

Strictly singular inclusions of rearrangement invariant spaces and Rademacher spaces

by

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Abstract. If G is the closure of L_∞ in $\exp L_2$, it is proved that the inclusion between rearrangement invariant spaces $E \subset F$ is strictly singular if and only if it is disjointly strictly singular and $E \not\supset G$. For any Marcinkiewicz space $M(\varphi) \subset G$ such that $M(\varphi)$ is not an interpolation space between L_∞ and G it is proved that there exists another Marcinkiewicz space $M(\psi) \subsetneq M(\varphi)$ with the property that the $M(\psi)$ and $M(\varphi)$ norms are equivalent on the Rademacher subspace. Applications are given and a question of Milman answered.

1. Introduction. A linear operator between two Banach spaces E and F is called *strictly singular* (SS for short), or *Kato*, if it fails to be an isomorphism on any infinite-dimensional subspace (cf. [LT1, 2.c.2]). The class of all strictly singular operators is a well-known closed operator ideal with important applications. A weaker notion for Banach lattices, introduced in [HR], is the following: a bounded operator A from a Banach lattice E to a Banach space F is said to be *disjointly strictly singular* (DSS for short) if there is no disjoint sequence of non-null vectors $\{x_n\}_{n=1}^\infty$ in E such that the restriction of A to the subspace $[x_n]$ spanned by the vectors $\{x_n\}$ is an isomorphism. This is a useful tool in comparing structures of rearrangement invariant spaces (cf. [HR], [GHSS]).

This paper deals with the strict singularity of inclusions $E \subset F$ between rearrangement invariant (r.i.) function spaces E and F on the interval $[0, 1]$. That means that the norms of E and F are non-equivalent on any (closed) infinite-dimensional subspace of E .

The canonical inclusion $L_\infty \subset E$ is always strictly singular for any r.i. space $E \neq L_\infty$ ([N]), and the case of L_p -spaces is Grothendieck's classical result. Furthermore, this property characterizes the space L_∞ among all r.i.

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spaces ([GHSS]). Concerning the right extreme inclusion $E \subset L_1$, its strict singularity has been characterized in [HNS] by the condition that the r.i. space E does not contain the space G , the closure of L_∞ in the exponential Orlicz space $\exp L_2$. Recall that Rodin and Semenov [RS] (see also [LT2]) proved that the condition $E \supset G$ determines precisely the r.i. spaces E for which the Rademacher function system $\{r_k\}$ is equivalent to the canonical basis of ℓ_2 .

One of the aims of this article is to give a complete characterization of the strict singularity of inclusions between arbitrary r.i. spaces in terms of disjoint strict singularity. More precisely, it is proved in Section 3 (Theorem 2) that the inclusion $E \subset F$ is strictly singular if and only if it is disjointly strictly singular and the norms of these spaces are not equivalent on the Rademacher subspace $[r_n]$. This extends some previous results given in [HNS].

In [RS] the following result was proved for the class of r.i. spaces contained in G : Under some additional assumptions, the equivalence of the norms in two r.i. spaces E and F of this class on the Rademacher subspace, i.e.,

$$\left\| \sum c_k r_k \right\|_E \asymp \left\| \sum c_k r_k \right\|_F,$$

implies the coincidence of E and F up to equivalence of norms, i.e., $E = F$. More recently in [A] this result was obtained under a weaker assumption: the r.i. spaces E and F have to be interpolation spaces between L_∞ and G . It turns out that this interpolation assumption is actually a necessary condition for the above statement to hold. Theorem 9 in Section 4 shows that for any Marcinkiewicz space $M(\varphi) \subset G$ such that $M(\varphi)$ is not an interpolation space between L_∞ and G , there exists another Marcinkiewicz space $M(\psi) \subsetneq M(\varphi)$ with the property that the $M(\psi)$ -norm and the $M(\varphi)$ -norm are equivalent on the Rademacher subspace $[r_n]$. Also a criterion for the strict singularity of inclusions between Lorentz spaces $\Lambda(\varphi)$ and Marcinkiewicz spaces $M(\psi)$ is given (Theorem 11). In particular, for the class of all proper Lorentz spaces $\Lambda(\varphi)$ which do not contain G , the norms in $\Lambda(\varphi)$ and in the associated Marcinkiewicz space $M(\varphi)$ on the Rademacher subspace are *never* equivalent (contrasting with the uniform behavior in the complementary class).

The last part of the paper contains some applications. In particular, we answer in the negative a question of V. Milman [Mi], showing that the r.i. spaces $E = L \log^\lambda L$ and $F = L_1$ satisfy the following conditions: the inclusion $E \subset F$ is not strictly singular and any infinite-dimensional subspace of E on which the norms of E and F are equivalent is an *uncomplemented* subspace of E (Theorem 16). We also prove that any disjointly strictly singular inclusion between r.i. spaces is weakly compact.

Some results of this paper have been announced in [SH].

2. Notation and definitions. Recall that a Banach function space E of measurable functions on $[0, 1]$ is called *rearrangement invariant* (r.i. for short) or symmetric (cf. [LT2, 2.a.1], [KPS, 2.4.1]) if

- $|x(t)| \leq |y(t)|$ for all $t \in [0, 1]$ and $y \in E$ imply $x \in E$ and $\|x\|_E \leq \|y\|_E$,
- if x and y are equimeasurable and $y \in E$, then $x \in E$ and $\|x\|_E = \|y\|_E$.

As usual we assume that every r.i. space E is separable or isomorphic to the conjugate space of some separable space. If E is an r.i. space then $L_\infty \subset E \subset L_1$ and $\|x\|_{L_1} \leq \|x\|_E \leq \|x\|_{L_\infty}$ for each $x \in L_\infty$, assuming $\|\chi_{(0,1)}\|_E = 1$.

Recall some important classes of r.i. spaces. If M is a positive convex function on $[0, \infty)$ with $M(0) = 0$, then the *Orlicz space* L_M consists of all measurable functions on $[0, 1]$ for which

$$\|x\|_{L_M} = \inf \left\{ s > 0 : \int_0^1 M(|x(t)|/s) dt \leq 1 \right\}.$$

A remarkable example is the Orlicz space L_N generated by the function $N(u) = e^{u^2} - 1$. The space L_N is non-separable and we will denote by G the closure of L_∞ in L_N . Another two special Orlicz spaces that will be considered here are generated by the functions $M(u) = e^{u^\lambda} - 1$ and $M(u) = u \log^\lambda(1 + u)$, for $\lambda > 0$, and denoted by $\exp L_\lambda$ and $L \log^\lambda L$.

Let Ω be the set of all increasing concave functions on $[0, 1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. Each $\varphi \in \Omega$ generates the *Lorentz space* $\Lambda(\varphi)$ endowed with the norm

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t) d\varphi(t),$$

and the *Marcinkiewicz space* $M(\varphi)$ with

$$\|x\|_{M(\varphi)} = \sup_{0 < \tau \leq 1} \frac{\varphi(\tau)}{\tau} \int_0^\tau x^*(t) dt,$$

where $x^*(t)$ is the decreasing rearrangement of $|x(t)|$. For any $\varphi \in \Omega$ we have $\Lambda(\varphi) \subset M(\varphi)$ and $\|x\|_{M(\varphi)} \leq \|x\|_{\Lambda(\varphi)}$ for every $x \in \Lambda(\varphi)$. The spaces $\Lambda(\varphi)$ and $M(\varphi)$ coincide up to equivalence of norms if and only if $\varphi(+0) > 0$ or $\lim_{t \rightarrow 0} \varphi(t)/t < \infty$.

Recall that the *fundamental function* φ_E of a r.i. space E is defined by $\varphi_E(t) = \|\chi_{[0,t]}\|_E$ for $0 \leq t \leq 1$. The function φ_E is quasi-concave, i.e., $\varphi_E(t)$ and $t/\varphi_E(t)$ increase on $(0, 1]$. Up to equivalence of norms, φ_E is a concave function. In that case $\Lambda(\varphi_E) \subset E \subset M(\varphi_E)$ and $\|x\|_{M(\varphi_E)} \leq \|x\|_E \leq \|x\|_{\Lambda(\varphi_E)}$ for every $x \in \Lambda(\varphi_E)$. The fundamental function $\varphi_E(t)$ is continuous for $t \in (0, 1]$. The condition $\varphi_E(+0) > 0$ is necessary and sufficient for the

coincidence of the spaces E and L_∞ up to equivalence of norms. If two r.i. spaces E and F coincide as sets then (by the closed graph theorem) the norms $\|\cdot\|_E$ and $\|\cdot\|_F$ are equivalent, and we write $E = F$.

Let $r_k(t) = \text{sign}(\sin 2^k \pi t)$, $k \in \mathbb{N}$, be the Rademacher functions on $[0, 1]$. It was proved in [RS] (see also [LT2, Thm. 2.b.4]) that for an r.i. space E the Khinchin inequality

$$\left\| \sum_{k=1}^{\infty} c_k r_k \right\|_E \leq M \|\{c_k\}\|_{\ell_2}$$

is valid, for some constant $M > 0$, if and only if $E \supset G$. It follows immediately that for r.i. spaces E and F with $E \subset F$ the inclusion $E \subset F$ is not SS provided that $E \supset G$.

The proofs of some statements of this article will make use of interpolation methods. Therefore we recall some concepts and results in the r.i. setting.

Let (E, F) be a pair of r.i. spaces and $x \in E + F$. The Peetre's K -functional is defined as

$$K(t, x, E, F) = \inf\{\|u\|_E + t\|v\|_F : x = u + v\}$$

for every $t > 0$. Every Banach lattice Φ on $[0, \infty)$ such that $\min(1, t) \in \Phi$ generates the space $(E, F)_{\Phi}^K$ of the real interpolation method endowed with the norm

$$\|x\|_{(E, F)_{\Phi}^K} := \|K(\cdot, x, E, F)\|_{\Phi}.$$

The space $(E, F)_{\Phi}^K$ has the interpolation property with respect to the pair (E, F) , i.e., every linear operator A bounded in E and F is also bounded in $(E, F)_{\Phi}^K$ and $\|A\|_{(E, F)_{\Phi}^K} \leq \max(\|A\|_E, \|A\|_F)$. In the classical case of Φ being the lattice on $[0, \infty)$ with the norm

$$\|z\|_{\Phi} = \left(\int_0^{\infty} (t^{-\theta} |z(t)|)^p \frac{dt}{t} \right)^{1/p},$$

where $\theta \in (0, 1)$ and $p \in [1, \infty]$ (with the usual modification for $p = \infty$), the interpolation spaces $(E, F)_{\Phi}^K$ are denoted by $(E, F)_{\theta, p}$.

We will denote by $I(E, F)$ the set of all interpolation spaces with respect to the pair (E, F) . If, for any $x, y \in E + F$ with $K(t, x, E, F) \leq K(t, y, E, F)$ for every $t > 0$, there exists a linear operator A bounded in E and F and such that $x = Ay$, then the set $I(E, F)$ is *described by the real interpolation method*, in the sense that. In that case for each space $Q \in I(E, F)$ there exists a Banach lattice Φ such that $\|x\|_Q = \|K(\cdot, x, E, F)\|_{\Phi}$ (cf. [BK]).

If $f(x)$ and $g(x)$ are positive functions on some set A , we shall write $f \asymp g$ if there exists $C > 0$ such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$ for every $x \in A$.

We refer to the monographs [LT2] and [KPS] for the above results on r.i. spaces and to [BK] and [BL] for those on interpolation spaces.

3. Strict singularity via disjoint strict singularity. Given an r.i. space E on $[0, 1]$, we denote by E_0 the closure of L_∞ in E . The space E_0 is always separable, except for $E = L_\infty$.

PROPOSITION 1. *Let E and F be r.i. spaces with $E \subset F$. Then the inclusion $E \subset F$ is disjointly strictly singular if and only if the inclusion $E_0 \subset F$ is disjointly strictly singular.*

Proof. The “only if” part is evident. Assume that the inclusion $E \subset F$ is not DSS. Thus there exist a disjoint sequence $\{x_k\}_{k=1}^\infty$ in E and $M > 0$ such that

$$\left\| \sum_{k=1}^n c_k x_k \right\|_E \leq M \left\| \sum_{k=1}^n c_k x_k \right\|_F$$

for every $n \in \mathbb{N}$ and $c_k \in \mathbb{R}$. We consider separately the cases (i) $E \subset F_0$ and (ii) $E \not\subset F_0$.

In case (i) we have $x_k \in F_0$ for $k \in \mathbb{N}$. Clearly we can assume $\|x_k\|_F = 1$ for $k \in \mathbb{N}$. It is well-known that $\lim_{\text{meas } A \rightarrow 0} \|x \chi_A\|_F = 0$ for any $x \in F_0$ (cf. [KPS, 2.4.5]). Hence there exists a sequence $\{A_k\}_{k=1}^\infty$ of subsets of $[0, 1]$ with $A_k \subset \text{supp } x_k$ such that $y_k = x_k \chi_{A_k} \in L_\infty \subset E_0$ for $k \in \mathbb{N}$ and

$$\sum_{k=1}^\infty \|x_k - y_k\|_F < \frac{1}{2}.$$

Now, by a stability result [LT1, Prop. 1.a.9], the sequences $\{x_k\}$ and $\{y_k\}$ are equivalent in F and

$$\left\| \sum_{k=1}^n c_k x_k \right\|_F \leq 2 \left\| \sum_{k=1}^n c_k y_k \right\|_F$$

for every $n \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Since $|\sum_{k=1}^n c_k y_k| \leq |\sum_{k=1}^n c_k x_k|$, we have

$$\left\| \sum_{k=1}^n c_k y_k \right\|_E \leq \left\| \sum_{k=1}^n c_k x_k \right\|_E \leq M \left\| \sum_{k=1}^n c_k x_k \right\|_F \leq 2M \left\| \sum_{k=1}^n c_k y_k \right\|_F$$

for any $n \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Therefore the norms $\|\cdot\|_E$ and $\|\cdot\|_F$ are equivalent on the span of $\{y_k\}$ in E_0 and the inclusion $E_0 \subset F$ is not DSS.

(ii) Consider now the case $E \not\subset F_0$. Since $E_0 \subset F_0$ we have $E \setminus E_0 \not\subset F_0$ and $(E \setminus E_0) \cap (F \setminus F_0) \neq \emptyset$. Choose $z = z^* \in (E \setminus E_0) \cap (F \setminus F_0)$. Then

$$d_E(z, L_\infty) = a > 0, \quad d_F(z, L_\infty) = b > 0$$

where $d_E(z, L_\infty) = \inf\{\|z - u\|_E : u \in L_\infty\}$. Since we have $\|z \chi_{(0, \tau)}\|_E =$

$\lim_{\varepsilon \rightarrow 0} \|z\chi_{(\varepsilon, \tau)}\|_E$ and $\|z\chi_{(0, \tau)}\|_E \geq a$ for $0 < \tau \leq 1$, it follows that

$$\lim_{\varepsilon \rightarrow 0} \|z\chi_{(\varepsilon, \tau)}\|_E \geq a.$$

Similarly we have $\lim_{\varepsilon \rightarrow 0} \|z\chi_{(\varepsilon, \tau)}\|_F \geq b$ for $0 < \tau \leq 1$. Hence we can construct a sequence $\tau_k \downarrow 0$ such that

$$\|z\chi_{(\tau_{k+1}, \tau_k)}\|_E \geq a/2 \quad \text{and} \quad \|z\chi_{(\tau_{k+1}, \tau_k)}\|_F \geq b/2$$

for every natural k . Let $z_k := z\chi_{(\tau_{k+1}, \tau_k)}$ for $k \in \mathbb{N}$. Clearly, $z_k \in L_\infty \subset E_0 \subset F_0$ and

$$\frac{a}{2} \max_{k \in \mathbb{N}} |c_k| \leq \left\| \sum_{k=1}^{\infty} c_k z_k \right\|_E \leq \|z\|_E \max_{k \in \mathbb{N}} |c_k|$$

and

$$\frac{b}{2} \max_{k \in \mathbb{N}} |c_k| \leq \left\| \sum_{k=1}^{\infty} c_k z_k \right\|_F \leq \|z\|_F \max_{k \in \mathbb{N}} |c_k|,$$

for any sequence $\{c_k\} \in c_0$. Hence the sequence $\{z_k\}_{k=1}^{\infty}$ is equivalent in E_0 and in F_0 to the canonical basis of c_0 . Consequently, the inclusion $E_0 \subset F$ is not DSS. ■

Recall that G is the closure of L_∞ in $\exp L_2$. We can now prove the main result of this section.

THEOREM 2. *Let E and F be r.i. spaces with $E \subset F$. The inclusion $E \subset F$ is strictly singular if and only if it is disjointly strictly singular and $E \not\subset G$.*

Proof. The case of E separable has been proved in [HNS, Theorem 5], so we assume that E is non-separable. Suppose that the inclusion $E \subset F$ is not SS and $E \not\subset G$. We have to prove that the inclusion $E \subset F$ is not DSS. Let Q denote the (closed) infinite-dimensional subspace of E on which the norms $\|\cdot\|_E$ and $\|\cdot\|_F$ are equivalent. Now, if the norms of E and L_1 were equivalent on Q , we would have $E \supset G$, by Theorem 1 of [HNS]. Therefore, we can assume that the norms of E and L_1 are not equivalent on Q .

We first deal with the case of F separable. Consider the real interpolation space $E_1 := (E, F)_{\theta, p}$ for some $0 < \theta < 1$ and $1 < p < \infty$. The separability of F implies $\lim_{\text{meas } A \rightarrow 0} \|x\chi_A\|_F = 0$ for any $x \in F$, so also for $x \in E$. Hence $K(t, x, E, F) = K(t, x, E_0, F)$ for $x \in F$, which implies that $E_1 = (E_0, F)_{\theta, p}$ (cf. [BL, Thm. 3.4.2]). Therefore E_1 is also separable and $E \subset E_1 \subset F$ with $E_1 \neq F$.

Now, since the norms of E, E_1 and F are equivalent on Q , the norms of E_1 and L_1 are not. Hence, applying the Kadec–Pełczyński method ([LT2], see [HNS, Thm. 5]) we can find a normalized sequence $\{x_n\}$ in Q and a sequence of disjoint measurable sets $A_n \subset \text{supp } x_n$, $n \in \mathbb{N}$, such that $y_n := x_n \chi_{A_n} \in L_\infty$ and the sequence $\{x_n\}$ is equivalent to $\{y_n\}$ in E_1 and in F .

Now, using that the fact $|\sum c_n y_n| \leq |\sum c_n x_n|$ and the equivalence of the norms of E and E_1 on $[x_n]$, we have

$$\begin{aligned} \left\| \sum_n c_n y_n \right\|_E &\leq \left\| \sum_n c_n x_n \right\|_E \\ &\leq M_1 \left\| \sum_n c_n x_n \right\|_{E_1} \leq M_2 \left\| \sum_n c_n y_n \right\|_{E_1} \leq M_3 \left\| \sum_n c_n y_n \right\|_E \end{aligned}$$

for any scalar sequence $\{c_n\}$ and for some constants $M_1, M_2, M_3 > 0$. Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ are also equivalent in E . Thus the norms of E and F are equivalent on $[y_n]$ and the inclusion $E \subset F$ is not DSS.

Finally, assume that E and F are non-separable. We distinguish two cases: $E \subset F_0$ and $E \not\subset F_0$. If $E \subset F_0$, this inclusion cannot be SS and since F_0 is separable, we deduce as earlier that the inclusion $E \subset F$ is not DSS. In the case of $E \not\subset F_0$, we get the same conclusion by proceeding as in the second part of the proof of Proposition 1. ■

Notice that Theorem 2 may be reformulated as follows: the inclusion $E \subset F$ is strictly singular if and only if it is disjointly strictly singular and the norms of E and F are not equivalent on the Rademacher subspace $[r_n]$.

COROLLARY 3. *Let E and F be r.i. spaces with $E \subset F$ and $E \not\subset G$. If the norms of E and F are equivalent on $[r_n]$ then there exists a disjoint sequence $\{x_n\}$ in E for which the norms of E and F are equivalent on $[x_n]$.*

COROLLARY 4. *Let E and F be r.i. spaces with $E \subset F$. The inclusion $E \subset F$ is strictly singular if and only if the inclusion $E_0 \subset F$ is strictly singular.*

Proof. The “only if” part is trivial. Suppose that the inclusion $E \subset F$ is not SS. It follows from Theorem 2 that either $E \supset G$, or the inclusion $E \subset F$ is not DSS. If $E \supset G$ then $E_0 \supset G$ since G is separable. And if the inclusion $E \subset F$ is not DSS then, by Proposition 1, neither is the inclusion $E_0 \subset F$; all the more, it is not SS. ■

4. Strict singularity and Rademacher spaces. In this section we study couples of r.i. spaces E and F “close” to L_∞ with equivalence of norms on the Rademacher subspace. For that we will make use of some interpolation results.

Given a r.i. space E , consider the sequence space $R(E)$ of Rademacher coefficients $\{a_k\}$ endowed with the norm

$$\|\{a_k\}\|_{R(E)} = \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_E.$$

It is easy to check that $R(E)$ is an interpolation space between ℓ_1 and ℓ_2 ,

i.e., $R(E) \in I(\ell_1, \ell_2)$. Moreover, it is known that the set $I(\ell_1, \ell_2)$ is described by the real interpolation method (cf. [LS]). Therefore there exists a Banach lattice F of measurable functions on $[0, \infty)$ with respect to the measure dt/t such that $\min(1, t) \in F$ and

$$(1) \quad R(E) = (\ell_1, \ell_2)_F^K = \{a \in \ell_2 : K(t, a, \ell_1, \ell_2) \in F\}$$

(cf. [BK, Thms. 4.4.5 and 4.4.38]).

We can consider the r.i. space $\tilde{E}$ associated to E defined by

$$\tilde{E} := (L_\infty, G)_F^K$$

with its canonical norm $\|x\|_{\tilde{E}} = \|K(\cdot, x, L_\infty, G)\|_F$, where G is the closure of L_∞ in $L_N \equiv \exp L_2$. It is known ([A, Thm. 1.4]) that $R(E) = R(\tilde{E})$, i.e.,

$$(2) \quad \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{\tilde{E}} \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_E.$$

Moreover, $E = \tilde{E}$ if and only if $E \in I(L_\infty, G)$ ([A, Thm. 1.5]).

Given $x \in L_N$, denote by Sx the function

$$(3) \quad Sx(t) := \log^{1/2} \frac{e}{t} \sup_{0 < u \leq t} x^*(u) \log^{-1/2} \frac{e}{u}$$

for $0 < t \leq 1$. The following statement gives a simple description of the r.i. space $\tilde{E}$.

PROPOSITION 5. *If E is an r.i. space then $\|x\|_{\tilde{E}} \asymp \|Sx\|_E$.*

Proof. It follows from [M, Cor. 2.2] that there exist absolute constants $C_1, C_2, \beta > 0$ such that for all $a \in \ell_2$,

$$\left(\sum_{k=1}^{\infty} a_k r_k \right)^*(t) \leq C_1 K(\log^{1/2}(e/t), a, \ell_1, \ell_2)$$

and

$$\left(\sum_{k=1}^{\infty} a_k r_k \right)^*(\beta t) \geq C_2 K(\log^{1/2}(e/t), a, \ell_1, \ell_2)$$

for every $0 < t \leq 1$. Hence

$$(4) \quad \|a\|_{R(E)} \asymp \|K(\log^{1/2}(e/t), a, \ell_1, \ell_2)\|_E.$$

The sets of K -functionals corresponding to the Banach pairs (L_∞, G) and (ℓ_1, ℓ_2) coincide up to equivalence ([A]). If $x \in \tilde{E}$, then $K(t, x, L_\infty, G) \in F$ and there exists $a \in \ell_2$ such that

$$(5) \quad K(t, x, L_\infty, G) \asymp K(t, a, \ell_1, \ell_2)$$

for $t > 0$. Hence $a \in R(E)$ and, by (4), we have $K(\log^{1/2}(e/t), a, \ell_1, \ell_2) \in E$. Using (5) we get $K(\log^{1/2}(e/t), x, L_\infty, G) \in E$.

Similar arguments show that the converse holds: if $K(\log^{1/2}(e/t), x, L_\infty, G) \in E$, then $x \in \tilde{E}$. Thus the space $\tilde{E}$ and the (Banach) space endowed with the norm $\|K(\log^{1/2}(e/t), \cdot, L_\infty, G)\|_E$ coincide as sets, so, by the closed graph theorem, both norms are equivalent, i.e.,

$$(6) \quad \|x\|_{\tilde{E}} \asymp \|K(\log^{1/2}(e/t), x, L_\infty, G)\|_E.$$

Finally, it is easy to see that

$$(7) \quad K(t, x, L_\infty, G) = K(t, x, L_\infty, L_N)$$

for any $x \in G$ and $t > 0$. And it is well-known ([Lo]) that

$$(8) \quad \|x\|_{L_N} \asymp \|x\|_{M(\varphi_0)}$$

for $x \in L_N$ where $\varphi_0(t) = \log^{-1/2}(e/t)$, and clearly, $L_\infty = M(\varphi_1)$ for the function $\varphi_1(t) = 1$. Therefore we can consider the Banach pair (L_∞, L_N) as a pair of Marcinkiewicz spaces and apply a formula for the K -functional from [CN]. Thus we have

$$(9) \quad K(\log^{1/2}(e/t), x, L_\infty, L_N) \asymp \log^{1/2}(e/t) \sup_{0 < u \leq t} x^*(u) \log^{-1/2}(e/u),$$

and the needed equivalence follows now from (6)–(9). ■

Note that $Sx(t) \geq x^*(t)$. Hence the above proposition implies that $\tilde{E} \subset E$ and $\|x\|_E \leq C\|x\|_{\tilde{E}}$ for every $x \in \tilde{E}$ and some constant $C > 0$. In particular, $\varphi_E(t) \leq C\varphi_{\tilde{E}}(t)$ for every $t \in [0, 1]$.

We can now give a characterization of the Lorentz and Marcinkiewicz spaces which are interpolation spaces between L_∞ and G .

PROPOSITION 6. *Let $\psi \in \Omega$.*

- (i) *A Lorentz space $\Lambda(\psi)$ belongs to the set $I(L_\infty, G)$ if and only if $\varphi_{\widetilde{\Lambda(\psi)}}(t) \leq C\psi(t)$ for some $C > 0$ and $0 \leq t \leq 1$.*
- (ii) *A Marcinkiewicz space $M(\psi)$ belongs to the set $I(L_\infty, L_N)$ if and only if $\varphi_{\widetilde{M(\psi)}}(t) \leq C\psi(t)$ for some $C > 0$ and $0 \leq t \leq 1$.*

Proof. (i) If $\Lambda(\psi) \in I(L_\infty, G)$ then $\widetilde{\Lambda(\psi)} = \Lambda(\psi)$ ([A]) and hence the functions $\varphi_{\widetilde{\Lambda(\psi)}}$ and ψ are equivalent.

Conversely, if $\varphi_{\widetilde{\Lambda(\psi)}}(t) \leq C\psi(t)$, then Proposition 5 implies that the quasi-linear operator S on $\Lambda(\psi)$ defined in (3) is uniformly bounded on the set of characteristic functions. Hence [KPS, Lemma 2.5.2] shows that S is bounded in $\Lambda(\psi)$. Therefore $\|Sx\|_{\Lambda(\psi)} \leq C\|x\|_{\Lambda(\psi)}$ for some $C > 0$, and Proposition 5 yields $\widetilde{\Lambda(\psi)} = \Lambda(\psi)$, and hence $\Lambda(\psi) \in I(L_\infty, G)$.

(ii) If $M(\psi) \in I(L_\infty, L_N)$ then $M(\psi) \cap G \in I(L_\infty, G)$. Indeed, since the set $I(L_\infty, L_N)$ can be described by the real interpolation method ([A]) we have $M(\psi) = (L_\infty, L_N)_F^K$ for some Banach lattice F on $[0, \infty)$. And, by (7),

we have $M(\psi) \cap G = (L_\infty, G)_F^K$. This means that $M(\psi) \cap G \in I(L_\infty, G)$ and $M(\psi) \cap G = \widetilde{M(\psi) \cap G}$. Moreover, since $M(\psi) \subset L_N$ the fundamental functions of the spaces $M(\psi)$ and $M(\psi) \cap G$ are equivalent, and

$$\left\| \sum a_k r_k \right\|_{M(\psi)} \asymp \left\| \sum a_k r_k \right\|_{M(\psi) \cap G},$$

therefore $\widetilde{M(\psi)} = \widetilde{M(\psi) \cap G}$. Hence,

$$\varphi_{\widetilde{M(\psi)}}(t) \asymp \varphi_{\widetilde{M(\psi) \cap G}}(t) \asymp \varphi_{M(\psi) \cap G}(t) \asymp \varphi_{M(\psi)}(t) = \psi(t).$$

Let us now prove the converse. Assume $\varphi_{\widetilde{M(\psi)}}(t) \leq C\psi(t)$. Then, by Proposition 5, there is $C > 0$ such that

$$(10) \quad \|S\chi_{(0,\tau)}\|_{M(\psi)} \leq C\psi(\tau)$$

for $\tau \in [0, 1]$. Now, since

$$(11) \quad S\chi_{(0,\tau)}(t) = \begin{cases} 1, & 0 < t \leq \tau \\ \left(\frac{\log(e/t)}{\log(e/\tau)} \right)^{1/2}, & \tau \leq t \leq 1, \end{cases}$$

and $\int_0^t \log^{1/2}(e/s) ds \asymp t \log^{1/2}(e/t)$, for $0 < t \leq 1$, we get

$$(12) \quad \frac{1}{t} \int_0^t S\chi_{(0,\tau)}(s) ds \asymp \chi_{(0,\tau)}(t) + \left(\frac{\tau}{t} + \left(\frac{\log(e/t)}{\log(e/\tau)} \right)^{1/2} \right) \chi_{(\tau,1)}(t) \\ \asymp S\chi_{(0,\tau)}(t).$$

Hence

$$(13) \quad \|S\chi_{(0,\tau)}\|_{M(\psi)} \asymp \left(\log \frac{e}{\tau} \right)^{-1/2} \sup_{\tau \leq t \leq 1} \psi(t) \log^{1/2} \frac{e}{t}.$$

Therefore (10) can be rewritten as

$$\frac{\psi(t)}{\psi(\tau)} \leq C_1 \left(\frac{\log(e/\tau)}{\log(e/t)} \right)^{1/2}, \quad 0 < \tau \leq t \leq 1.$$

Since $\varphi_{L_N}(t) = \log^{-1/2}(e/t)$ the above inequality proves, by [S], that $M(\psi) \in I(L_\infty, L_N)$. ■

COROLLARY 7. *Let E be an r.i. space with $E \in I(L_\infty, G)$. Then the Marcinkiewicz space $M(\varphi_E)$ belongs to $I(L_\infty, L_N)$.*

Proof. We have $E \subset M(\varphi_E)$ (cf. [KPS, Thm. 2.5.7]) and, using [A] once more, we get $E = \widetilde{E} \subset \widetilde{M(\varphi_E)} \subset M(\varphi_E)$. Now, since E and $M(\varphi_E)$ have the same fundamental functions, the spaces $\widetilde{M(\varphi_E)}$ and $M(\varphi_E)$ have equivalent fundamental functions. This implies, by Proposition 6 above, that $M(\varphi_E) \in I(L_\infty, L_N)$. ■

COROLLARY 8. *Given a Marcinkiewicz space $M(\varphi)$, there exists a Marcinkiewicz space $M(\psi)$ with $M(\psi) \in I(L_\infty, L_N)$ such that $\widetilde{M(\varphi)} = M(\psi) \cap G$.*

Proof. Let ψ denote the fundamental function of $\widetilde{M(\varphi)}$. Since $\widetilde{M(\varphi)} \in I(L_\infty, G)$, Corollary 7 yields $M(\psi) \in I(L_\infty, L_N)$. Now, from the inclusions $\widetilde{M(\varphi)} \subset M(\psi) \cap G \subset M(\varphi)$ and (2), we deduce that

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{\widetilde{M(\varphi)}} \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{M(\psi) \cap G} \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{M(\varphi)}$$

for **all** sequences $a \in \ell_2$. Since it was proved in Proposition 6 that $M(\psi) \cap G \in I(L_\infty, G)$, an application of [A, Thm. 1.5] shows that $\widetilde{M(\varphi)} = M(\psi) \cap G$. ■

We are now in a position to present one of the main results of this section.

THEOREM 9. *If a Marcinkiewicz space $M(\varphi) \subset G$ does not belong to $I(L_\infty, G)$, then there exists another Marcinkiewicz space $M(\psi)$ such that $M(\psi) \subsetneq M(\varphi)$ and $R(M(\varphi)) = R(M(\psi))$.*

Proof. We have $\widetilde{M(\varphi)} \subset M(\varphi) \subset G$. Hence, by Corollary 8, if ψ is the fundamental function of $\widetilde{M(\varphi)}$ then $\widetilde{M(\varphi)} = M(\psi) \cap G = M(\psi)$ and $M(\psi) \in I(L_\infty, G)$. Moreover, the Marcinkiewicz spaces $M(\varphi)$ and $M(\psi)$ do not coincide because $M(\varphi) \notin I(L_\infty, G)$. Finally, by (2), we have $R(M(\varphi)) = R(M(\psi))$. ■

An analogous result is also valid for Lorentz spaces.

THEOREM 10. *If a Lorentz space $\Lambda(\varphi) \subset G$ does not belong to $I(L_\infty, G)$, then there exists another Lorentz space $\Lambda(\psi)$ such that $\Lambda(\psi) \subsetneq \Lambda(\varphi)$ and $R(\Lambda(\varphi)) = R(\Lambda(\psi))$.*

Proof. Let $X := \widetilde{\Lambda(\varphi)}$. Since $X \in I(L_\infty, G)$ and $\Lambda(\varphi) \notin I(L_\infty, G)$, we have $X \subsetneq \Lambda(\varphi)$. It is easily checked that the Köthe dual X' satisfies $X' \supsetneq M(t/\varphi)$. Therefore, there exists a positive decreasing function $a(\cdot) \in X'$ such that

$$a(t) \geq \varphi'(t) \quad (0 < t \leq 1) \quad \text{and} \quad \limsup_{t \rightarrow +0} \frac{1}{\varphi(t)} \int_0^t a(s) ds = \infty.$$

Define $\psi(t) := \int_0^t a(s) ds$ ($0 \leq t \leq 1$). Since for every $x \in X$ we have

$$\int_0^1 x^*(t) d\psi(t) = \int_0^1 x^*(t) a(t) dt < \infty,$$

it follows that $X \subset \Lambda(\psi)$. Moreover, $\Lambda(\psi) \subsetneq \Lambda(\varphi)$, so the equality $R(\Lambda(\varphi)) = R(\Lambda(\psi))$ follows from (2). ■

In particular, the above inclusions $M(\psi) \subset M(\varphi)$ and $\Lambda(\psi) \subset \Lambda(\varphi)$ are not strictly singular and, by Corollary 3, not disjointly strictly singular either. By contrast, we have the following:

THEOREM 11. *Let $\varphi, \psi \in \Omega$ be such that $\Lambda(\varphi) \subset M(\psi)$. The inclusion $\Lambda(\varphi) \subset M(\psi)$ is strictly singular if and only if $\Lambda(\varphi) \not\supset G$ and $\psi(+0) = 0$.*

Proof. The necessity part is well-known. If $\Lambda(\varphi) \supset G$ then $R(\Lambda(\varphi)) = R(M(\psi)) = \ell_2$ ([RS]). In the case when $\psi(+0) > 0$ we have $M(\psi) = L_\infty = \Lambda(\varphi)$.

Conversely, since $\psi(+0) = 0$ we have $M(\psi) \neq L_\infty$, and clearly $\Lambda(\varphi) \neq L_1$. Hence the spaces $\Lambda(\varphi)$ and $M(\psi)$ do not coincide. Thus the statement is known for the left extreme case of $\Lambda(\varphi) = L^\infty$ ([N]) and also for the right extreme case of $M(\psi) = L^1$ since $\Lambda(\varphi) \not\supset G$ ([HNS, Thm. 1]). Now, using the fact that any normalized disjoint sequence in $\Lambda(\varphi)$ (resp. $M(\psi)$) contains a subsequence equivalent to the canonical basis of ℓ_1 [FJT] (resp. of c_0 , cf. [Se]) we deduce that the inclusion $\Lambda(\varphi) \subset M(\psi)$ is DSS. Hence, by Theorem 2, it is also SS. ■

In particular: *the canonical inclusion $\Lambda(\varphi) \subset M(\varphi)$ is strictly singular if and only if $\Lambda(\varphi) \not\supset G$ and $\varphi(+0) = 0$.*

A direct consequence is

COROLLARY 12. *Let $\varphi \in \Omega$. Then $R(\Lambda(\varphi)) = R(M(\varphi))$ if and only if $\Lambda(\varphi) \supset G$ or $\varphi(+0) > 0$.*

5. Applications. In this section we give some applications of the main results.

PROPOSITION 13. *Let E and F be r.i. spaces with $E \not\supset G$. If*

$$(14) \quad \int_0^1 \left(\frac{t}{\varphi_E(t)} \right)' \varphi'_F(t) dt < \infty,$$

then $E \subset F$ and this inclusion is strictly singular.

Proof. It was proved in Theorem 3.1 of [GHSS] that condition (14) implies the inclusion $E \subset F$ and that this inclusion is DSS. Hence, using Theorem 2, we get the statement. ■

COROLLARY 14. *Let E and F be r.i. spaces such that $\varphi_E(t) \geq a \log^{-\alpha}(e/t)$ and $\varphi_F(t) \leq b \log^{-\beta}(e/t)$ for some $0 < \alpha < \min(\beta, 1/2)$ and constants $a, b > 0$. Then the inclusion $E \subset F$ is strictly singular.*

Proof. We may assume that the functions $t/\varphi_E(t)$ and φ_F are concave on $(0, 1]$. Then $(t/\varphi_E(t))'$ and φ'_F decrease on $(0, 1]$. Now, applying twice the

property 2.2.19 in [KPS], we get

$$\begin{aligned} \int_0^1 \left(\frac{t}{\varphi_E(t)} \right)' \varphi_F'(t) dt &\leq \int_0^1 \left(\frac{t}{a \log^{-\alpha}(e/t)} \right)' \left(b \log^{-\beta} \frac{e}{t} \right)' dt \\ &\leq \frac{b\beta}{a} \int_0^1 \log^{\alpha-\beta-1} \frac{e}{t} \frac{dt}{t} = \frac{b\beta}{a(\beta-\alpha)} < \infty. \end{aligned}$$

By [KPS, Thm. 2.5.7]), the assumption $\alpha < 1/2$ implies $E \subsetneq G$. Hence the statement follows from the above proposition. ■

PROPOSITION 15. *Let E and F be r.i. spaces with $E \subset F$. If the inclusion $E \subset F$ is disjointly strictly singular then the inclusion operator is weakly compact.*

Proof. We can assume that $E \subset F_0$. Indeed, otherwise, reasoning as in the proof of Proposition 1 we construct a disjoint sequence $\{z_k\}$ in E_0 which is equivalent in E_0 and in F_0 to the canonical basis of c_0 . So the inclusion $E \subset F$ is not DSS.

Now, let $E \subset F$ with F separable, hence order continuous. Assume that $E \subset F$ is not weakly compact. Consider the real interpolation space $(E, F)_{\theta, p}$ for $0 < \theta < 1$, $1 < p < \infty$, which is not reflexive by [B, Thm. 3.1]. Hence the lattice $(E, F)_{\theta, p}$ contains a subspace Q isomorphic to ℓ_1 or to c_0 (cf. [LT2]). Now if Q is isomorphic to ℓ_1 , we find, by [B, Prop. 2.3.3], that the inclusion $E \hookrightarrow F$ preserves an ℓ_1 -isomorphic copy. In the case of Q isomorphic to c_0 , an analogous statement is also true [Ma, Cor. 4.1]. Now, using ([Me, Thms. 3.4.11–3.4.17]), we deduce that $E \subset F$ also preserves a disjoint ℓ_1 -sequence or a disjoint c_0 -sequence. ■

V. Milman [Mi] posed the following question: Given two Banach spaces E and F and a non-strictly singular operator A from E into F , does there exist a *complemented* subspace Q in E such that the restriction of the operator A to Q is an isomorphism?

We give a negative answer to this question using the above results. First note that the inclusions $L \log^\lambda L \subset L_1$ are not strictly singular for $\lambda > 0$ because the Rademacher spaces satisfy $R(L \log^\lambda L) = R(L_1) = \ell_2$.

Recall that an operator $A : E \rightarrow F$ between two Banach spaces E and F is said to be *strictly cosingular* (or *Pełczyński*) if there is no infinite-dimensional space H and onto operators $h : E \rightarrow H$ and $g : F \rightarrow H$ such that $h = gA$. Note that this class of operators is somewhat related by duality to strictly singular operators ([P]).

THEOREM 16. *Let $0 < \lambda < 1/2$. If Q is a subspace of $L \log^\lambda L$ on which the $L \log^\lambda L$ -norm and the L_1 -norm are equivalent, then Q is not complemented in $L \log^\lambda L$.*

Proof. Suppose the contrary and denote by P a projection from $L \log^\lambda L$ onto Q . There exists a reflexive r.i. space E with $L \log^\lambda L \subset E \subset L_1$ ([FS]). Therefore Q is a reflexive subspace of L_1 . It follows from Rosenthal's theorem [R, Thm. 8] that Q embeds isomorphically into L_p for some $p > 1$, i.e., there exists an operator $T : (Q, \|\cdot\|_{L_1}) \rightarrow L_p$ which is an isomorphism onto its image. Set $Z = T(Q)$.

Now, consider the inclusion operator $i : L_p \hookrightarrow L \log^\lambda L$ which is not strictly cosingular since there exist onto operators $R = TPi : L_p \rightarrow Z$ and $TP : L \log^\lambda L \rightarrow Z$ with $TPi = R$. On the other hand, by Corollary 14, the adjoint operator $i^* : \exp L_\mu \hookrightarrow L_{p'}$ is SS because $\mu > 2$ (here $\mu = 1/\lambda$ and $p' = p/(p-1)$). Hence, using [P, Prop. 1] we conclude that the inclusion operator i is strictly cosingular, which gives a contradiction. Thus the subspace Q cannot be complemented in $L \log^\lambda L$ (and hence not in L_1 either). ■

Note that the assumption $0 < \lambda < 1/2$ is essential since the Rademacher subspace $[r_n]$ is complemented in $L \log^\lambda L$ for $\lambda \geq 1/2$ (cf. [LT2, Prop. 2.b.4]).

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