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Rearrangement invariance of Rademacher multiplicator spaces

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Abstract

Let X be a rearrangement invariant function space on $[0,1]$. We consider the Rademacher multiplicator space $\Lambda(\mathcal{R},X)$ of all measurable functions x such that $x\cdot h\in X$ for every a.e. converging series $h=\sum a_n r_n\in X$, where (r_n) are the Rademacher functions. We study the situation when $\Lambda(\mathcal{R},X)$ is a rearrangement invariant space different from L^∞. Particular attention is given to the case when X is an interpolation space between the Lorentz space $\Lambda(\varphi)$ and the Marcinkiewicz space $M(\varphi)$. Consequences are derived regarding the behaviour of partial sums and tails of Rademacher series in function spaces.

Keywords: Rademacher functions; Rearrangement invariant spaces

Introduction

In this paper we study the behaviour of the Rademacher functions (r_n) in function spaces. Let $\mathcal{R}$ denote the set of all functions of the form $\sum a_n r_n$, where the series converges a.e. For a rearrangement invariant (r.i.) space X on $[0,1]$, let $\mathcal{R}(X)$ be the closed linear subspace of X given by $\mathcal{R}\cap X$. The *Rademacher multiplicator space* of X is the space $\Lambda(\mathcal{R},X)$ of all measurable functions $x:[0,1]\to\mathbb{R}$ such that $x\sum a_n r_n\in X$, for every $\sum a_n r_n\in\mathcal{R}(X)$. It is a

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Banach function space on [0, 1] when endowed with the norm

$$\|x\|_{\Lambda(\mathcal{R},X)} = \sup\left\{\left\|x\sum a_n r_n\right\|_X : \sum a_n r_n \in X,\ \left\|\sum a_n r_n\right\|_X \leqslant 1\right\}.$$

The space $\Lambda(\mathcal{R}, X)$ can be viewed as the space of operators from $\mathcal{R}(X)$ into the whole space X given by multiplication by a measurable function.

The Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$ was firstly considered in [8] where it was shown that for a broad class of classical r.i. spaces X the space $\Lambda(\mathcal{R}, X)$ is not r.i. This result was extended in [2] to include all r.i. spaces such that the lower dilation index γ_{φ_X} of their fundamental function φ_X satisfies $\gamma_{\varphi_X} > 0$. This result motivated the study the symmetric kernel $\mathrm{Sym}(\mathcal{R}, X)$ of the space $\Lambda(\mathcal{R}, X)$, that is, the largest r.i. space embedded into $\Lambda(\mathcal{R}, X)$. The space $\mathrm{Sym}(\mathcal{R}, X)$ was studied in [2], where it was shown that, if X is an r.i. space satisfying the Fatou property and $X \supset L_N$, where L_N is the Orlicz space with $N(t) = \exp(t^2) - 1$, then $\mathrm{Sym}(\mathcal{R}, X)$ is the r.i. space with the norm $\|x\| := \|x^*(t)\log^{1/2}(2/t)\|_X$. It was also shown that any space X which has the Fatou property and is an interpolation space for the couple $(L\log^{1/2} L, L^\infty)$ can be realized as the symmetric kernel of a certain r.i. space. The opposite situation is when the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$ is r.i. The simplest case of this situation is when $\Lambda(\mathcal{R}, X) = L^\infty$. In [1] it was shown that $\Lambda(\mathcal{R}, X) = L^\infty$ holds for all r.i. spaces X which are interpolation spaces for the couple (L^∞, L_N). It was shown in [3] that $\Lambda(\mathcal{R}, X) = L^\infty$ if and only if the function $\log^{1/2}(2/t)$ does not belong to the closure of L^∞ in X.

In this paper we investigate the case when the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$ is an r.i. space different from L^∞. Examples of this situation were considered in [2,8,9]. In all cases they were spaces X consisting of functions with exponential growth.

The paper is organized as follows. Section 1 is devoted to the preliminaries. In Section 2 we study technical conditions on an r.i. space X and its fundamental function φ. In Section 3 we present a sufficient condition for $\Lambda(\mathcal{R}, X)$ being r.i. (Theorem 3.4). For this, two results are needed. Firstly, that the symmetric kernel $\mathrm{Sym}(\mathcal{R}, X)$ is a maximal space (Proposition 3.1), and secondly, a condition, of independent interest, on the behaviour of logarithmic functions on an r.i. space (Proposition 3.3). Section 4 is devoted to the study of necessary conditions for $\Lambda(\mathcal{R}, X)$ being an r.i. space. This is done by separately studying conditions on partial sums and tails of Rademacher series (Propositions 4.1 and 4.2). Theorem 4.4 addresses the case when X in an interpolation space for the couple $(\Lambda(\varphi), M(\varphi))$, where $\Lambda(\varphi)$ and $M(\varphi)$ are, respectively, the Lorentz and Marcinkiewicz spaces with the fundamental function φ. Theorem 4.5 specializes the previous result for the case of $X = M(\varphi)$. We end presenting, in Section 5, examples which highlight certain features of the previous results.

1. Preliminaries

Throughout the paper a rearrangement invariant (r.i.) space X is a Banach space of classes of measurable functions on [0, 1] such that if $y^* \leqslant x^*$ and $x \in X$ then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$. Here x^* is the decreasing rearrangement of x, that is, the right continuous inverse of its distribution function: $n_x(\tau) = \lambda\{t \in [0, 1]\colon |x(t)| > \tau\}$, where λ is the Lebesgue measure on [0, 1]. Functions x and y are said to be equimeasurable if $n_x(\tau) = n_y(\tau)$, for all $\tau > 0$. The associated space (or Köthe dual) of X is the space X' of all functions y such that $\int_0^1 |x(t)y(t)|\,dt < \infty$, for every $x \in X$. It is an r.i. space. The space X' is a subspace of the topological dual X^*. If X' is a norming subspace of X^*, then X is isometric to a subspace of the space $X'' = (X')'$. The space X is maximal when $X = X''$. We denote by X_0 the closure of L^∞ in X. If X is not L^∞, then X_0 coincides with the absolutely continuous part of X, that is, the set of all functions $x \in X$ such that $\lim_{\lambda(A)\to 0} \|x\chi_A\|_X = 0$. Here and next, χ_A is the characteristic function of the set $A \subset [0,1]$. The fundamental function of X is the function $\varphi_X(t) := \|\chi_{[0,t]}\|_X$.

Important examples of r.i. spaces are Marcinkiewicz, Lorentz and Orlicz spaces. Let $\varphi : [0,1] \to [0,+\infty)$ be a quasi-concave function, that is, φ increases, $\varphi(t)/t$ decreases and $\varphi(0) = 0$. The Marcinkiewicz space $M(\varphi)$ is the space of all measurable functions x on $[0,1]$ for which the norm

$$\|x\|_{M(\varphi)} = \sup_{0<t\leqslant 1} \frac{\varphi(t)}{t} \int_0^t x^*(s)\,ds < \infty.$$

If $\varphi : [0,1] \to [0,+\infty)$ is an increasing concave function, $\varphi(0) = 0$, then the Lorentz space $\Lambda(\varphi)$ consists of all measurable functions x on $[0,1]$ such that

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(s)\,d\varphi(s) < \infty.$$

Let M be an Orlicz function, that is, an increasing convex function on $[0,\infty)$ with $M(0) = 0$. The norm of the Orlicz space L_M is defined as follows

$$\|x\|_{L_M} = \inf\left\{\lambda > 0\colon \int_0^1 M\left(\frac{|x(s)|}{\lambda}\right) ds \leqslant 1\right\}.$$

The fundamental functions of these spaces are $\varphi_{M(\varphi)}(t) = \varphi_{\Lambda(\varphi)}(t) = \varphi(t)$, and $\varphi_{L_M}(t) = 1/M^{-1}(1/t)$, respectively.

The Marcinkiewicz $M(\varphi)$ and Lorentz $\Lambda(\varphi)$ spaces are, respectively, the largest and the smallest r.i. spaces with fundamental function φ, that is, if the fundamental function of an r.i. space X is equal to φ, then $\Lambda(\varphi) \subset X \subset M(\varphi)$.

If ψ is a positive function defined on $[0,1]$, then its lower dilation index is

$$\gamma_\psi := \lim_{t\to 0^+} \frac{\log(\sup_{0<s\leqslant 1} \frac{\psi(st)}{\psi(s)})}{\log t},$$

and its upper dilation index is

$$\delta_\psi := \lim_{t\to +\infty} \frac{\log(\sup_{0<s\leqslant 1/t} \frac{\psi(st)}{\psi(s)})}{\log t}.$$

If a quasi-concave function φ satisfies $\delta_\varphi < 1$, then we have the following equivalence for the norm in the Marcinkiewicz space $M(\varphi)$

$$\|x\|_{M(\varphi)} \asymp \sup_{0<t\leqslant 1} \varphi(t)x^*(t), \tag{1.1}$$

[11, Theorem II.5.3]. Here, and throughout the paper, $A \asymp B$ means that there exist constants $C > 0$ and $c > 0$ such that $c \cdot A \leqslant B \leqslant C \cdot A$.

For each $t > 0$, the dilation operator $\sigma_t x(s) := x(s/t)\chi_{[0,1]}(s/t)$, $s \in [0, 1]$, is bounded on any r.i. space X.

Given Banach spaces X_0 and X_1 continuously embedded in a common Hausdorff topological vector space, a Banach space X is an interpolation space with respect to the couple (X_0, X_1) if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and for every linear operator T with $T : X_i \to X_i$ $(i = 0, 1)$ continuously, we have $T : X \to X$.

The Rademacher functions are $r_n(t) := \operatorname{sign}\sin(2^n\pi t)$, $t \in [0, 1]$, $n \geqslant 1$. We have already defined $\mathcal{R}(X) := \mathcal{R} \cap X$ where $\mathcal{R}$ is the set of all a.e. converging series $\sum a_n r_n$, that is, $(a_n) \in \ell^2$ [15, Theorem V.8.2]. For $X = L^p$, $1 \leqslant p < \infty$, Khintchin inequality shows that $\mathcal{R}(X)$ is isomorphic to ℓ^2. If $X = L^\infty$, then $\mathcal{R}(X) = \ell^1$. The Orlicz space L_N, for $N(t) = \exp(t^2) - 1$, will be of major importance in our study. A result of Rodin and Semenov shows that $\mathcal{R}(X) \approx \ell^2$ if and only if $(L_N)_0 \subset X$, [14]. Hence, for spaces X satisfying this condition we have

$$\Bigl\|\sum a_n r_n\Bigr\|_X \asymp \bigl\|(a_n)\bigr\|_2. \tag{1.2}$$

The fundamental function of L_N is (equivalent to) $\varphi(t) = \log^{-1/2}(2/t)$. Since $N(t)$ increases very rapidly, L_N coincides with the Marcinkiewicz space with fundamental function φ, [13]. This, together with $\delta_\varphi = 0 < 1$, gives

$$\|x\|_{L_N} \asymp \sup_{0<t\leqslant 1} x^*(t)\log^{-1/2}(2/t).$$

In particular, for every $0 < t \leqslant 1$ we have

$$x^*(t) \leqslant C\|x\|_{L_N}\log^{1/2}(2/t). \tag{1.3}$$

Hence, for an r.i. space X, $L_N \subset X$ is equivalent to $\log^{1/2}(2/t) \in X$.

We denote the dyadic intervals of $[0, 1]$ by $\Delta_n^k := [\frac{k-1}{2^n}, \frac{k}{2^n})$ for $n \in \mathbb{N}$ and $k = 1, \dots, 2^n$. The set of all dyadic step functions is $D = \bigcup_n D_n$, where, for $n \in \mathbb{N}$, D_n is the set of all dyadic step functions of order n:

$$f(t) = \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k}(t), \quad c_k \in \mathbb{R}.$$

For convenience of computations, all logarithms will be considered with base 2.

For any undefined notion regarding function spaces, r.i. spaces, and interpolation of linear operators, we refer the reader to the monographs [6,7,11,12].

2. Conditions on r.i. spaces

In this section we collect together the conditions and results of technical nature on an r.i. space X and its fundamental function φ that will be needed in the following sections.

Definition 2.1. Let X be an r.i. space on $[0, 1]$ and φ be a function defined on $[0, 1]$.

(a) The operator Q is defined, over adequate functions $x(t)$ on $[0, 1]$, as follows

$$Qx(t) := x\big(t^2\big), \quad 0 \leqslant t \leqslant 1.$$

(b) φ satisfies the Δ^2-*condition* if it is nonnegative, increasing, concave, and there exists $C > 0$ such that

$$\varphi(t) \leqslant C \cdot \varphi\big(t^2\big), \quad 0 < t \leqslant 1. \tag{2.1}$$

(c) X satisfies the *log-condition* if there exists $C > 0$ such that, for all $u \in (0, 1)$,

$$\left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(0,u]}\right\|_X \leqslant C\left\|\log^{1/2}\left(\frac{2u}{t}\right)\chi_{(0,u]}\right\|_X. \tag{2.2}$$

(d) If φ is increasing and quasi-concave, we define the function

$$\bar{\varphi}(t) := t^{-1/2}\varphi\big(2^{1-1/t}\big), \quad 0 < t \leqslant 1.$$

(e) $\varphi(t)$ satisfies the $\sqrt{2}$-*condition* if it is increasing, quasi-concave, and there exists $n_0 \in \mathbb{N}$ such that

$$c_\varphi := \sup_{n \geqslant n_0} \frac{\varphi(2^{-2n})}{\varphi(2^{-n})} \leqslant \frac{1}{\sqrt{2}}. \tag{2.3}$$

The following proposition is rather elementary. However, since we will refer next to it many times, we include the proofs of the less trivial implications (3) and (4).

Proposition 2.2. *Let X be an r.i. space on* $[0, 1]$ *with fundamental function $\varphi(t)$.*

(1) *If Q is bounded on X, then $\varphi \in \Delta^2$.*
(2) *If $\varphi \in \Delta^2$ and X is an interpolation space for the couple $(\Lambda(\varphi), M(\varphi))$, then Q is bounded on X.*
(3) *If $\log^{1/2}(2/t) \in X$ and $\varphi \in \Delta^2$, then X satisfies the log-condition.*
(4) *If the lower dilation index of $\bar{\varphi}$ satisfies $\gamma_{\bar{\varphi}} > 0$, then*

$$C_0 := \sup_{n=1,2,\ldots} \frac{1}{\sqrt{n}\varphi(2^{-n})} \sum_{k=n}^{\infty} \frac{\varphi(2^{-k})}{\sqrt{k}} < \infty.$$

(5) *If the function $\log^{1/2}(2/t)\varphi(t)$ is increasing on some interval $(0, t_0)$, for $t_0 > 0$, then φ satisfies the $\sqrt{2}$-condition.*

Proof. (3) For $0 < u \leqslant 1$, we have

$$\left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(0,u]}\right\|_X \leqslant \left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(0,u^2]}\right\|_X + \left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(u^2,u]}\right\|_X. \tag{2.4}$$

If $0 < t \leqslant u^2$, then $\sqrt{t} \geqslant t/u$. Therefore,

$$\left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(0,u^2]}\right\|_X \leqslant \sqrt{2}\left\|\log^{1/2}\left(\frac{2}{\sqrt{t}}\right)\chi_{(0,u^2]}\right\|_X \leqslant \sqrt{2}\left\|\log^{1/2}\left(\frac{2u}{t}\right)\chi_{(0,u]}\right\|_X.$$

By (2.1),

$$\begin{aligned}\left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(u^2,u]}\right\|_X &\leqslant \sqrt{2}\log^{1/2}\left(\frac{2}{u}\right)\|\chi_{(u^2,u]}\|_X \leqslant \sqrt{2}C\log^{1/2}\left(\frac{2}{u}\right)\varphi\left(u^2\right)\\ &= \sqrt{2}C\log^{1/2}\left(\frac{2u}{u^2}\right)\varphi\left(u^2\right) \leqslant \sqrt{2}C\left\|\log^{1/2}\left(\frac{2u}{t}\right)\chi_{(0,u]}\right\|_X.\end{aligned}$$

The last two formulas and (2.4) imply the log-condition (2.2).

(4) If $\gamma_{\bar{\varphi}} > 0$, then there are $\delta > 0$ and $C > 0$ such that for all $0 < u \leqslant 1$ and $t \geqslant 1$ we have $\bar{\varphi}(tu) \leqslant Ct^{\delta}\bar{\varphi}(u)$, that is,

$$\varphi\left(2^{1-1/(tu)}\right) \leqslant Ct^{\delta+1/2}\varphi\left(2^{1-1/u}\right).$$

Fix $n \in \mathbb{N}$, set $u = 1/n$, and apply the previous inequality with $t = 2^{-j}$, where $j = 0, 1, 2, \ldots$. This, together with the quasi-concavity of φ, implies that

$$\varphi\left(2^{-2^j n}\right) \leqslant C2^{-(\delta+1/2)j}\varphi\left(2^{-n}\right), \quad j = 0, 1, 2, \ldots.$$

Therefore,

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{\varphi(2^{-k})}{\sqrt{k}} &= \sum_{j=0}^{\infty}\sum_{k=2^j n}^{2^{j+1}n-1}\frac{\varphi(2^{-k})}{\sqrt{k}} \leqslant \sum_{j=0}^{\infty}2^{j/2}\sqrt{n}\varphi\left(2^{-2^j n}\right)\\ &\leqslant C\sqrt{n}\varphi\left(2^{-n}\right)\sum_{j=0}^{\infty}2^{-(\delta+1/2)j}2^{j/2} = C\sqrt{n}\varphi\left(2^{-n}\right)\frac{2^{\delta}}{2^{\delta}-1}. \qquad \square\end{aligned}$$

Remark 2.3. (a) Definition 2.1(b) was introduced in [4] in connection with extrapolation of operators in r.i. spaces.

(b) Due to φ being concave and increasing, the Δ^2-condition (2.1) is equivalent to its discrete analog: there exist $\beta > 1$ and $C > 0$ such that

$$\varphi\left(2^{-n}\right) \leqslant C\varphi\left(2^{-\beta n}\right), \quad n \in \mathbb{N}. \tag{2.5}$$

(c) The log-condition (2.2) is equivalent to its discrete analog: there exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(0,2^{-n}]}\right\|_X \leqslant C\left\|\log^{1/2}\left(\frac{2}{2^n t}\right)\chi_{(0,2^{-n}]}\right\|_X. \tag{2.6}$$

(d) The same proof as of Proposition 2.2(3) shows that if inequality (2.1) holds for $u \in E$, for some $E \subset [0, 1]$, then inequality (2.2) also holds for $u \in E$.

(e) Note that the condition $\gamma_{\bar{\varphi}} > 0$ implies that $\varphi(t) \leqslant C \log^{-1/2-\delta}(2/t)$ $(0 < t \leqslant 1)$ with some $C > 0$ and $\delta > 0$. Therefore, $\log^{1/2}(2/t) \in \Lambda(\varphi)$. The condition $\gamma_{\bar{\varphi}} > 0$ means, roughly speaking, that the distance from φ to the fundamental function $\log^{-1/2}(2/t)$ of the space L_N is sufficiently large.

3. Sufficient conditions for $\Lambda(R, X)$ being an r.i. space

We begin with a sharpening of results of §2 of [2], that will be needed for the main result of this section.

Proposition 3.1. *Let X be an r.i. space on* $[0, 1]$. *Then* $\mathrm{Sym}(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X'')$. *In particular,* $\mathrm{Sym}(\mathcal{R}, X)$ *is a maximal r.i. space.*

Proof. First, suppose $\log^{1/2}(2/t) \notin X_0$. Since $(X'')_0 = X_0$, then $\log^{1/2}(2/t) \notin (X'')_0$. From [2, Theorem 3.2], $\mathrm{Sym}(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X'') = L_\infty$.

Suppose now that $\log^{1/2}(2/t) \in X_0$. Since $X \subset X''$, then $\mathrm{Sym}(\mathcal{R}, X) \subset \mathrm{Sym}(\mathcal{R}, X'')$, [2, Corollary 3.4]. Thus, we only have to prove that $\mathrm{Sym}(X'') \subset \mathrm{Sym}(\mathcal{R}, X)$. By [2, Corollary 2.11], we have

$$\|x\|_{\mathrm{Sym}(\mathcal{R}, X'')} \asymp \left\| x^*(t) \log^{1/2}(2/t) \right\|_{X''}.$$

Therefore, if $x \in \mathrm{Sym}(\mathcal{R}, X'')$ then $x^*(t) \log^{1/2}(2/t) \in X''$. Let $a = (a_k) \in \ell_2$. It is well known that the function $x_a := \sum a_k r_k \in (L_N)_0$, which implies, by [11, Lemma II.5.4], that

$$\lim_{t \to 0^+} \log^{-1/2}\left(\frac{2}{t}\right) \frac{1}{t} \int_0^t x_a^*(s)\, ds = 0,$$

whence

$$\lim_{t \to 0^+} x_a^*(t) \log^{-1/2}\left(\frac{2}{t}\right) = 0. \tag{3.1}$$

For $0 < h \leqslant 1$ and $0 < t \leqslant 1$, we have

$$x^*(t) x_a^*(t) \chi_{[0,h]}(t) \leqslant \sup_{0 < s \leqslant h} \left(x_a^*(s) \log^{-1/2}(2/s) \right) \cdot x^*(t) \log^{1/2}(2/t),$$

whence

$$\left\| x^*(t) x_a^*(t) \chi_{[0,h]}(t) \right\|_{X''} \leqslant \sup_{0 < s \leqslant h} \left(x_a^*(s) \log^{-1/2}(2/s) \right) \cdot \left\| x^*(t) \log^{1/2}(2/t) \right\|_{X''}.$$

Since $x^*(t) \log^{1/2}(2/t) \in X''$, by (3.1),

$$\left\| x^*(t) x_a^*(t) \chi_{[0,h]}(t) \right\|_{X''} \to 0 \quad \text{as } h \to 0^+.$$

This means that $x^*(t)x_a^*(t) \in (X'')_0 = X_0$. Since this holds for every $a \in \ell_2$, then [2, Corollary 2.5] implies that $x \in \mathrm{Sym}(\mathcal{R}, X_0) \subset \mathrm{Sym}(\mathcal{R}, X)$. The maximality of $\mathrm{Sym}(\mathcal{R}, X)$ follows from [2, Proposition 2.6]. $\square$

The following corollary is a straightforward consequence of Proposition 3.1 and [2, Corollary 2.11].

Corollary 3.2. *For every r.i. space X on $[0,1]$ such that $\log^{1/2}(2/t) \in X$*

$$\|x\|_{\mathrm{Sym}(\mathcal{R},X)} \asymp \left\| x^*(t) \log^{1/2}(2/t) \right\|_{X''}.$$

The next result gives a useful sufficient condition for the rearrangement invariance of $\Lambda(\mathcal{R}, X)$.

Proposition 3.3. *Let X be an r.i. space on $[0,1]$ such that $\log^{1/2}(2/t) \in X_0$. Suppose there exists $A > 0$ such that for $n \in \mathbb{N}$*

$$\left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\left(\frac{2}{t}\right) \right\|_X \leqslant A \left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\left(\frac{2}{2^n t + 1 - k}\right) \right\|_X, \tag{3.2}$$

for every $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{2^n} \geqslant 0$. Then $\Lambda(\mathcal{R}, X)$ is an r.i. space.

Proof. Since $\mathrm{Sym}(\mathcal{R}, X) \subseteq \Lambda(\mathcal{R}, X)$, we just have to show the opposite inclusion.

Let $f \in D_n$, that is, $f = \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k}$, with $c_k \in \mathbb{R}$. Since $\mathrm{Sym}(\mathcal{R}, X)$ is r.i., we can assume that the sequence $(c_k)_{k=1}^{2^n}$ is nonnegative and decreasing. In this case, by Corollary 3.2,

$$\|f\|_{\mathrm{Sym}(\mathcal{R},X)} \asymp \left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\left(\frac{2}{t}\right) \right\|_{X''}. \tag{3.3}$$

On the other hand, by the definition of the norm in $\Lambda(\mathcal{R}, X)$,

$$\|f\|_{\Lambda(\mathcal{R},X)} \geqslant \sup\left\{ \left\| f \cdot \sum_{k=n+1}^{\infty} a_k r_k \right\|_X : \left\| \sum_{k=n+1}^{\infty} a_k r_k \right\|_X \leqslant 1 \right\}.$$

Since $\log^{1/2}(2/t) \in X_0$, we have $\mathcal{R}(X) \approx \ell^2$ so, by (1.2), for all $n, m \in \mathbb{N}$,

$$\left\| \sum_{k=n+1}^{n+m} \frac{1}{\sqrt{m}} r_k \right\|_X \leqslant C_1.$$

Hence,

$$\|f\|_{\Lambda(\mathcal{R},X)} \geqslant C_1^{-1} \sup_{m\in\mathbb{N}} \left\{ \left\| f \cdot \sum_{k=n+1}^{n+m} \frac{1}{\sqrt{m}} r_k \right\|_X \right\}. \tag{3.4}$$

Note that, for $n \in \mathbb{N}$ and $k = 1, \dots, 2^n$, we have

$$\left(\chi_{\Delta_n^k} \cdot \sum_{i=n+1}^{n+m} r_i\right)^*(t) = \left(\sum_{i=1}^{m} r_i\right)^*\big(2^n t\big), \quad 0 \leqslant t \leqslant 2^{-n}.$$

Therefore, since X is r.i., we get

$$\begin{aligned}\left\| f \cdot \frac{1}{\sqrt{m}} \sum_{i=n+1}^{n+m} r_i \right\|_X &= \left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \frac{1}{\sqrt{m}} \sum_{i=n+1}^{n+m} r_i \right\|_X \\ &= \left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \frac{1}{\sqrt{m}} \left(\sum_{i=1}^{m} r_i\right)^* \big(2^n t + 1 - k\big) \right\|_X.\end{aligned}$$

Using the central limit theorem (see [14] or [12, Theorem 2.b.4]), we have

$$\lim_{m \to \infty} \left(\frac{1}{\sqrt{m}} \sum_{i=1}^{m} r_i\right)^*(t) \geqslant \log^{1/2}\left(\frac{2}{e\sqrt{2\pi}\,t}\right), \quad 0 < t \leqslant \frac{2}{e\sqrt{2\pi}}. \tag{3.5}$$

This and inequality (3.4) imply that

$$\|f\|_{\Lambda(\mathcal{R},X)} \geqslant M \left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\left(\frac{2}{2^n t + 1 - k}\right) \right\|_{X''}.$$

By assumption, $\log^{1/2}(2/t) \in X_0$. Therefore, since $(X'')_0 = X_0$ isometrically, we can change the norm of X'' by the norm of X, i.e., we have

$$\|f\|_{\Lambda(\mathcal{R},X)} \geqslant M \left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\left(\frac{2}{2^n t + 1 - k}\right) \right\|_{X}.$$

This inequality, (3.3), and (3.2) imply that for $f \in D$ we have

$$\|f\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant C \|f\|_{\Lambda(\mathcal{R},X)}. \tag{3.6}$$

Next, we extend inequality (3.6) to L_∞. Since $\log^{1/2}(2/t) \in X_0$, we have $\mathrm{Sym}(\mathcal{R}, X) \neq L_\infty$ [2, Theorem 3.2]. In particular, the fundamental function of $\mathrm{Sym}(\mathcal{R}, X)$ tends to 0 as $t \to 0^+$. By Lusin's theorem, we deduce that for every $g \in L_\infty$ there is a sequence $\{f_n\} \subset D$ such that $\|f_n - g\|_{\mathrm{Sym}(\mathcal{R},X)} \to 0$. The embedding $\mathrm{Sym}(\mathcal{R}, X) \subset \Lambda(\mathcal{R}, X)$ implies that $\|f_n - g\|_{\Lambda(\mathcal{R},X)} \to 0$. By (3.6), we have that $\|f_n\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant C \|f_n\|_{\Lambda(\mathcal{R},X)}$. Consequently

$$\|g\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant C \|g\|_{\Lambda(\mathcal{R},X)}, \quad g \in L_\infty. \tag{3.7}$$

Let $h \in \Lambda(\mathcal{R}, X)$ be a nonnegative function. There exists a sequence $\{g_n\} \subset L_\infty$ such that $0 \leqslant g_n \uparrow h$ a.e. on $[0, 1]$. Then, (3.7) implies that

$$\|g_n\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant C \|g_n\|_{\Lambda(\mathcal{R},X)} \leqslant C \|h\|_{\Lambda(\mathcal{R},X)}.$$

Since, by Proposition 3.1, $\mathrm{Sym}(\mathcal{R}, X)$ is maximal, we deduce that $h \in \mathrm{Sym}(\mathcal{R}, X)$ and $\|h\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant C\|h\|_{\Lambda(\mathcal{R},X)}$. □

We can now prove the main result of this section.

Theorem 3.4. *If X is an r.i. space on $[0,1]$ such that the operator $Qx(s) = x(s^2)$ is bounded in X, then*

$$\Lambda(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X).$$

Proof. First we note that if $\log^{1/2}(2/t) \notin X_0$ then, by [3], $\Lambda(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X) = L_\infty$. Thus, we may assume that $\log^{1/2}(2/t) \in X_0$. In view of Proposition 3.3, we just have to establish (3.2).

Denote

$$g(t) = \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\left(\frac{2}{t}\right) \quad \text{and} \quad h(t) = \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \log^{1/2}\left(\frac{2}{2^n t + 1 - k}\right),$$

where $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{2^n} \geqslant 0$. We write g in the form $g = g_1 + g_2$,

$$g_1(t) = c_1 \chi_{\Delta_n^1}(t) \cdot \log^{1/2}(2/t) \quad \text{and} \quad g_2(t) = g(t) - g_1(t). \tag{3.8}$$

Since Q is bounded in X, by Proposition 2.2(1), (3), we have

$$\|g_1\|_X \leqslant Cc_1 \left\| \chi_{\Delta_n^1} \cdot \log^{1/2}\left(\frac{2}{2^n t}\right) \right\|_X \leqslant C\|h\|_X. \tag{3.9}$$

For estimating $\|g_2\|_X$, we consider the function

$$\bar{g}_2(t) := \sum_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} c_i \chi_{\Delta_n^i}(t) \log^{1/2}\left(\frac{2}{2^{j-n}}\right).$$

It is easily seen that

$$\left|g_2(t)\right| \leqslant \left|\bar{g}_2(t)\right|, \quad 0 < t \leqslant 1. \tag{3.10}$$

Let Q^{-1} be the inverse operator to Q, i.e., $Q^{-1}x(t) := x(\sqrt{t})$. A straightforward calculation shows that

$$Q^{-1}\bar{g}_2(t) = \sum_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} c_i \chi_{[\frac{(i-1)^2}{2^{2n}}, \frac{i^2}{2^{2n}}]}(t) \log^{1/2}\left(\frac{2}{2^{j-n}}\right).$$

Thus, in order to prove the inequality

$$\left(Q^{-1}\bar{g}_2\right)^*(t) \leqslant h^*(t), \quad 0 < t \leqslant 1, \tag{3.11}$$

it suffices to show that, for every $j = 1, 2, \dots, n$ and i satisfying $2^{j-1} < i \leqslant 2^j$, we have

$$\lambda\left\{t \in [0,1]:\ \left|h(t)\right| \geqslant c_i \log^{1/2}\left(\frac{2}{2^{j-n}}\right)\right\} \geqslant \sum_{k=1}^{i} \frac{2k-1}{2^{2n}} = \frac{i^2}{2^{2n}}.$$

Indeed, let $k = 1, 2, \dots, i$. Since $c_1 \geqslant c_2 \geqslant \dots \geqslant c_{2^n} \geqslant 0$, we have

$$\begin{aligned}
&\lambda\left\{t \in \Delta_n^k:\ \left|h(t)\right| \geqslant c_i \log^{1/2}\left(\frac{2}{2^{j-n}}\right)\right\} \\
&\quad \geqslant \lambda\left\{t \in \left(0, 2^{-n}\right):\ \log^{1/2}\left(\frac{2}{2^n t}\right) \geqslant \log^{1/2}\left(\frac{2}{2^{j-n}}\right)\right\} \\
&\quad = 2^{j-2n}.
\end{aligned}$$

Hence,

$$\lambda\left\{t \in [0,1]:\ \left|h(t)\right| \geqslant c_i \log^{1/2}\left(\frac{2}{2^{j-n}}\right)\right\} \geqslant i \cdot 2^{j-2n} \geqslant i^2 \cdot 2^{-2n}.$$

So, inequality (3.11) is proved.

By (3.10) and (3.11) we get

$$\|g_2\|_X \leqslant \left\|Q\left(Q^{-1}\bar{g}_2\right)\right\|_X \leqslant \|Q\| \left\|Q^{-1}\bar{g}_2\right\|_X \leqslant \|Q\| \|h\|_X.$$

This inequality together with (3.9) yields (3.2) so, the result is proved. □

Combining Theorem 3.4 and Proposition 2.2(2) we have the following.

Corollary 3.5. *If X is an interpolation space for the couple $(\Lambda(\varphi), M(\varphi))$ and $\varphi \in \Delta^2$, then $\Lambda(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X)$. In particular, X may be the Lorentz space $\Lambda(\varphi)$ or the Marcinkiewicz space $M(\varphi)$.*

Remark 3.6. Boundedness of the operator Q in X is not a necessary condition for the equality $\Lambda(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X)$. Indeed, for every increasing concave function $\varphi \in \Delta^2$ there is an r.i. space X such that Q is not bounded on X [5, Example 2.12]. Actually, X is a subspace of the Marcinkiewicz space $M(\varphi)$ and therefore $X'' = M(\varphi)$. Then, Proposition 3.1, Corollary 3.5, and [3, Corollary 3] imply that

$$\mathrm{Sym}(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X'') = \mathrm{Sym}\left(\mathcal{R}, M(\varphi)\right) = \Lambda\left(\mathcal{R}, M(\varphi)\right) \supset \Lambda(\mathcal{R}, X).$$

This, together with $\Lambda(\mathcal{R}, X) \supset \mathrm{Sym}(\mathcal{R}, X)$, gives $\Lambda(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X)$.

Remark 3.7. The proofs of Proposition 3.3 and Theorem 3.4, and [3, Remark 5] show that the following *tail-assertion* holds: if X is an r.i. space on [0, 1] such that the operator Q is bounded on X, then there exists $A > 0$ such that for every $f \in D_n$ we have

$$\|f\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant A \sup\left\{\left\|f \cdot \sum_{i=n+1}^{\infty} c_i r_i\right\|_X :\ \left\|\sum_{i=n+1}^{\infty} c_i r_i\right\|_X \leqslant 1\right\}.$$

4. Necessary conditions for $\Lambda(R, X)$ being an r.i. space

Now we pass to the consideration of the opposite problem: finding necessary conditions for an r.i. space X satisfying $\Lambda(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X)$.

We require some specific sets formed by unions of dyadic intervals. Consider the matrix $\mathcal{A} = (\theta_{i,j})$, where $\theta_{i,j}$ is the value of the function r_j on the interval Δ^i_{2n}, where $n \in \mathbb{N}$, $j = 1, 2, \dots, 2n$, and $i = 1, 2, \dots, 2^{2n}$. Let $\Omega_n \subset \{1, 2, \dots, 2^{2n}\}$ be the set of all rearrangements of signs on $\{1, 2, \dots, 2^n\}$ such that for every $i \in \Omega_n$ we have

$$\theta_{i,j+n} = \theta_{i,j}, \quad j = 1, 2, \dots, n.$$

Denote by $\mathcal{A}(\Omega_n)$ the submatrix of $\mathcal{A}$ corresponding to the set Ω_n. We will consider $\mathcal{A}(\Omega_n)$ as an operator acting from ℓ_2^{2n} into $\ell_2^{2^n}$. Define

$$U_n = \bigcup_{i \in \Omega_n} \Delta^i_{2n}.$$

Since $\operatorname{card} \Omega_n = 2^n$, then $\lambda(U_n) = 2^{-n}$.

Now, we prove a first result assuming certain *tail-estimates*.

Proposition 4.1. *Suppose that X is an r.i. space on* $[0, 1]$ *with fundamental function φ such that $\gamma_{\bar\varphi} > 0$. Let $I \subset \mathbb{N}$ be such that for every $n \in I$ we have*

$$\|\chi_{[0,2^{-n}]}\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant A \sup\left\{ \left\| \chi_{[0,2^{-n}]} \cdot \sum_{i=n+1}^{\infty} c_i r_i \right\|_X : \left\| \sum_{i=n+1}^{\infty} c_i r_i \right\|_X \leqslant 1 \right\},$$

where $A > 0$ does not depend on n. Then, there exist $\beta > 1$ and $C > 0$ such that, for all $n \in I$, we have (2.5)*, that is,*

$$\varphi(2^{-n}) \leqslant C\varphi(2^{-\beta n}), \quad n \in I.$$

Proof. Note that, by Remark 2.3(e), $\log^{1/2}(2/t) \in X_0$. We only have to consider the case when I is infinite. In view of the hypothesis and arguing as in the proof of Proposition 3.3, we get

$$\left\| \log^{1/2}\left(\frac{2}{t}\right) \chi_{[0,2^{-n}]} \right\|_X \leqslant C_1 \left\| \log^{1/2}\left(\frac{2}{2^n t}\right) \chi_{[0,2^{-n}]} \right\|_X, \quad n \in I.$$

Since $\Lambda(\varphi) \subset X \subset M(\varphi)$ [11, Theorems 2.5.5, 2.5.7], then

$$\left\| \log^{1/2}\left(\frac{2}{t}\right) \chi_{[0,2^{-n}]} \right\|_{M(\varphi)} \leqslant C_1 \left\| \log^{1/2}\left(\frac{2}{2^n t}\right) \chi_{[0,2^{-n}]} \right\|_{\Lambda(\varphi)}, \quad n \in I. \tag{4.1}$$

The right-hand side of (4.1) is equal to

$$\sum_{k=n}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \log^{1/2}\left(\frac{2}{2^n t}\right) d\varphi(t) \leqslant 2 \sum_{k=n}^{\infty} \log^{1/2}\left(\frac{2}{2^{n-k}}\right) \left(\varphi(2^{-k}) - \varphi(2^{-k-1})\right).$$

For $j > n$, by applying the Abel transformation, we have

$$\begin{aligned}
&\sum_{k=n}^{j} \log^{1/2}\left(\frac{2}{2^{n-k}}\right)\left(\varphi\left(2^{-k}\right) - \varphi\left(2^{-k-1}\right)\right) \\
&\quad = \sum_{k=n}^{j-1} \varphi\left(2^{-k-1}\right)\left(\sqrt{k+2-n} - \sqrt{k+1-n}\,\right) + \varphi\left(2^{-n}\right) - \varphi\left(2^{-j-1}\right)\sqrt{j+1-n} \\
&\quad \leqslant \sum_{k=n+1}^{j} \frac{\varphi(2^{-k})}{\sqrt{k-n}} + \varphi\left(2^{-n}\right).
\end{aligned}$$

Therefore,

$$\left\| \log^{1/2}\left(\frac{2}{2^n t}\right)\chi_{[0,2^{-n}]}\right\|_{\Lambda(\varphi)} \leqslant 2C_1 \cdot \left(\sum_{k=n+1}^{\infty} \frac{\varphi(2^{-k})}{\sqrt{k-n}} + \varphi\left(2^{-n}\right)\right).$$

On the other hand,

$$\left\| \log^{1/2}\left(\frac{2}{t}\right)\chi_{[0,2^{-n}]}\right\|_{M(\varphi)} \geqslant \sqrt{n}\varphi\left(2^{-n}\right).$$

Thus (4.1) implies that

$$\sqrt{n}\varphi\left(2^{-n}\right) \leqslant 2C_1\left(\sum_{k=n+1}^{\infty} \frac{\varphi(2^{-k})}{\sqrt{k-n}} + \varphi\left(2^{-n}\right)\right).$$

Hence, if $n \in I$ is large enough, we have

$$\sqrt{n}\varphi\left(2^{-n}\right) \leqslant 3C_1 \sum_{k=n+1}^{\infty} \frac{\varphi(2^{-k})}{\sqrt{k-n}}. \tag{4.2}$$

Let $\varepsilon > 0$ to be chosen later. Then

$$\begin{aligned}
\sum_{k=n+1}^{[(1+\varepsilon)n]} \frac{\varphi(2^{-k})}{\sqrt{k-n}} &\leqslant \varphi\left(2^{-n}\right) \sum_{k=1}^{[(1+\varepsilon)n]-n} \frac{1}{\sqrt{k}} \\
&\leqslant 2\varphi\left(2^{-n}\right)\sqrt{\left[(1+\varepsilon)n\right] - n} \\
&\leqslant 2\sqrt{\varepsilon}\varphi\left(2^{-n}\right)\sqrt{n}.
\end{aligned} \tag{4.3}$$

Since $\gamma_{\bar{\varphi}} > 0$, by Proposition 2.2(4), we have

$$\sum_{k=2n}^{\infty} \frac{\varphi(2^{-k})}{\sqrt{k-n}} \leqslant \sqrt{2} \sum_{k=2n}^{\infty} \frac{\varphi(2^{-k})}{\sqrt{k}} \leqslant 2C_0\varphi\left(2^{-2n}\right)\sqrt{n}. \tag{4.4}$$

We choose ε so that $0 < \varepsilon \leqslant (12C')^{-2}$. Then inequalities (4.2)–(4.4) imply that

$$\sqrt{n}\varphi\big(2^{-n}\big) \leqslant 6C_1\left(\sum_{k=[(1+\varepsilon)n]+1}^{2n-1} \frac{\varphi(2^{-k})}{\sqrt{k-n}} + 2C_0\varphi\big(2^{-2n}\big)\sqrt{n}\right).$$

Since

$$\sum_{k=[(1+\varepsilon)n]+1}^{2n-1} \frac{\varphi(2^{-k})}{\sqrt{k-n}} \leqslant \varphi\big(2^{-[(1+\varepsilon)n]}\big)\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \leqslant 2\varphi\big(2^{-[(1+\varepsilon)n]}\big)\sqrt{n},$$

we then get

$$\sqrt{n}\varphi\big(2^{-n}\big) \leqslant 12C_1\max(1, C_0)\sqrt{n}\varphi\big(2^{-[(1+\varepsilon)n]}\big)$$

or

$$\varphi\big(2^{-n}\big) \leqslant C\varphi\big(2^{-[(1+\varepsilon)n]}\big).$$

Choosing $\beta \in (1, 1+\varepsilon)$, we have that $[(1+\varepsilon)n] > \beta n$ for large enough $n \in \mathbb{N}$. Therefore, inequality (2.5) holds for large enough $n \in I$ for such β. Changing C, if necessary, we obtain (2.5) for all $n \in I$. $\square$

We now assume certain *head-estimates*. This requires the intervals $[0, 2^{-n}]$ to be replaced with the sets U_n defined previously (see Example 5.1 below).

Proposition 4.2. *Suppose that X is an r.i. space on* $[0,1]$ *such that* $\log^{1/2}(2/t) \in X_0$. *Let $I \subset \mathbb{N}$ be such that, for every $n \in I$,*

$$\|\chi_{U_n}\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant A\sup\left\{\left\|\chi_{U_n}\cdot\sum_{j=1}^{2n} c_i r_i\right\|_X : \left\|\sum_{j=1}^{2n} c_i r_i\right\|_X \leqslant 1\right\},$$

where $A > 0$ does not depend on n. Then, there exist $\beta > 1$ and $C > 0$ such that, for all $n \in I$, we have (2.5)*, that is,*

$$\varphi\big(2^{-n}\big) \leqslant C\varphi\big(2^{-\beta n}\big), \quad n \in I.$$

Proof. Since $\log^{1/2}(2/t) \in X_0$, we have $\mathcal{R}(X) \approx \ell^2$. Thus

$$\sup\left\{\left\|\chi_{U_n}\cdot\sum_{j=1}^{2n} c_i r_i\right\|_X : \left\|\sum_{j=1}^{2n} c_i r_i\right\|_X \leqslant 1\right\} \asymp \sup\left\{\left\|\chi_{U_n}\cdot\sum_{j=1}^{2n} c_i r_i\right\|_X : \big\|(c_j)\big\|_2 \leqslant 1\right\}.$$

For $c = (c_j)_{j=1}^{2n}$ and $\theta_{i,j}$ the value of r_j on Δ_{2n}^i, we have

$$\chi_{U_n}\cdot\sum_{j=1}^{2n} c_j r_j = \sum_{i\in\Omega_n}\left(\sum_{j=1}^{2n} c_j\theta_{i,j}\right)\cdot\chi_{\Delta_n^i} = \sum_{i\in\Omega_n} b_i\chi_{\Delta_n^i}, \tag{4.5}$$

where $b_i := (\mathcal{A}(\Omega_n)c)_i$ for $i \in \Omega_n$. Then, if $\|(c_j)\|_2 \leqslant 1$,

$$|b_i| = \left|\sum_{j=1}^{2n} c_j \theta_{i,j}\right| \leqslant \big\|(c_j)\big\|_2 \sqrt{2n} \leqslant \sqrt{2n}, \quad i \in \Omega_n. \tag{4.6}$$

Moreover, the choice of Ω_n implies that

$$b_i = \sum_{j=1}^{2n} c_j \theta_{i,j} = \sum_{j=1}^{n} (c_j + c_{j+n}) \varepsilon_j^i, \quad i \in \Omega_n,$$

for some rearrangement of signs $(\varepsilon_j^i)_{j=1}^n$. Note that, since $\|(c_j)\|_2 \leqslant 1$,

$$\left(\sum_{j=1}^{n} (c_j + c_{j+n})^2\right)^{1/2} \leqslant \sqrt{2}. \tag{4.7}$$

Given $\varepsilon > 0$, to be chosen later, consider the set

$$B_n := \big\{k \in \Omega_n\colon\ |b_k| \geqslant \varepsilon\sqrt{n}\,\big\}.$$

We estimate the size of B_n. For this, we use the exponential estimate of Rademacher sums, [10, Theorem II.2.5], and (4.7):

$$\begin{aligned}
\operatorname{card} B_n &= \operatorname{card}\left\{\varepsilon_k = \pm 1\colon\ \left|\sum_{k=1}^{n} (c_k + c_{k+n})\varepsilon_k\right| \geqslant \varepsilon\sqrt{n}\right\} \\
&= 2^n \lambda\left\{t \in [0,1]\colon\ \left|\sum_{k=1}^{n} (c_k + c_{k+n}) r_k(t)\right| \geqslant \varepsilon\sqrt{n}\right\} \\
&\leqslant 2^n \lambda\left\{t \in [0,1]\colon\ \left|\sum_{k=1}^{n} (c_k + c_{k+n}) r_k(t)\right| \geqslant \frac{\varepsilon\sqrt{n}}{\sqrt{2}}\left(\sum_{j=1}^{n} (c_j + c_{j+n})^2\right)^{1/2}\right\} \\
&\leqslant 2 \cdot 2^n \cdot e^{-\varepsilon^2 n/8},
\end{aligned}$$

whence

$$\operatorname{card} B_n \leqslant 2 \cdot 2^{n(1-\alpha\varepsilon^2)}, \quad \text{for } \alpha = (\log_2 e)/8.$$

Using this estimate, equality (4.5), and (4.6), we have, for the fundamental function φ of X,

$$\begin{aligned}
\left\|\chi_{U_n} \cdot \sum_{j=1}^{2n} c_j r_j\right\|_X &\leqslant \left\|\sum_{i \in B_n} b_i \chi_{\Delta_n^i}\right\|_X + \left\|\sum_{i \in \Omega_n \setminus B_n} b_i \chi_{\Delta_n^i}\right\|_X \\
&\leqslant \sqrt{2n} \cdot \|\chi_{[0, \operatorname{card} B_n \cdot 2^{-2n}]}\|_X + \varepsilon\sqrt{n}\|\chi_{U_n}\|_X \\
&\leqslant \sqrt{2n} \cdot \varphi\big(2 \cdot 2^{-n(1+\alpha\varepsilon^2)}\big) + \varepsilon\sqrt{n}\varphi\big(2^{-n}\big).
\end{aligned}$$

Consequently,

$$\sup\left\{\left\|\chi_{U_n}\cdot\sum_{j=1}^{2n}c_ir_i\right\|_X:\ \left\|\sum_{j=1}^{2n}c_ir_i\right\|_X\leqslant 1\right\}\leqslant C_1\sqrt{n}\left(\varphi\left(2\cdot 2^{-n(1+\alpha\varepsilon^2)}\right)+\varepsilon\varphi\left(2^{-n}\right)\right).$$

On the other hand, since $\log^{1/2}(2/t)\in X_0$, then by [2, Theorem 2.8]

$$\|\chi_{U_n}\|_{\mathrm{Sym}(\mathcal{R},X)}\asymp\left\|\chi_{[0,2^{-n}]}\log^{1/2}(2/t)\right\|_X\geqslant\sqrt{n}\varphi\left(2^{-n}\right),\quad n\in\mathbb{N}.$$

From the last two inequalities and the hypothesis it follows that, for every $n\in I$,

$$\varphi\left(2^{-n}\right)\leqslant C_2\left(\varphi\left(2\cdot 2^{-n(1+\alpha\varepsilon^2)}\right)+\varepsilon\varphi\left(2^{-n}\right)\right).$$

For $\varepsilon>0$ small enough, we get

$$\varphi\left(2^{-n}\right)\leqslant C_3\varphi\left(2^{-\beta n}\right),$$

where $C_3>0$ does not depend on $n\in I$ and $\beta=1+\alpha\varepsilon^2>1$. $\square$

Combining Propositions 4.1 and 4.2 we obtain the following result.

Theorem 4.3. *Let X be an r.i. space on $[0,1]$ such that $\log^{1/2}(2/t)\in X_0$. Suppose that there exists $A>0$ such that, for every $n\in\mathbb{N}$,*

$$\|\chi_{U_n}\|_{\mathrm{Sym}(\mathcal{R},X)}\leqslant A\|\chi_{U_n}\|_{\Lambda(\mathcal{R},X)}. \tag{4.8}$$

Then X satisfies the discrete log-condition (2.6)*, that is, there exists $C>0$ such that for all $n\in\mathbb{N}$*

$$\left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(0,2^{-n}]}\right\|_X\leqslant C\left\|\log^{1/2}\left(\frac{2}{2^nt}\right)\chi_{(0,2^{-n}]}\right\|_X.$$

If, in addition, $\gamma_{\bar\varphi}>0$, where φ is the fundamental function of X, then $\varphi\in\Delta^2$.

Proof. Denote by I_1 the set of all $n\in\mathbb{N}$ such that

$$\|\chi_{U_n}\|_{\mathrm{Sym}(\mathcal{R},X)}\leqslant 2A\sup\left\{\left\|\chi_{U_n}\cdot\sum_{i=2n+1}^{\infty}c_ir_i\right\|_X:\ \left\|\sum_{i=2n+1}^{\infty}c_ir_i\right\|_X\leqslant 1\right\}. \tag{4.9}$$

The definition of $\Lambda(\mathcal{R},X)$ and the properties of Rademacher functions yield

$$\begin{aligned}\|\chi_{U_n}\|_{\Lambda(\mathcal{R},X)}\leqslant{}&\sup\left\{\left\|\chi_{U_n}\cdot\sum_{i=1}^{2n}c_ir_i\right\|_X:\ \left\|\sum_{i=1}^{2n}c_ir_i\right\|_X\leqslant 1\right\}\\&+\sup\left\{\left\|\chi_{U_n}\cdot\sum_{i=2n+1}^{\infty}c_ir_i\right\|_X:\ \left\|\sum_{i=2n+1}^{\infty}c_ir_i\right\|_X\leqslant 1\right\}.\end{aligned}$$

Set $I_2 := \mathbb{N} \setminus I_1$. By (4.8) and (4.9) we have, for every $n \in I_2$,

$$\|\chi_{U_n}\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant 2A \sup\left\{\left\|\chi_{U_n} \cdot \sum_{i=1}^{2n} c_i r_i\right\|_X : \left\|\sum_{i=1}^{2n} c_i r_i\right\|_X \leqslant 1\right\}. \tag{4.10}$$

It is clear that $\|\chi_{U_n}\|_{\mathrm{Sym}(\mathcal{R},X)} = \|\chi_{[0,2^{-n}]}\|_{\mathrm{Sym}(\mathcal{R},X)}$ and

$$\begin{aligned} &\sup\left\{\left\|\chi_{U_n} \cdot \sum_{i=2n+1}^{\infty} c_i r_i\right\|_X : \left\|\sum_{i=2n+1}^{\infty} c_i r_i\right\|_X \leqslant 1\right\} \\ &\quad = \sup\left\{\left\|\chi_{[0,2^{-n}]} \cdot \sum_{i=n+1}^{\infty} c_i r_i\right\|_X : \left\|\sum_{i=n+1}^{\infty} c_i r_i\right\|_X \leqslant 1\right\}, \end{aligned}$$

since the functions $\chi_{U_n} \cdot \sum_{i=2n+1}^{\infty} c_i r_i$ and $\chi_{[0,2^{-n}]} \cdot \sum_{i=n+1}^{\infty} c_i r_i$ are equimeasurable. Therefore, (4.9) is equivalent to

$$\|\chi_{[0,2^{-n}]}\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant 2A \sup\left\{\left\|\chi_{[0,2^{-n}]} \cdot \sum_{i=n+1}^{\infty} c_i r_i\right\|_X : \left\|\sum_{i=n+1}^{\infty} c_i r_i\right\|_X \leqslant 1\right\}, \tag{4.11}$$

that is (as in the proof of Proposition 3.3), to the inequality

$$\left\|\log^{1/2}\left(\frac{2}{t}\right)\chi_{(0,2^{-n}]}\right\|_X \leqslant C' \left\|\log^{1/2}\left(\frac{2}{2^n t}\right)\chi_{(0,2^{-n}]}\right\|_X, \quad n \in I_1.$$

By Proposition 4.2, inequality (4.10) implies that there exist $\beta_1 > 1$ and $C_1 > 0$ such that inequality (2.5) holds for $n \in I_2$. This, together with Proposition 2.2(3) and Remark 2.3(d), give (2.6) for $n \in I_2$. Consequently, (2.6) holds for all $n \in \mathbb{N}$.

If $\gamma_{\tilde{\varphi}} > 0$ then, by (4.11) and Proposition 4.1, we have (2.5) for some $\beta_2 > 1$ and $C_2 > 0$ and all $n \in I_1$. Combining this with Proposition 4.2 and (4.10) we obtain (2.5) for all $n \in \mathbb{N}$. From Remark 2.3(b), we conclude that $\varphi \in \Delta^2$. □

Theorem 4.3, Corollary 3.5, Proposition 4.1, and Remark 3.7 yield the following result.

Theorem 4.4. *Let X be an interpolation space for the couple $(\Lambda(\varphi), M(\varphi))$, where the function φ satisfies the condition: $\gamma_{\tilde{\varphi}} > 0$. The following conditions are equivalent:*

(i) $\Lambda(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X)$.
(ii) *There exists $A > 0$ such that*

$$\|\chi_{U_n}\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant A \|\chi_{U_n}\|_{\Lambda(\mathcal{R},X)}, \quad n \in \mathbb{N}.$$

(iii) $\varphi \in \Delta^2$.
(iv) *There exists $A > 0$ such that for all $n \in \mathbb{N}$ and for every $f \in D_n$ we have*

$$\|f\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant A \sup\left\{\left\|f \cdot \sum_{k=n+1}^{\infty} a_k r_k\right\|_X : \left\|\sum_{k=n+1}^{\infty} a_k r_k\right\|_X \leqslant 1\right\}.$$

(v) *There exists* $A > 0$ *such that for every* $n \in \mathbb{N}$

$$\|\chi_{[0,2^{-n}]}\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant A \sup\left\{ \left\| \chi_{[0,2^{-n}]} \cdot \sum_{k=n+1}^{\infty} a_k r_k \right\|_X : \left\| \sum_{k=n+1}^{\infty} a_k r_k \right\|_X \leqslant 1 \right\}.$$

The previous result can be improved for Marcinkiewicz spaces.

Theorem 4.5. *Suppose that* X *is a Marcinkiewicz space* $M(\varphi)$ *with* $\log^{1/2}(2/t) \in X_0$ *and* $\delta_\varphi < 1$. *The following conditions are equivalent*:

(i) $\Lambda(\mathcal{R}, X) = \mathrm{Sym}(\mathcal{R}, X)$.
(ii) X *satisfies the discrete log-condition* (2.6)*, that is, there exists* $C > 0$ *such that*

$$\left\| \log^{1/2}\left(\frac{2}{t}\right) \chi_{(0,2^{-n}]} \right\|_X \leqslant C \left\| \log^{1/2}\left(\frac{2}{2^n t}\right) \chi_{(0,2^{-n}]} \right\|_X, \quad n \in \mathbb{N}.$$

(iii) *There exists* $C > 0$ *such that*

$$\sqrt{n} \cdot \varphi\left(2^{-n}\right) \leqslant C \sup_{k \geqslant n} \sqrt{k-n} \cdot \varphi\left(2^{-k}\right), \quad n \in \mathbb{N}.$$

Moreover, if φ *satisfies the* $\sqrt{2}$*-condition* (2.3)*, then conditions* (i)–(iii) *are equivalent to*:

(iv) $\varphi \in \Delta^2$.

Proof. We start proving the equivalence between (ii) and (iii). Since

$$\int_0^u \log^{1/2}\left(\frac{2}{t}\right) dt \asymp u \log^{1/2}\left(\frac{2}{u}\right), \quad 0 < u \leqslant 1,$$

we have, by the quasi-concavity of $\varphi(t)$,

$$\begin{aligned} \left\| \log^{1/2}\left(\frac{2}{2^n t}\right) \chi_{[0,2^{-n}]} \right\|_X &= \sup_{0<t\leqslant 2^{-n}} \frac{\varphi(t)}{t} \int_0^t \log^{1/2}\left(\frac{2}{2^n s}\right) ds \\ &= \sup_{0<t\leqslant 2^{-n}} \frac{\varphi(t)}{2^n t} \int_0^{2^n t} \log^{1/2}\left(\frac{2}{s}\right) ds \\ &\asymp \sup_{0<t\leqslant 2^{-n}} \varphi(t) \log^{1/2}\left(\frac{2}{2^n t}\right) \\ &\asymp \sup_{k\geqslant n} \sqrt{k-n} \cdot \varphi\left(2^{-k}\right), \end{aligned}$$

for every $n \in \mathbb{N}$. Similarly,

$$\left\| \log^{1/2}\left(\frac{2}{t}\right)\chi_{[0,2^{-n}]} \right\|_X \asymp \sup_{k \geqslant n} \sqrt{k} \cdot \varphi\left(2^{-k}\right).$$

Therefore, inequality (2.6) is equivalent to

$$\sup_{k \geqslant n} \sqrt{k} \cdot \varphi\left(2^{-k}\right) \leqslant C \sup_{k \geqslant n} \sqrt{k-n} \cdot \varphi\left(2^{-k}\right),$$

which, in turn, is equivalent to (iii).

The implication (i) $\Rightarrow$ (ii) is a straightforward consequence of Theorem 4.3. Let us prove the opposite statement. Assume that (ii) is fulfilled. From Proposition 3.3, it suffices to prove (3.2), that is,

$$\|x\|_X \leqslant C\|y\|_X,$$

where

$$x(t) := \sum_{k=1}^{2^n} a_k \chi_{\Delta_n^k}(t) \log^{1/2}\left(\frac{2}{t}\right), \qquad y(t) := \sum_{k=1}^{2^n} a_k \chi_{\Delta_n^k}(t) \log^{1/2}\left(\frac{2}{2^n t + 1 - k}\right),$$

the constant $C > 0$ does not depend on $n \in \mathbb{N}$, and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{2^n} \geqslant 0$.

Since $x(t)$ is a decreasing function and $\delta_\varphi < 1$, by (1.1) we have

$$\|x\|_X \asymp \sup_{0<t\leqslant 1} x(t)\varphi(t).$$

Hence, using the quasi-concavity of $\varphi(t)$, we have

$$\|x\|_X \asymp \sup_{1\leqslant k\leqslant 2^n} a_k \log^{1/2}\left(\frac{2}{k2^{-n}}\right)\varphi\left(\frac{k}{2^n}\right) \asymp \sup_{0\leqslant j\leqslant n} a_{2^j}\sqrt{n-j+1}\,\varphi\left(2^{j-n}\right). \tag{4.12}$$

Since

$$a_{2^j}\sqrt{n-j+1}\,\varphi\left(2^{j-n}\right) \leqslant \left\| a_{2^j} \log^{1/2}\left(\frac{2}{t}\right)\chi_{[0,2^{j-n}]} \right\|_X,$$

by (ii), we have

$$a_{2^j}\sqrt{n-j+1}\,\varphi\left(2^{j-n}\right) \leqslant C \left\| a_{2^j} \log^{1/2}\left(\frac{2}{2^{n-j}t}\right)\chi_{[0,2^{j-n}]} \right\|_X. \tag{4.13}$$

Note that, for σ_a the dilation operator by $a > 0$,

$$\log^{1/2}\left(\frac{2}{2^{n-j}t}\right)\chi_{[0,2^{j-n}]}(t) = \sigma_{2^j}\left(\log^{1/2}\left(\frac{2}{2^n t}\right)\chi_{[0,2^{-n}]}(t)\right).$$

This implies that the functions

$$\log^{1/2}\left(\frac{2}{2^{n-j}t}\right)\chi_{[0,2^{j-n}]}(t) \quad \text{and} \quad \sum_{i=1}^{2^j}\chi_{\Delta_n^i}(t)\log^{1/2}\left(\frac{2}{2^n t+1-i}\right)$$

have the same distribution function. Therefore, since $a_1 \geqslant \cdots \geqslant a_{2^j} \geqslant 0$, we get

$$\left\| a_{2^j}\log^{1/2}\left(\frac{2}{2^{n-j}t}\right)\chi_{[0,2^{j-n}]}\right\|_X \leqslant \left\|\sum_{i=1}^{2^j} a_i\chi_{\Delta_n^i}(t)\log^{1/2}\left(\frac{2}{2^n t+1-i}\right)\right\|_X \leqslant \|y\|_X.$$

The result follows from the previous inequality together with (4.12) and (4.13).

We prove that (iii), together with the $\sqrt{2}$-condition, implies (iv). By Remark 2.3(b), we only have to show that there exist $\beta > 1$ and $C > 0$ such that (2.5) holds, that is,

$$\varphi(2^{-n}) \leqslant C\varphi(2^{-\beta n}), \quad n \in \mathbb{N}.$$

By assumption, for every $n \in \mathbb{N}$ there exists $k_n \geqslant n$ such that

$$\sqrt{n}\varphi(2^{-n}) \leqslant C_1\sqrt{k_n - n}\varphi(2^{-k_n}). \tag{4.14}$$

Let us show that we can assume, if n is large enough, that k_n can be chosen satisfying

$$\frac{C_1^2+1}{C_1^2}n \leqslant k_n \leqslant 2n. \tag{4.15}$$

The first inequality follows directly from (4.14) since, as $k_n \geqslant n$, we have

$$\sqrt{n} \leqslant C_1\sqrt{k_n - n}.$$

For the second inequality in (4.15), suppose that $k_n \geqslant 2n$. Then, for some $m \in \mathbb{N}$, we have $2^m n \leqslant k_n < 2^{m+1}n$. This, together with the $\sqrt{2}$-condition (2.3), implies that, for large enough n, we have

$$\begin{aligned}\sqrt{k_n - n}\varphi(2^{-k_n}) &\leqslant 2^{\frac{m+1}{2}}\sqrt{n}\varphi(2^{-2^m n}) \\ &\leqslant c_\varphi^{-1}(\sqrt{2}c_\varphi)^m\sqrt{2n}\varphi(2^{-2n}) \\ &\leqslant c_\varphi^{-1}\sqrt{2n}\varphi(2^{-2n}).\end{aligned}$$

But, then (4.14) holds also for $k_n = 2n$ (with the constant $c_\varphi^{-1}\sqrt{2}C_1$ instead of C_1). Thus, (4.15) holds. Combining (4.14) and (4.15), we conclude that

$$\varphi(2^{-n}) \leqslant C\varphi(2^{-\beta n}),$$

where $n \in \mathbb{N}$ is large enough, $\beta := (C_1^2 + 1)/C_1^2 > 1$, and the constant $C > 0$ does not depend on n. By changing, if necessary, the constant C, we have the above inequality for all $n \in \mathbb{N}$.

Since, by Proposition 2.2(3), (iv) implies (ii) the proof is completed. □

Remark 4.6. The proof of Theorem 4.5 and [3, Remark 5] show that the following *tail-assertion* holds: if X is a Marcinkiewicz space $M(\varphi)$ such that $\delta_\varphi < 1$, then $\Lambda(\mathcal{R}, X)$ is r.i. if and only if there exists $A > 0$ such that for every $f \in D_n$ we have

$$\|f\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant A \sup\left\{\left\|f \cdot \sum_{i=n+1}^{\infty} c_i r_i\right\|_X : \left\|\sum_{i=n+1}^{\infty} c_i r_i\right\|_X \leqslant 1\right\}.$$

5. Examples

We end with two examples which highlight certain features of the previous results.

Example 5.1. The sets U_n appearing in Proposition 4.2 are, in some sense, necessary since they cannot be replaced by the simpler sets, of equal measure, $[0, 2^{-n}]$. This is so even in the case when the r.i. space X is "very close" to L_∞.

To see this, define

$$\psi(t) := 2^{1-\log^{1/2}(2/t)}, \quad 0 < t \leqslant 1.$$

It can be easily checked that, for sufficiently small $t > 0$, $\psi'(t) > 0$ and $\psi''(t) < 0$. Therefore, there exists a nonnegative increasing concave function $\varphi(t)$ on $[0, 1]$ such that $\varphi(t) = \psi(t)$ for small enough $t > 0$.

Let X be the Marcinkiewicz space $M(\varphi)$. Since the upper dilation index of φ satisfies $\delta_\varphi = 0$, we have $X \subset L_p$ for all $p < \infty$. Since $\lim_{t\to 0^+} \psi(t)\log^{1/2}(2/t) = 0$, we have $\log^{1/2}(2/t) \in X_0$. Therefore,

$$\begin{aligned}
&\sup\left\{\left\|\chi_{[0,2^{-n}]} \cdot \sum_{i=1}^{n} c_i r_i\right\|_X : \left\|\sum_{i=1}^{n} c_i r_i\right\|_X \leqslant 1\right\} \\
&\quad\asymp \sup\left\{\left\|\chi_{[0,2^{-n}]} \cdot \sum_{i=1}^{n} c_i r_i\right\|_X : \left\|(a_i)\right\|_2 \leqslant 1\right\} \\
&\quad\geqslant \left\{\left\|\chi_{[0,2^{-n}]} \cdot \sum_{i=1}^{n} \frac{1}{\sqrt{n}} r_i\right\|_X\right\} \asymp \sqrt{n}2^{-\sqrt{n}}.
\end{aligned}$$

From [2, Corollary 2.11] and [11, Theorem 2.5.3] it follows that

$$\|\chi_{[0,2^{-n}]}\|_{\mathrm{Sym}(\mathcal{R},X)} \asymp \sup_{0<t<2^{-n}} \log^{1/2}(2/t) \cdot 2^{1-\log^{1/2}(2/t)} \leqslant C\sqrt{n}2^{-\sqrt{n}}.$$

Thus, for all $n \in \mathbb{N}$, we have

$$\|\chi_{[0,2^{-n}]}\|_{\mathrm{Sym}(\mathcal{R},X)} \leqslant C \sup\left\{\left\|\chi_{[0,2^{-n}]} \cdot \sum_{i=1}^{n} c_i r_i\right\|_X : \left\|\sum_{i=1}^{n} c_i r_i\right\|_X \leqslant 1\right\}.$$

However, since

$$\frac{\varphi(2^{-2n})}{\varphi(2^{-n})} \asymp 2^{-\sqrt{n}} \to 0 \quad \text{if } n \to \infty,$$

it follows that $\varphi \notin \Delta^2$.

This example also allows to see that the discrete log-condition in Theorem 4.5(ii) cannot be replaced with the weaker condition

$$\|\chi_{[0,2^{-n})}\|_{\mathrm{Sym}(M(\psi))} \asymp \|\chi_{[0,2^{-n})}\|_{\Lambda(\mathcal{R},M(\psi))}, \quad n \in \mathbb{N}.$$

To see this, note that $\psi(t)$ satisfies the $\sqrt{2}$-condition (due to Proposition 2.2(5)) and $\delta_\psi = 0$. However, since $\psi \notin \Delta^2$, we conclude that $\Lambda(\mathcal{R}, M(\psi)) \neq \mathrm{Sym}(\mathcal{R}, M(\psi))$.

Example 5.2. The implication (i) $\Rightarrow$ (iv) in Theorem 4.5 is not valid in general, that is, the $\sqrt{2}$-condition is essential.

To see this, let $n_k = 2^{2^k}$, $t_k = 2^{-n_k}$, and $\gamma_k = 2^{k-1-2^k}$ for $k \in \mathbb{N}$. Note that $n_{k+1} = n_k^2$ and $n_k\gamma_k = 2^{k-1}$. Define a continuous function $\varphi(t)$ on the interval $(0, 2^{-4}]$ as follows: $\varphi(t) = \frac{1}{k}t^{\gamma_k}$ if $t_k^2 < t \leqslant t_k$ and $\varphi(t)$ is linear on the interval $(t_{k+1}, t_k^2]$, $k \in \mathbb{N}$. Direct calculation shows that

$$\varphi(t_k) = \frac{1}{k} n_k^{-1/2} \quad \text{and} \quad \varphi\big(t_k^2\big) = \frac{1}{k} n_{k+1}^{-1/2}, \quad k \in \mathbb{N}. \tag{5.1}$$

The function $\varphi(t)$ is quasi-concave and $\lim_{t\to 0^+} \varphi(t) = 0$.

Let us check that $X = M(\varphi)$ and φ satisfy the conditions of Theorem 4.5. Condition $\log^{1/2}(2/t) \in (M(\varphi))_0$ is equivalent to

$$\lim_{i\to\infty} \sqrt{i+1}\varphi\big(2^{-i}\big) = 0. \tag{5.2}$$

To see this, let $n_k \leqslant i < 2n_k$, then, by (5.1),

$$\sqrt{i+1}\varphi\big(2^{-i}\big) \leqslant \sqrt{2n_k+1}\varphi(t_k) = \sqrt{2n_k+1}\frac{1}{k}n_k^{-1/2} < \frac{2}{k}.$$

If $2n_k \leqslant i < n_{k+1}$, again by (5.1), we have

$$\sqrt{i+1}\varphi\big(2^{-i}\big) \leqslant \sqrt{n_{k+1}+1}\varphi\big(t_k^2\big) = \sqrt{n_{k+1}+1}\frac{1}{k}n_{k+1}^{-1/2} < \frac{2}{k}.$$

This gives (5.2).

In order to calculate the upper dilation index δ_φ, we consider the dilation function $\mathcal{M}_\varphi$ of φ. Since

$$\mathcal{M}_\varphi\big(2^{n_k}\big) = \sup_{n=0,1,\dots} \frac{\varphi(2^{-n})}{\varphi(2^{-n-n_k})} = 2^{n_k\gamma_k},$$

then $\delta_\varphi \leqslant \gamma_k \to 0$, whence $\delta_\varphi = 0$. Moreover, $\varphi \notin \Delta^2$ since, by (5.1),

$$\frac{\varphi(2^{-2n_k})}{\varphi(2^{-n_k})} = \frac{\varphi(t_k^2)}{\varphi(t_k)} = 2^{-2^{k-1}} \to 0 \quad \text{as } k \to \infty.$$

Lastly, we show that $\Lambda(\mathcal{R}, M(\varphi)) = \mathrm{Sym}(M(\varphi))$. In view of the proof of Theorem 4.5, it suffices to prove that there exists $C > 0$ such that

$$\sqrt{n} \cdot \varphi\bigl(2^{-n}\bigr) \leqslant C \sup_{k \geqslant n} \sqrt{k-n} \cdot \varphi\bigl(2^{-k}\bigr), \quad n \in \mathbb{N}.$$

For this, it suffices to find a constant $C > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$\varphi\bigl(2^{-n}\bigr) \leqslant C 2^{m/2} \varphi\bigl(2^{-2^m n}\bigr). \tag{5.3}$$

Suppose first that $n_k \leqslant n < 2n_k$. Set $m := [\log(n_{k+1}/n)]$. Since

$$\frac{n_{k+1}}{n} \geqslant 2^m \geqslant \frac{n_{k+1}}{2n} \geqslant \frac{n_{k+1}}{4n_k} = \frac{1}{4} n_k,$$

by using (5.1), we get

$$\begin{aligned}\varphi\bigl(2^{-2^m n}\bigr) \geqslant \varphi\bigl(2^{-n_{k+1}}\bigr) &= \frac{1}{k+1} n_{k+1}^{-1/2} = \frac{1}{k+1} n_k^{-1} \\ &\geqslant \frac{k}{k+1} n_k^{-1/2} \varphi\bigl(2^{-n_k}\bigr) \geqslant \frac{1}{4} 2^{-m/2} \varphi\bigl(2^{-n}\bigr).\end{aligned}$$

If $2n_k \leqslant n < n_{k+1}$, set $m := [\log(n_{k+2}/n)]$. Then,

$$\frac{n_{k+2}}{n} \geqslant 2^m \geqslant \frac{n_{k+2}}{2n} \geqslant \frac{n_{k+2}}{2n_{k+1}} = \frac{1}{2} n_{k+1}.$$

Therefore, again by (5.1),

$$\begin{aligned}\varphi\bigl(2^{-2^m n}\bigr) \geqslant \varphi\bigl(2^{-n_{k+2}}\bigr) &= \frac{1}{k+2} n_{k+2}^{-1/2} = \frac{1}{k+2} n_{k+1}^{-1} \\ &\geqslant \frac{k}{k+2} n_{k+1}^{-1/2} \varphi\bigl(2^{-2n_k}\bigr) \geqslant \frac{1}{3\sqrt{2}} 2^{-m/2} \varphi\bigl(2^{-n}\bigr).\end{aligned}$$

Thus (5.3) is proved and so, $\Lambda(\mathcal{R}, M(\varphi)) = \mathrm{Sym}(M(\varphi))$.

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