p}\} \leq e^{-p}, \quad p \geq 1. \tag{8}$$

In addition, it follows from the last relation and the definition of β_f that

$$\mathsf{P}'\{g \geq z\} = 0, \qquad \text{if} \quad z \geq Be\beta_f. \tag{9}$$

Let us define two functionals

$$E(s) := \sup\{p \ge 1 : (\mathsf{E}f^p)^{1/p} \le s\}, \quad G(s) := \inf\{p \ge 1 : (\mathsf{E}f^p)^{1/p} \ge s\}, \qquad s > 0$$

(we assume that $\sup\varnothing = \inf\varnothing = 1$). We can easily see that these functionals are increasing and $G(s) \le E(s)$, $s > 0$.

If $0 < z < \beta_f/3$, then, by the definition of β_f,

$$U := \{p \ge 1 : \mathsf{E}f^p \ge (3z)^p\} \ne \varnothing$$

and, in view of (6), for each $p \in U$, we have

$$\mathsf{P}\{f > z\} \ge \mathsf{P}\biggl\{f > \frac{1}{3}(\mathsf{E}f^p)^{1/p}\biggr\} \ge (2A)^{-2p};$$

hence, by the definition of G,

$$\mathsf{P}\{f > z\} \ge (2A)^{-2G(3z)}.$$

Thus,

$$\mathsf{P}\{f > z\} \ge e^{-C_1 G(3z)}, \qquad \text{if} \quad 0 < z < \frac{\beta_f}{3}, \tag{10}$$

where $C_1 := 2\ln(2A) \ge 1$.

Let us show that

$$C_1 G(3z) \le E(3A^{[C_1]+1}z), \qquad z > 0,$$

where $[C_1]$ is the integer part of the number C_1. Indeed, because $M := 2^{[C_1]+1} \ge C_1$, in view of (3) and of the definition of G, we can write

$$(\mathsf{E}f^{C_1G(3z)})^{1/(C_1G(3z))} \le (\mathsf{E}f^{MG(3z)})^{1/(MG(3z))} \le A^{[C_1]+1}(\mathsf{E}f^{G(3z)})^{1/G(3z)} = 3zA^{[C_1]+1};$$

hence the required inequality follows from the definition of the functional E. Thus, (10) can be rewritten as

$$\mathsf{P}\{f > z\} \ge e^{-E(C_2z)}, \qquad \text{if} \quad 0 < z < \frac{\beta_f}{3}, \tag{11}$$

where $C_2 := 3A^{[C_1]+1}$.

Further, if $z > Be\alpha_f$, then, by the definition of α_f,

$$V := \biggl\{p \ge 1 : (\mathsf{E}f^p)^{1/p} \le \frac{z}{Be}\biggr\} \ne \varnothing$$

and, in view of (8), for each $p \in V$, we can write

$$\mathsf{P}'\{g \ge z\} \le \mathsf{P}'\{g \ge Be(\mathsf{E}f^p)^{1/p}\} \le e^{-p}.$$

Thus, by the definition of E, we have

$$\mathsf{P}'\{g \ge z\} \le e^{-E(B^{-1}e^{-1}z)}, \qquad \text{if} \quad z > Be\alpha_f. \tag{12}$$

Let us show that inequality (5) holds with the constant $C := \max(BeC_2, (2A)^2)$. First, suppose that $0 < z \le Be\alpha_f$. Since $A \ge 1$, we have $C_2 \ge 3$ and $C \ge 3Be$. Therefore, in this case, $z/C \le \alpha_f/3$, and hence it follows from (7) and the choice of C that

$$\mathsf{P}'\{g > z\} \le 1 \le (2A)^2\mathsf{P}\biggl\{f > \frac{z}{C}\biggr\} \le C\mathsf{P}\biggl\{f > \frac{z}{C}\biggr\}.$$

If $Be\alpha_f < z < Be\beta_f$, then $z/C < \beta_f/3$, and so, in view of (11), (12) and the inequality $Be \le C/C_2$, we have

$$\mathsf{P}'\{g > z\} \le e^{-E(B^{-1}e^{-1}z)} \le e^{-E(C_2z/C)} \le \mathsf{P}\biggl\{f > \frac{z}{C}\biggr\}.$$

Finally, if $z \ge Be\beta_f$, then (9) implies $\mathsf{P}'\{g > z\} = 0$,, and thus inequality (5) again holds. $\square$

Remark 1. Theorem 2 is stronger than Theorem 1, because its assumptions (3) and (4) are consequences of the corresponding assumptions of Theorem 1, i.e., of inequalities (1) and (2). Indeed, first, it is readily verified [7, Proposition 2] that, with a constant independent of the random variable f and $n \in \mathbb{N}$,

$$\Big\|\max_{i=1,\dots,n} f_i\Big\|_p \asymp n\bigg(\int_0^{n^{-p}} (f^*(t))^p\,dt\bigg)^{1/p},$$

where $f^*(t)$ is a nonincreasing permutation of $|f|$ with respect to P, i.e.,

$$f^*(t) = \inf\big\{\tau \ge 0 : \mathsf{P}\{\omega \in \Omega : |f(\omega)| > \tau\} < t\big\}, \qquad 0 < t \le 1.$$

Therefore, conditions (1) and (2) are equivalent to the relations

$$\bigg(\int_0^{1/n^2} (f^*(t))^2\,dt\bigg)^{1/2} \le C_1 \int_0^{1/n} f^*(t)\,dt, \qquad \int_0^{1/n} g^*(t)\,dt \le C_2 \int_0^{1/n} f^*(t)\,dt$$

respectively, where the constants C_1 and C_2 are independent of $n \in \mathbb{N}$. In turn, using the well-known expressions for the $\mathscr{K}$-functionals of the pairs of the L_p-spaces [8, Theorem 5.2.1], we can rewrite the last inequalities as follows:

$$\mathscr{K}(t, f^*; L_2, L_\infty) \le 2C_1 \mathscr{K}(t, f^*; L_1, L_\infty), \qquad 0 \le t \le 1, \tag{13}$$

$$\mathscr{K}(t, g^*; L_1, L_\infty) \le 2C_2 \mathscr{K}(t, f^*; L_1, L_\infty), \qquad 0 \le t \le 1, \tag{14}$$

respectively. Since, with constants independent of $p \in [1, \infty)$,

$$(L_1, L_\infty)_{1/p', p} = L_p \qquad \text{and} \qquad (L_2, L_\infty)_{1/p', p} = L_{2p},$$

where $p' = p/(p-1)$, while $(\,\cdot\,,\,\cdot\,)_{\theta,p}$ is a functor of the real $\mathscr{K}$-method [8, Theorem 5.2.1]; hence we immediately we obtain relations (3) and (4). At the same time, we can show that inequality (13) for $\mathscr{K}$-functionals equivalent to condition (1) from Theorems 1, is not necessarily, in general, a consequence of inequalities (3) and (4) (compare [9, Proposition 2]).

Let us apply the theorem to the comparison of systems of random variables with the "model" Rademacher system $\{r_n\}_{n=1}^\infty$, where, as usual, $r_n(t) = \operatorname{sign}\sin 2^n \pi t$, $0 \le t \le 1$.

Corollary 1. *Suppose that $\{g_n\}_{n=1}^\infty$ is a sequence of random variables given on a probability space $(\Omega, \Sigma, \mathsf{P})$ such that*

$$\bigg\|\bigg\|\sum_{n=1}^m x_n g_n\bigg\|_F\bigg\|_p \le B\bigg\|\bigg\|\sum_{n=1}^m x_n r_n\bigg\|_F\bigg\|_p,$$

where the constant B is independent of the Banach space F, $m \in \mathbb{N}$, $(x_n)_{n=1}^m \subset F$, and $p \in [1, \infty)$. Then there exists a $C > 0$ such that, for an arbitrary Banach space F and all $m \in \mathbb{N}$, $(x_n)_{n=1}^m \subset F$, and $z > 0$, the following inequality holds:

$$\mathsf{P}\bigg\{\bigg\|\sum_{n=1}^m x_n g_n(\omega)\bigg\|_F > z\bigg\} \le C\mu\bigg\{\bigg\|\sum_{n=1}^m x_n r_n(t)\bigg\|_F > \frac{z}{C}\bigg\}.$$

Proof. It is well known that, for the random variable

$$f(t) = \bigg\|\sum_{k=1}^m x_k r_k(t)\bigg\|_F, \qquad 0 \le t \le 1,$$

inequality (3) holds with a constant A not depending on F, $m \in \mathbb{N}$, $(x_n)_{n=1}^m \subset F$, and $p \in [1, \infty)$ [10, Theorem 1.e.13]. Therefore, the statement is a direct consequence of Theorem 2. □

3. SYSTEMS OF RANDOM VARIABLES EQUIVALENT IN DISTRIBUTION TO THE RADEMACHER SYSTEM

First, let us present a few definitions. Systems of random variables $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ given on probability spaces $(\Omega, \Sigma, \mathsf{P})$ and $(\Omega', \Sigma', \mathsf{P}')$, respectively, are said to be *equivalent in distribution in the vector sense* if there exists a $C > 0$ such that, for any Banach space F and arbitrary $m \in \mathbb{N}$, $x_n \in F$, $n = 1, 2, \dots, m$, and $z > 0$,

$$C^{-1}\mathsf{P}\biggl\{\biggl\|\sum_{n=1}^{m} x_n f_n(\omega)\biggr\|_F > Cz\biggr\} \le \mathsf{P}'\biggl\{\biggl\|\sum_{n=1}^{m} x_n g_n(\omega')\biggr\|_F > z\biggr\} \le C\mathsf{P}\biggl\{\biggl\|\sum_{n=1}^{m} x_n f_n(\omega)\biggr\|_F > C^{-1}z\biggr\}.$$

An important role in different branches of calculus, such as operator interpolation theory, approximation theory, etc., is played by the Peetre $\mathscr{K}$-functional defined as follows: if (X_0, X_1) is an arbitrary Banach couple (i.e., two Banach spaces linearly and continuously embedded in one separable linear topological space), then

$$\mathscr{K}(t, x; X_0, X_1) = \inf\{||x_0||_{X_0} + t||x_1||_{X_1} : x = x_0 + x_1,\ x_i \in X_i\},$$

where $x \in X_0 + X_1$ and $t > 0$. Further, we shall need the $\mathscr{K}$-functional

$$\mathscr{K}_{1,2}(t, a) := \mathscr{K}(t, a; l_1, l_2), \qquad a = (a_n)_{n=1}^{\infty} \in l_2, \quad t > 0,$$

constructed from the Banach couple (l_1, l_2). Let us recall the most well-known of its approximations, which is obtained by applying the so-called Holmstedt formula (see [11] or [8, Theorem 3.6.1]):

$$\mathscr{K}_{1,2}(t, a) \asymp \sum_{i=1}^{[t^2]} a_i^* + t\biggl[\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\biggr]^{1/2}, \qquad t > 0,$$

where $(a_i^*)_{i=1}^{\infty}$ is a nonincreasing rearrangement of the sequence $(|a_n|)_{n=1}^{\infty}$.

Further, if $1 \le p < \infty$, and F is a Banach space, then a sequence $(x_n) \subset X$ is said to be *weakly p-summable* under the condition that, for each $x^* \in F^*$, the scalar sequences $(x^*(x_n))$ belongs to l_p. The set of all weakly p-summable sequences forms a Banach space denoted by $l_p^w(F)$ with norm

$$\|(x_n)\| = \sup_{\|x^*\|_{F^*} \le 1} \|(x^*(x_n))\|_{l_p}.$$

If $(x_n) \in l_2^w(F)$, then, by definition,

$$\mathscr{K}_{1,2}^w(t, (x_n)) = \sup_{\|x^*\|_{F^*} \le 1} \mathscr{K}_{1,2}(t, x^*(x_n)).$$

Now we can state and prove the following criterion for the equivalence, in distribution in the vector sense, of an arbitrary sequence of random variables to the Rademacher system.

Theorem 3. *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of random variables given on a probability space $(\Omega, \Sigma, \mathsf{P})$. The following conditions are equivalent*:

1) *the systems $\{f_n\}$ and $\{r_n\}$ are equivalent in distribution in the vector sense*;

2) *with a constant independent of the Banach space F, $(x_n)_{n=1}^{m} \subset F$, $m \in \mathbb{N}$, and $p \in [1, \infty)$,*

$$\biggl\|\biggl\|\sum_{n=1}^{m} x_n f_n\biggr\|_F\biggr\|_p \asymp \biggl\|\biggl\|\sum_{n=1}^{m} x_n r_n\biggr\|_F\biggr\|_p;$$

3) *with a constant independent of the Banach space F, $(x_n)_{n=1}^{m} \subset F$, $m \in \mathbb{N}$, and $p \in [1, \infty)$,*

$$\biggl\|\biggl\|\sum_{n=1}^{m} x_n f_n\biggr\|_F\biggr\|_p \asymp \|\mathscr{R}\|_1 + \mathscr{K}_{1,2}^w(\sqrt{p}, (x_n)), \tag{15}$$

where

$$\mathscr{R}(t)=\Big\|\sum_{k=1}^{m}x_kr_k(t)\Big\|_F.$$

Proof. It was proved in [3] that, with a constant independent of the Banach space F,

$$m\in\mathbb{N},\qquad (x_n)_{n=1}^m\subset F,\qquad \text{and}\qquad p\in[1,\infty),$$

we have

$$\bigg\|\Big\|\sum_{n=1}^{m}x_nr_n\Big\|_F\bigg\|_p\asymp\|\mathscr{R}\|_1+\mathscr{K}_{1,2}^{w}(\sqrt{p},(x_n)).\tag{16}$$

Thus, the equivalence 2) $\Leftrightarrow$ 3) is obvious. Since the implication 1) $\Rightarrow$ 2) follows from the definition of equivalence of systems in distribution, it only remains to prove the implication 3) $\Rightarrow$ 1).

The inequality

$$\mathsf{P}\bigg\{\Big\|\sum_{n=1}^{m}x_nf_n(\omega)\Big\|_F>z\bigg\}\le C\mu\bigg\{\Big\|\sum_{n=1}^{m}x_nr_n(t)\Big\|_F>z/C\bigg\},\qquad z>0,$$

with a constant $C>0$ follows immediately from Corollary 1. Let us show that the reverse inequality also holds.

For any sequence $(x_n)_{n=1}^m\subset F$, we define

$$\kappa(t,(x_n))=\|\mathscr{R}\|_1+\mathscr{K}_{1,2}^{w}(\sqrt{t},(x_n)),\qquad t>0,$$

$$\mathscr{F}(\omega)=\Big\|\sum_{n=1}^{m}x_nf_n(\omega)\Big\|_F,\qquad \omega\in\Omega.$$

Note that in view of the concavity of the $\mathscr{K}$-functional,

$$\kappa(2t,(x_n))\le\sqrt{2}\kappa(t,(x_n)),\qquad t>0.$$

Therefore, since condition (15) with a constant $C_1>0$ implies

$$C_1^{-1}\kappa(t,(x_n))\le(\mathsf{E}\mathscr{F}^t)^{1/t}\le C_1\kappa(t,(x_n)),\qquad t\ge 1,$$

it follows that

$$(\mathsf{E}\mathscr{F}^{2t})^{1/(2t)}\le C_1\kappa(2t,(x_n))\le\sqrt{2}C_1\kappa(t,(x_n))\le\sqrt{2}C_1^2(\mathsf{E}\mathscr{F}^t)^{1/t},\qquad t\ge 1.$$

Thus, for the random variables $\mathscr{F}(\omega)$, $\omega\in\Omega$, and $\mathscr{R}(t)$, $t\in[0,1]$, the assumptions of Theorem 2 hold. Using the theorem, we obtain the required inequality. $\square$

Consider some somewhat more specific applications of the results obtained.

Suppose that G is a compact Abelian group with Haar probability measure θ, and Γ is a group of characters given on G, and $C(G)$ is the space of continuous complex-valued functions on G. A set $E\subset\Gamma$ is called a *Sidon set* if, for some $C_E>0$,

$$\sum_{\gamma\in\Gamma}|\widehat{f}(\gamma)|\le C_E\|f\|_\infty$$

for any $f\in C(G)$ such that

$$\widehat{f}(\gamma):=\int_G f(s)\gamma(-s)\,d\theta=0,\qquad \text{if}\quad \gamma\notin E.$$

Pisier proved the following theorem illustrating the close relationship between an arbitrary Sidon system of characters and the Rademacher system [4, Theorem 2.1].

Theorem 4. *Suppose that $E=\{\gamma_n\}_{n=1}^{\infty}\subset\Gamma$ is a Sidon set, F is a Banach space, and the points $x_1,\dots,x_m$ belong to F. Then there exists a constant $C>0$ depending only on the Sidon constant C_E such that, for all $p\in[1,\infty)$,*

$$C^{-1}\biggl(\mathsf{E}\biggl\|\sum_{n=1}^{m}x_nr_n\biggr\|_F^p\biggr)^{1/p}\le\biggl(\int_G\biggl\|\sum_{n=1}^{m}x_n\gamma_n\biggr\|_F^p\,d\theta\biggr)^{1/p}\le C\biggl(\mathsf{E}\biggl\|\sum_{n=1}^{m}x_nr_n\biggr\|_F^p\biggr)^{1/p}.$$

This result, as well as Theorem 3, immediately yields the following statement.

Theorem 5. *Suppose that $E=\{\gamma_n\}_{n=1}^{\infty}\subset\Gamma$ is a Sidon set and F is a Banach space. Then there exists a constant $B>0$ depending only on the Sidon constant C_E such that, for all $m\in\mathbb{N}$, $x_1,\dots,x_m\in F$ and all $z>0$,*

$$B^{-1}\mu\biggl\{\biggl\|\sum_{n=1}^{m}x_nr_n(t)\biggr\|_F>Bz\biggr\}\le\theta\biggl\{\biggl\|\sum_{n=1}^{m}x_n\gamma_n(g)\biggr\|_F>z\biggr\}\le B\mu\biggl\{\biggl\|\sum_{n=1}^{m}x_nr_n(t)\biggr\|_F>B^{-1}z\biggr\}.$$

Remark 2. The assertion in Theorem 5 was obtained earlier in [1] on the basis of Theorem 1 given in the introduction and Theorem 4. Note that its proof given by the authors in [1] is not simple at all: a wide range of additional auxiliary assertions were needed.

Now consider a somewhat different situation. Let us recall [12] that an increasing sequence $(\lambda_k)_{k=1}^{\infty}$ of positive numbers is called a *topological Sidon set* if there exists a compact set $K\subset\mathbb{R}$ and $C>0$ such that, for each sequence of numbers $(a_k)_{k=1}^{\infty}$,

$$C^{-1}\sum_{k=1}^{\infty}|a_k|\le\sup_{t\in K}\biggl|\sum_{k=1}^{\infty}a_ke^{i\lambda_kt}\biggr|.$$

An important example of the topological Sidon set is a sequence $(\lambda_k)_{k=1}^{\infty}$ satisfying the Hadamard condition i.e., such that [12]

$$\inf_k\lambda_{k+1}\lambda_k^{-1}\ge q>1.$$

Pełczyński proved the following result [5, Theorem 3].

Theorem 6. *Suppose that $(\lambda_k)_{k=1}^{\infty}\subset\mathbb{R}$ is a topological Sidon set. Then, for each interval $\Delta=[a,b]$, $-\infty<a<b<+\infty$, there exists a constant $C=C(\Delta)$ such that, for an arbitrary Banach space F, all $x_1,\dots,x_m\in F$, and $1\le p<\infty$, the following inequalities hold*:

$$C^{-1}\biggl(\mathsf{E}\biggl\|\sum_{n=1}^{m}x_nr_n\biggr\|_F^p\biggr)^{1/p}\le\biggl(\int_a^b\biggl\|\sum_{n=1}^{m}x_ne^{i\lambda_nt}\biggr\|_F^p\,dt\biggr)^{1/p}\le C\biggl(\mathsf{E}\biggl\|\sum_{n=1}^{m}x_nr_n\biggr\|_F^p\biggr)^{1/p}.$$

Using this theorem and Theorem 3, we obtain the following statement.

Theorem 7. *Suppose that $(\lambda_k)_{k=1}^{\infty}\subset\mathbb{R}$ is a topological Sidon set. Then, for each interval $\Delta=[a,b]$, $-\infty<a<b<+\infty$, there exists a constant $B=B(\Delta)$ such that, for an arbitrary Banach space F, all $x_1,\dots,x_m\in F$, and $z>0$, the following inequalities hold*:

$$\begin{aligned}B^{-1}\mu\biggl\{t\in[0,1]:\biggl\|\sum_{n=1}^{m}x_nr_n(t)\biggr\|_F>Bz\biggr\}&\le\mu\biggl\{t\in[a,b]:\biggl\|\sum_{n=1}^{m}x_ne^{i\lambda_nt}\biggr\|_F>z\biggr\}\\&\le B\mu\biggl\{t\in[0,1]:\biggl\|\sum_{n=1}^{m}x_nr_n(t)\biggr\|_F>B^{-1}z\biggr\}.\end{aligned}$$

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