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Kruglov Operator and Operators Defined by Random Permutations*

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Abstract. The Kruglov property and the Kruglov operator play an important role in the study of geometric properties of r. i. function spaces. We prove that the boundedness of the Kruglov operator in an r. i. space is equivalent to the uniform boundedness on this space of a sequence of operators defined by random permutations. It is also shown that there is no minimal r. i. space with the Kruglov property.

Key words: rearrangement invariant (r. i.) space, Orlicz, Marcinkiewicz, Lorentz spaces, Kruglov property, Kruglov operator, independent functions.

1. Introduction

Let f be a random variable (measurable function) on the interval $[0,1]$. By $\pi(f)$ we denote a random variable on $[0,1]$ with the same distribution as $\sum_{i=1}^{N} f_i$, where the f_i are independent copies of f and N is a Poisson random variable with parameter 1 independent of the sequence $\{f_i\}$.

Definition. An r. i. function space E (see Section 2) on the interval $[0,1]$ is said to have the *Kruglov property* ($E \in \mathbb{K}$) if

$$f \in E \iff \pi(f) \in E.$$

This property was introduced and studied by Braverman [1], who exploited some probabilistic constructions and ideas from Kruglov's article [2]. An operator approach to the study of this property was introduced in [3] (see also [4]).

Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint measurable subsets of $[0,1]$ such that $\operatorname{mes} B_n = 1/(e\,n!)$. (Here mes is the Lebesgue measure on $[0,1]$.) If $f \in L_1[0,1]$, then we set

$$Kf(\omega_0,\omega_1,\dots) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} f(\omega_k)\chi_{B_n}(\omega_0). \tag{1}$$

Here and throughout the following, χ_B stands for the characteristic function of a set B. Then K is a positive linear operator, $K\colon L_1[0,1] \to L_1(\Omega,P)$, where $(\Omega,P) = \prod_{n=0}^{\infty}([0,1],\mathrm{mes})$. Since Kf is equidistributed with $\pi(f)$ (see [3]), we can treat Kf as an explicit representation of $\pi(f)$. In particular, an r. i. space E has the Kruglov property if and only if K (boundedly) maps E into $E(\Omega,P)$ (see [3]).

By Kf we also denote another random variable defined on $[0,1]$ and having the same distribution as the variable introduced in (1). Let f^* be the nonincreasing rearrangement of $|f|$; that is, $f^*(t)$ is nonincreasing on $[0,1]$ and equimeasurable with $|f(t)|$. If $f \in L_1[0,1]$, $\{B_n\}$ is the same sequence of subsets of $[0,1]$ as above, and, for each $n \in \mathbb{N}$, $f_{n,1},\dots,f_{n,n}$ and χ_{B_n} form a set of independent functions such that $f_{n,k}^* = f^*$ for every $k = 1,\dots,n$, then $Kf(t)$ is defined as some rearrangement of the function

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{n,k}(t)\chi_{B_n}(t) \qquad (0 \leqslant t \leqslant 1). \tag{2}$$

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We point out that K is a linear operator from $L_1[0,1]$ to $L_1(\Omega, P)$. By saying that K is bounded in a r. i. space X on $[0,1]$, we mean that K is bounded as a linear mapping of $X[0,1]$ into $X(\Omega, P)$. It follows from the definitions of an r. i. space and the operator K that $\|Kf\|_E \geqslant e^{-1}\|f\|_E$ $(f \in E)$ for every r. i. space E (see also [1, 1.6, p. 11]). It was shown in [3] that the operator K plays an important role in estimating the norms of sums of independent random variables via the norms of sums of their disjoint copies. (These estimates have numerous applications.) The well-known results due to Johnson and Schechtman [5] were strengthened along these lines in [3].

It is well known (e.g., see [2] or [1]) that the Orlicz space $\exp L_1$ defined by the function $e^t - 1$ satisfies the Kruglov property. It follows that the same is true for its separable part $(\exp L_1)_0$. Indeed, since K is bounded in $\exp L_1$, we have $K((\exp L_1)_0) \subset \overline{K(L_\infty)}$. (The closure is taken with respect to the norm in $\exp L_1$.) At the same time, $K(L_\infty) \subset (\exp L_1)_0$ [3, Theorem 4.4]. Since $(\exp L_1)_0$ is a closed subset of $\exp L_1$, we conclude that the operator K is bounded in $(\exp L_1)_0$. All previously known r. i. spaces E with the Kruglov property satisfy the inclusion $E \supset (\exp L_1)_0$. This, together with some results in [3] (e.g. Theorem 7.2), suggests that $(\exp L_1)_0$ is the minimal r. i. space with the Kruglov property. However, in the first part of the paper we show that this conjecture fails. Moreover, we show that for each r. i. space $E \in \mathbb{K}$ there exists a Marcinkiewicz space satisfying the Kruglov property such that $M_\psi \subsetneqq E$ (see Corollary 3). In a sense, the situation is quite different in the subclass of Lorentz spaces. Indeed, every Lorentz space satisfying the Kruglov property necessarily contains $\exp L_1$ (see Theorem 4).

In [6], Kwapien and Schütt studied the properties of random permutations and then applied them to the geometry of Banach spaces. These results were generalized in [7] and [8] via an operator approach. The following family of operators was introduced there. Let $n \in \mathbb{N}$, and let S_n be the set of all permutations of the numbers $1, \dots, n$. From now on, the sets S_n and $\{1, \dots, n!\}$ will be identified (in an arbitrary manner). First, we define an operator A_n acting from $\mathbb{R}^n$ into $\mathbb{R}^{n!}$: if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then the component of $A_n x$ corresponding to an arbitrary permutation $\pi \in S_n$ (identified with an element of $\{1, \dots, n!\}$) is defined by the formula

$$(A_n x)_\pi := \sum_{i:\, \pi(i)=i} x_i\,. \tag{3}$$

For every $x \in L_1[0,1]$, we define a vector $B_n x \in \mathbb{R}^n$ with coordinates $(B_n x)_i = n \int_{(i-1)/n}^{i/n} x(t)\,dt$, $i = 1, \dots, n$. The operator B_n has a right inverse C_n $(B_n C_n x = x,\ x \in \mathbb{R}^n)$ that takes each vector to a function constant on the intervals $((i-1)/n, i/n]$, $i = 1, \dots, n$. Finally, we define the operator

$$T_n = C_{n!} A_n B_n.$$

For every $n \in \mathbb{N}$, this is a positive linear operator from $L_1[0,1]$ to the space of step functions. One can readily show that

$$\|T_n x\|_{L_1} = \|x\|_{L_1} \tag{4}$$

for every positive $x \in L_1[0,1]$. Sometimes, we shall also use the notation T_n for the operator $C_{n!}A_n$ defined in a similar way on $\mathbb{R}^n$. (This will not result in a misunderstanding.) If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and E is an arbitrary r. i. space, then $\|x\|_E$ is always understood as

$$\|C_n x\|_E = \left\| \sum_{k=1}^n x_k \chi_{\left(\frac{k-1}{n}, \frac{k}{n}\right)} \right\|_E.$$

The operators generated by random permutations and defined on the set of square matrices were considered in [8], where it was established that such operators are uniformly bounded provided that so are the operators T_n. Although there is apparently no connection between the operators K and T_n, the criteria for the boundedness of the operator K in any Lorentz space Λ_φ [3] and for the uniform boundedness of the operators T_n $(n \in \mathbb{N})$ in the same Λ_φ [8] coincide. More precisely, both criteria are equivalent to the condition

$$M := \sup_{0<t\leqslant 1} \frac{1}{\varphi(t)} \sum_{k=1}^{\infty} \varphi\left(\frac{t^k}{k!}\right) < \infty. \tag{5}$$

In this connection, the conjecture arises that the boundedness of K in an arbitrary r. i. space E is equivalent to the condition $\sup_n \|T_n\|_E < \infty$. The second part of the present paper deals with the proof of this assertion. The proof is based on combinatorial arguments and involves obtaining estimates for the corresponding distribution functions. The result has a number of useful corollaries for the operator K as well as for the operators T_n. In particular, Corollary 13 strengthens Theorem 19 in [8] by showing that the uniform boundedness of T_n in the Orlicz spaces $\exp L_p$ is equivalent to the condition $p \leqslant 1$.

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2. Definitions and Notation

A Banach space E of measurable functions on $[0, 1]$ is said to be *rearrangement invariant* (*r. i.*), or *symmetric*, if the following conditions hold:

1. If $|x(t)| \leqslant |y(t)|$ for $t \in [0, 1]$ and $y \in E$, then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$.
2. If two functions x and $y \in E$ are equimeasurable, that is, satisfy the condition

$$\operatorname{mes}\{t \in [0, 1] : |x(t)| > \tau\} = \operatorname{mes}\{t \in [0, 1] : |y(t)| > \tau\} \qquad (\tau > 0),$$

then $x \in E$ and $\|x\|_E = \|y\|_E$.

If E is an r. i. space, then $L_\infty \subset E \subset L_1$, and these inclusions are continuous. Moreover, if $\|\chi_{(0,1)}\|_E = 1$, then $\|x\|_{L_1} \leqslant \|x\|_E \leqslant \|x\|_{L_\infty}$ for every $x \in L_\infty$. For every $\tau > 0$, the dilation operator σ_τ defined by $\sigma_\tau x(t) := x(t/\tau)\chi_{[0,1]}(t/\tau)$ $(0 \leqslant t \leqslant 1)$ boundedly maps E into itself, and $\|\sigma_\tau\|_E \leqslant \max(1, \tau)$.

The space E' of measurable functions x on $[0, 1]$ such that

$$\|x\|_{E'} = \sup_{\|y\|_E \leqslant 1} \int_0^1 x(t)y(t)\, dt < \infty$$

is called the Köthe dual space of E; it is also an r. i. space. Following [9, 2.a.1], we assume that E is either separable or coincides with the second Köthe dual space E''. In both cases, E is contained in E'' as a closed subspace, and the inclusion $E \subset E''$ is an isometry. If E is separable, then E' coincides with the dual space E^*. The closure E_0 of L_∞ in E is called the *separable part* of E. The space E_0 coincides with L_∞ if and only if $E = L_\infty$ and is separable if $E \neq L_\infty$.

Recall that the weak convergence of the distributions of measurable functions x_n on $[0, 1]$ to the distribution of a function x ($x_n \Rightarrow x$) means that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} y(t)\, d\operatorname{mes}\{s : x_n(s) < t\} = \int_{-\infty}^{\infty} y(t)\, d\operatorname{mes}\{s : x(s) < t\}$$

for every bounded continuous function y on $(-\infty, \infty)$. If E is an r. i. space, $x_n \in E$ $(n \in \mathbb{N})$, $\limsup_{n\to\infty} \|x_n\|_E = C < \infty$, and $x_n \Rightarrow x$, then $x \in E''$ and $\|x\|_{E''} \leqslant C$ [1, Proposition 1.5].

The following semi-order relation on the set of integrable functions plays an important role in the theory of r. i. spaces. We write $x \prec y$ if

$$\int_0^\tau x^*(t)\, dt \leqslant \int_0^\tau y^*(t)\, dt$$

for all $\tau \in [0, 1]$. Here and below, $x^*(t)$ is the nonincreasing left continuous rearrangement of the function $|x(t)|$; i.e.,

$$x^*(t) = \inf\{\tau \geqslant 0 : \operatorname{mes}\{s \in [0, 1] : |x(s)| > \tau\} < t\} \qquad (0 < t \leqslant 1).$$

If $x \prec y$ and y belongs to some r. i. space E, then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$.

Let us give the most important examples of r. i. spaces. Let M be an increasing convex function on $[0, \infty)$ such that $M(0) = 0$. By L_M we denote the Orlicz space L_M with the norm

$$\|x\|_{L_M} = \inf\left\{\lambda > 0 : \int_0^1 M\left(\frac{|x(t)|}{\lambda}\right) dt \leqslant 1\right\}.$$

The function $M_p(u) = e^{u^p} - 1$ is convex for $p \geqslant 1$ and is equivalent to some convex function for $0 < p < 1$. We denote L_{M_p} by $\exp L_p$.

Let $\varphi(t)$ be an increasing concave function on $[0, 1]$ such that $\varphi(0) = 0$; then Λ_φ is the Lorentz space with the norm

$$\|x\|_{\Lambda_\varphi} = \int_0^1 x^*(t)\, d\varphi(t)$$

and M_φ is the Marcinkiewicz space with the norm

$$\|x\|_{M_\varphi} = \sup_{0<t\leqslant 1} \frac{1}{\varphi(t)} \int_0^t x^*(s)\, ds.$$

All above-mentioned facts about r. i. spaces can be found in the books [9] and [10].

In what follows, $\operatorname{supp} f$ is the support of a function f, i.e., the set $\{t : f(t) \neq 0\}$. We write $F \asymp G$ if $C^{-1}F \leqslant G \leqslant CF$, where $C > 0$ is a constant. Finally, $|A|$ stands for the number of elements in a finite set A.

3. Lorentz and Marcinkiewicz Spaces "near" $\exp L_1$

Theorem 1. *There exists a family $\{M_{\psi_\varepsilon}\}_{0<\varepsilon<1}$ of Marcinkiewicz spaces such that $M_{\psi_\varepsilon} \subset M_{\psi_\delta}$ for $0 < \varepsilon \leqslant \delta < 1$ and*

1. $M_{\psi_\varepsilon} \in \mathbb{K}$, $0 < \varepsilon < 1$.
2. *For every r. i. space $E \in \mathbb{K}$, one has $M_{\psi_\varepsilon} \subset E$ provided that ε is sufficiently small.*
3. *The functions ψ_ε are pairwise nonequivalent; more precisely,*

$$\lim_{t\to 0} \frac{\psi_\varepsilon(t)}{\psi_\delta(t)} = 0 \quad \textit{if } 0 < \varepsilon < \delta < 1. \tag{6}$$

4. $M_{\psi_\varepsilon} \subsetneqq (\exp L_1)_0$ *for sufficiently small $\varepsilon > 0$.*

To prove the theorem, we need the following simple assertion.

Lemma 2. *For every $f \in L_1[0, 1]$,*

$$\lim_{n\to\infty} \operatorname{mes}(\operatorname{supp} K^n f) = 0.$$

Proof. Since the operator K is positive, we can assume that $f \geqslant 0$ and $\operatorname{mes}(\operatorname{supp} f) = 1$. Now if $a_n := \operatorname{mes}\{t : K^n f(t) = 0\}$ $(n \in \mathbb{N})$, then, by the definition of the operator K (see relation (2)), $a_1 = 1/e$ and

$$a_{n+1} = \frac{1}{e} + \frac{1}{e}\sum_{k=1}^{\infty} \frac{a_n^k}{k!} = e^{a_n - 1} \qquad (n = 1, 2, \dots).$$

Obviously, the sequence $\{a_n\}$ increases, and $a_n \in [0, 1]$. Since the function $f(x) := e^{x-1} - x$ decreases on $[0, 1]$, it follows that the function e^{x-1} has the only fixed point $x = 1$ on $[0, 1]$. Hence $\lim_{n\to\infty} a_n = 1$, which proves the lemma. □

Proof of Theorem 1. Consider the functions $h_n = (K^n 1)^*$, $n \geqslant 0$. Since the operator K takes equimeasurable functions to equimeasurable ones, we have

$$(Kh_n)^* = h_{n+1}. \tag{7}$$

By Lemma 2, $\operatorname{mes}(\operatorname{supp} h_n) \to 0$ as $n \to \infty$. Hence the series

$$g_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n h_n \tag{8}$$

converges everywhere on the interval $(0,1]$ for every $\varepsilon > 0$, and the function g_ε decreases. Moreover, it follows from the definition of K (see (2)) that $\|K\|_{L_1} = 1$. Hence if $0 < \varepsilon < 1$, then the series (8) converges in L_1 and $g_\varepsilon \in L_1$. Now let us show that the assertions of the theorem hold for the family $\{M_{\psi_\varepsilon}\}_{\varepsilon>0}$, where $\psi_\varepsilon(t) = \int_0^t g_\varepsilon(s)\,ds$ $(0 \leqslant t \leqslant 1)$.

1. Let us prove that the operator K is bounded in M_{ψ_ε}. The extreme points of the unit ball in this space are equimeasurable with g_ε [11], and hence it suffices to show that $Kg_\varepsilon \in M_{\psi_\varepsilon}$. Since K is bounded in L_1, we have

$$Kg_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n K h_n \prec \sum_{n=0}^{\infty} \varepsilon^n h_{n+1} \leqslant \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n h_n = \frac{1}{\varepsilon}\, g_\varepsilon,$$

where the first inequality follows from (7) and the well-known Hardy–Littlewood semiorder property (e.g., see [10, Sec. 2.2]). Thus, $Kg_\varepsilon \in M_{\psi_\varepsilon}$.

2. Now assume that $E \in \mathbb{K}$. As was mentioned earlier, $C = \|K\|_{E\to E} < \infty$. Obviously, $\|h_n\|_E \leqslant C^n \|1\|_E$. Therefore, the series (8) converges in E for every $\varepsilon < C^{-1}$, and $g_\varepsilon \in E$. Since either the space E is separable or $E = E''$, we see that the conditions $x \in E$ and $y \prec x$ imply that $y \in E$ and $\|y\|_E \leqslant \|x\|_E$. Hence the unit ball of M_{ψ_ε}, together with g_ε, is contained in E. It follows that $M_{\psi_\varepsilon} \subset E$.

3. As before, let the function g_ε be defined by (8), and let $0 < \varepsilon < \delta$. Since

$$\lim_{t\to 0} \frac{h_{n+1}(t)}{h_n(t)} = \infty$$

by [3, Theorem 7.2], it follows that

$$\begin{aligned}\limsup_{t\to 0} \frac{g_\varepsilon(t)}{g_\delta(t)} &= \limsup_{t\to 0} \left(\sum_{n=1}^{\infty} \varepsilon^n h_n(t)\right) \cdot \left(\sum_{n=1}^{\infty} \delta^n h_n(t)\right)^{-1} = \cdots \\ &= \limsup_{t\to 0} \left(\sum_{n=m}^{\infty} \varepsilon^n h_n(t)\right) \cdot \left(\sum_{n=m}^{\infty} \delta^n h_n(t)\right)^{-1} \leqslant \left(\frac{\varepsilon}{\delta}\right)^m.\end{aligned}$$

for all $m = 1, 2, \ldots$. Therefore, $\lim_{t\to 0} \frac{g_\varepsilon(t)}{g_\delta(t)} = 0$, and (6) follows.

4. As was indicated in the introduction, the operator K acts boundedly in the space $(\exp L_1)_0$. Hence it suffices to apply the second and third assertions. □

Let $\varphi_n(t) := \int_0^t h_n(s)\,ds$ $(0 \leqslant t \leqslant 1)$, and let M_{φ_n} be the corresponding Marcinkiewicz space. Then $M_{\varphi_n} \subset M_{\varphi_{n+1}} \subset (\exp L_1)_0$ $(n = 1, 2, \ldots)$ and, in a sense, the spaces M_{φ_n}, $n \geqslant 1$, can be viewed as "approximations" to the space $(\exp L_1)_0$. By [3, Theorem 7.2], $M_{\varphi_n} \subset E$ for every r. i. space $E \in \mathbb{K}$ and every $n = 1, 2, \ldots$. This suggests the rather natural conjecture that $(\exp L_1)_0$ is the least r. i. space with the Kruglov property. However, the following corollary of Theorem 1 shows that this conjecture is not true.

Corollary 3. *For every r. i. space $E \in \mathbb{K}$, there exists an r. i. space $F \in \mathbb{K}$ such that $F \subsetneqq E$.*

In contrast to Marcinkiewicz spaces, all Lorentz spaces with the Kruglov property lie "on one side" of the space $\exp L_1$.

Theorem 4. *Let φ be an increasing concave function on $[0,1]$ such that $\varphi(0) = 0$. If $\Lambda_\varphi \in \mathbb{K}$, then $\Lambda_\varphi \supset \exp L_1$.*

Here we also start from a lemma.

Lemma 5. *Let φ be an increasing function on the interval $[0,1]$, and let $\varphi(0)=0$. If φ satisfies condition* (5)*, then*

$$\sum_{k=1}^{\infty} \varphi(2^{-k}) \leqslant A\varphi(1). \tag{9}$$

Here $A>0$ depends only on M in (5).

Proof. By (5), for each $i \in \mathbb{N}$ one has

$$\sum_{j=1}^{\infty} \varphi(2^{-ij} j^{-j}) \leqslant M\varphi(2^{-i}),$$

or, equivalently,

$$\sum_{j=1}^{\infty} \varphi(2^{-j(i+[\log_2 j])}) \leqslant M\varphi(2^{-i}). \tag{10}$$

Straightforward estimates show that the numbers

$$\alpha_n := |\{(i,j) \in \mathbb{N}^2 : j(i+[\log_2 j]) \leqslant n\}|$$

satisfy the condition $\lim_{n\to\infty} n^{-1}\alpha_n = \infty$. Hence $\alpha_n \geqslant (M+1)n$ for some $m \in \mathbb{N}$ and for all $n \geqslant m$. Since φ is increasing, it follows from this and from (10) that

$$(M+1)\sum_{n=m}^{l} \varphi(2^{-n}) \leqslant \sum_{i=1}^{l}\sum_{j=1}^{\infty} \varphi(2^{-j(i+[\log_2 j])}) \leqslant M\sum_{i=1}^{l} \varphi(2^{-i})$$

for every $l > m$, and so

$$\sum_{n=m}^{l} \varphi(2^{-n}) \leqslant M\sum_{i=1}^{m-1} \varphi(2^{-i}).$$

Now inequality (9) is a straightforward consequence of the fact that $l > m$ is arbitrary and m is independent of φ. □

Proof of Theorem 4. As was already noted in the introduction, condition (5) is equivalent to the condition $\Lambda_\varphi \in \mathbb{K}$ [3]. Therefore, Lemma 5 implies that inequality (9) holds. Moreover, by [12],

$$\|x\|_{\exp L_1} \asymp \sup_{0<t\leqslant 1} x^*(t) \log_2^{-1}(2/t),$$

and therefore, to prove the embedding $\Lambda_\varphi \supset \exp L_1$ it suffices to prove that $\log_2(2/t) \in \Lambda_\varphi$. The latter follows from the estimates

$$\begin{aligned}\|\log_2(2/t)\|_{\Lambda_\varphi} &= \int_0^1 \log_2(2/t)\, d\varphi(t) = \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{-k+1}} \log_2(2/t)\, d\varphi(t) \\ &\leqslant \sum_{k=1}^{\infty} (k+1)(\varphi(2^{-k+1}) - \varphi(2^{-k})) = 2\varphi(1) + \sum_{k=1}^{\infty} \varphi(2^{-k}) < \infty. \end{aligned}$$ □

4. Estimates of Distribution Functions

We shall use the following approximation to Kf, where f is an arbitrary measurable function on the interval $[0,1]$.

Let $m \in \mathbb{N}$, let $g_m(t) = \sigma_{1/m} f(t)$, and let $\{h_{m,i}\}_{i=1}^m$ be a set of independent functions equimeasurable with g_m. Then the sequence

$$H_m f(t) = \sum_{i=1}^{m} h_{m,i}(t) \qquad (0 \leqslant t \leqslant 1) \tag{11}$$

weakly converges in distribution to Kf as $m \to \infty$ (see [1, 1.6, p. 11]) or [3, Theorem 3.5]).

In particular, if $n \in \mathbb{N}$, $a_k \geqslant 0$ $(1 \leqslant k \leqslant n)$, and

$$f_a(t) = \sum_{k=1}^{n} a_k \chi_{(\frac{k-1}{n}, \frac{k}{n})}(t) \qquad (0 \leqslant t \leqslant 1), \tag{12}$$

then

$$g_m(t) = \sigma_{\frac{1}{m}} f_a(t) = \sum_{k=1}^{n} a_k \chi_{(\frac{k-1}{nm}, \frac{k}{nm})}(t) \qquad (m \in \mathbb{N}).$$

In this case, we set

$$H_m a(t) := H_m f_a(t) = \sum_{i=1}^{m} h_{m,i}(t). \tag{13}$$

In addition, let $\operatorname{Ch}(r)$ be the number of permutations π of the set $\{1, \dots, r\}$ such that $\pi(i) \neq i$ for every $i = 1, \dots, r$. It is well known (e.g., see [13, p. 20]) that

$$\frac{1}{3} r! \leqslant \operatorname{Ch}(r) \leqslant r! \qquad (r \in \mathbb{N}). \tag{14}$$

First, let us compare the functions $H_m a$ and $T_{nm} b$, where

$$b = (\underbrace{a_1, \dots, a_1}_{m}, \underbrace{a_2, \dots, a_2}_{m}, \dots, \underbrace{a_n, \dots, a_n}_{m}).$$

Lemma 6. *For every $n, m \in \mathbb{N}$ and every $\tau > 0$, one has*

$$\operatorname{mes}\{t : H_m a(t) > \tau\} \leqslant 3 \operatorname{mes}\{t : T_{nm} b(t) > \tau\}.$$

Proof. The function $H_m a(t)$ (respectively, $T_{nm} b(t)$) only takes values of the form $\sum_{i=1}^{n} k_i a_i$, where $k_i \in \mathbb{Z}$, $k_i \geqslant 0$ for all $i = 1, \dots, n$, and $\sum_{i=1}^{n} k_i \leqslant m$ (respectively, $\sum_{i=1}^{n} k_i \leqslant mn$). Therefore, it suffices to prove that

$$\operatorname{mes}\left\{t : H_m a(t) = \sum_{i=1}^{n} k_i a_i\right\} \leqslant 3 \operatorname{mes}\left\{t : T_{nm} b(t) = \sum_{i=1}^{n} k_i a_i\right\}$$

for any choice of $k_i \in \mathbb{N}$, $\sum_{i=1}^{n} k_i = q \leqslant m$. Note also that it suffices to consider the case in which

$$\sum_{i=1}^{n} k_i a_i \neq \sum_{i=1}^{n} k_i' a_i \ \text{ whenever } \ (k_1, \dots, k_n) \neq (k_1', \dots, k_n').$$

Hence $H_m a(t)$ is equal to $\sum_{i=1}^{n} k_i a_i$ if and only if exactly k_i (respectively, $m - q$) of the functions $h_{m,j}(t)$ $(j = 1, \dots, m)$ take the value a_i (respectively, 0). Since the $h_{m,j}$ are pairwise independent, we obtain

$$\begin{aligned} \operatorname{mes}\left\{t : H_m a(t) = \sum_{i=1}^{n} k_i a_i\right\} &= C_m^{m-q, k_1, \dots, k_n} \left(1 - \frac{1}{m}\right)^{m-q} \left(\frac{1}{mn}\right)^{k_1 + \dots + k_n} \\ &\leqslant C_m^{m-q, k_1, \dots, k_n} \left(\frac{1}{mn}\right)^{q}, \end{aligned} \tag{15}$$

where

$$C_m^{m-q, k_1, \dots, k_n} = \frac{m!}{(m-q)!\, k_1! \cdots k_n!}.$$

On the other hand, it follows from (3) and (14) that

$$\begin{aligned} \operatorname{mes}\left\{t : T_{mn} b(t) = \sum_{i=1}^{n} k_i a_i\right\} &= C_m^{k_1} \cdots C_m^{k_n} \operatorname{Ch}(mn - q) \frac{1}{(mn)!} \\ &\geqslant \frac{(m!)^n (mn - q)!}{3(m-k_1)! \cdots (m-k_n)!\, k_1! \cdots k_n!\, (mn)!}. \end{aligned}$$

Since $(m-k_1)!\cdots(m-k_n)! \leqslant (m!)^{n-1}(m-q)!$ and $(mn-q)!/(mn)! \geqslant 1/(mn)^q$, we have

$$\operatorname{mes}\left\{t : T_{mn}b(t) = \sum_{i=1}^{n} k_i a_i\right\} \geqslant \frac{m!\,(mn-q)!}{3k_1!\cdots k_n!\,(m-q)!\,(mn)!}$$
$$\geqslant \frac{m!}{3(m-q)!\,k_1!\cdots k_n!}\cdot\frac{1}{(mn)^q}\,.$$

The assertion of the lemma now follows from this inequality and inequality (15). □

Lemma 7. *If $n,k\in\mathbb{N}$, $n\geqslant 4$, and $k\leqslant n$, then*

$$\frac{(n-k)!}{n!} \leqslant 2\,\frac{(k-1)!}{n^k}\,.$$

Proof. Since $j(n-j) > n$ for $2\leqslant j\leqslant n-2$, we have

$$\frac{n^k(n-k)!}{n!\,(k-1)!} = \prod_{j=1}^{k-1}\frac{n}{j(n-j)} \leqslant \left(\frac{n}{n-1}\right)^2 < 2. \qquad \square$$

Now let us continue the study, initiated in Lemma 6, of the connections between the distribution functions of $T_n a$ and $H_m a$ (see relations (3) and (13)). While the estimate obtained in Lemma 6 holds for every m and n, the opposite inequality holds only asymptotically as $m\to\infty$.

Lemma 8. *Let $n\in\mathbb{N}$, $a=(a_1,\ldots,a_n)\geqslant 0$, and $\tau>0$. Then*

$$\operatorname{mes}\{t : T_n a(t) > \tau\} \leqslant 12\operatorname{mes}\{t : 2H_m a(t) > \tau\}$$

for sufficiently large $m\in N$.

Proof. Assume first that $n\geqslant 4$. Let $A=\{1,\ldots,n\}$ and $S(U) := \sum_{j\in U} a_j$ for every $U\subset A$. Without loss of generality, we can assume that $n=2s$ ($s\in\mathbb{N}$), $a_i>0$, and $S(U_1)\neq S(U_2)$ if $U_1\neq U_2$. Let $\mathscr{A}_i$ be the set of all $U\subset A$ such that $|U|=i$ ($i=1,\ldots,n$). Hence $\mathscr{A}=\bigcup_{i=1}^n \mathscr{A}_i$ is the set of all nonempty subsets of A. Let us represent the set $\mathscr{A}$ in a somewhat different way.

Let $U\in\mathscr{A}_k$ for some $k=1,\ldots,s$. Consider the set $\mathscr{A}_U$ (respectively, $\mathscr{B}_U$) of all $V\subset A$ such that $V\supset U$, $V\in\mathscr{A}_{2k}$ (respectively, $V\in\mathscr{A}_{2k-1}$), and $S(V\setminus U)\leqslant S(U)$. Since

$$\bigcup_{U\in\mathscr{A}_k}\mathscr{A}_U = \mathscr{A}_{2k} \quad\text{and}\quad \bigcup_{U\in\mathscr{A}_k}\mathscr{B}_U = \mathscr{A}_{2k-1} \qquad (k=1,\ldots,s),$$

we see that

$$\mathscr{A} = \bigcup_{k=1}^{s}\bigcup_{U\in\mathscr{A}_k}(\mathscr{A}_U\cup\mathscr{B}_U). \tag{16}$$

It follows from the definition of $\mathscr{A}_U$ and $\mathscr{B}_U$ that

$$S(U)\leqslant S(V)\leqslant 2S(U) \tag{17}$$

for $V\in\mathscr{A}_U\cup\mathscr{B}_U$.

Note that $T_n a(t)$ is a step function with values $S(V)$, $V\in\mathscr{A}$. If $|V|=r$, then

$$\operatorname{mes}\{t : T_n a(t) = S(V)\} = \frac{\mathrm{Ch}(n-r)}{n!} \leqslant \frac{(n-r)!}{n!}$$

by (14). Furthermore, if $|U|=k$ ($k=1,\ldots,s$), then

$$|\mathscr{A}_U| \leqslant C_{n-k}^{k} = \frac{(n-k)!}{k!\,(n-2k)!}$$

and, in a similar way,

$$|\mathscr{B}_U| \leqslant C_{n-k}^{k-1} = \frac{(n-k)!}{(k-1)!\,(n-2k+1)!}\,.$$

Therefore, (16) and (17) imply that

$$\begin{aligned}
\operatorname{mes}\{t : T_n a(t) > \tau\} &\leqslant \sum_{k=1}^{s} \sum_{U\in\mathscr{A}_k} \Biggl(\sum_{V\in\mathscr{A}_U,\, S(V)>\tau} \operatorname{mes}\{t : T_n a(t) = S(V)\} \\
&\qquad\qquad + \sum_{V\in\mathscr{B}_U,\, S(V)>\tau} \operatorname{mes}\{t : T_n a(t) = S(V)\} \Biggr) \\
&\leqslant \sum_{k=1}^{s} \sum_{U\in\mathscr{A}_k,\, S(U)>\tau/2} \biggl(\frac{(n-2k)!}{n!} \cdot \frac{(n-k)!}{k!\,(n-2k)!} \\
&\qquad\qquad + \frac{(n-2k+1)!}{n!} \cdot \frac{(n-k)!}{(k-1)!\,(n-2k+1)!} \biggr) \\
&\leqslant 2 \sum_{k=1}^{s} \sum_{U\in\mathscr{A}_k,\, S(U)>\tau/2} \frac{(n-k)!}{(k-1)!\,n!}\,. \qquad (18)
\end{aligned}$$

Now let us estimate the distribution function of $H_m a(t)$ from below. For every $U \in \mathscr{A}_k$ with $S(U) > \tau/2$, let F_U be the set of all $t \in [0,1]$ such that there exists a set $W \subset \{1,\dots,m\}$ and a bijection $\sigma\colon W \to U$ such that $|W| = k$ (we assume that $m \geqslant n$), $h_{m,j}(t) = a_{\sigma(j)}$ if $j \in W$, and $h_{m,j}(t) = 0$ if $j \notin W$. Thus,

$$H_m a(t) = \sum_{j=1}^{m} h_{m,j}(t) = S(U) > \frac{\tau}{2} \qquad (19)$$

for $t \in F_U$. The independence of the functions $h_{m,j}(t)$ $(j = 1,\dots,m)$ implies that

$$\operatorname{mes}(F_U) = C_m^k k!\, \frac{1}{(mn)^k} \Bigl(1 - \frac{1}{m}\Bigr)^{m-k} = \frac{m(m-1)\cdots(m-k+1)}{m^k} \Bigl(1 - \frac{1}{m}\Bigr)^{m-k} \frac{1}{n^k}\,.$$

Since

$$\lim_{m\to\infty} \frac{m(m-1)\cdots(m-k+1)}{m^k} = 1 \quad \text{and} \quad \lim_{m\to\infty} \Bigl(1 - \frac{1}{m}\Bigr)^{m-k} = \frac{1}{e} > \frac{1}{3},$$

we obtain

$$\operatorname{mes}(F_U) > \frac{1}{3} \cdot \frac{1}{n^k} \qquad (20)$$

for all sufficiently large $m \in \mathbb{N}$ and all $k \leqslant s$.

Note that $F_U \cap F_{U'} = \varnothing$ if $U \neq U'$. Indeed, let $i \in U \setminus U'$. Then for each $t \in F_U$ there exists a $j \in \{1,\dots,m\}$ such that $h_{m,j}(t) = a_i$. At the same time, if $t \in F_{U'}$, then either $h_{m,j}(t) = a_l \neq a_i$ or $h_{m,j}(t) = 0 \neq a_i$. Hence the estimates (20) and (18) and Lemma 7 imply that

$$\begin{aligned}
\operatorname{mes}\{t : 2H_m a(t) > \tau\} &= \sum_{k=1}^{s} \sum_{U\in\mathscr{A}_k,\, S(U)>\tau/2} \operatorname{mes}(F_U) \geqslant \frac{1}{3} \sum_{k=1}^{s} \sum_{U\in\mathscr{A}_k,\, S(U)>\tau/2} \frac{1}{n^k} \\
&\geqslant \frac{1}{6} \sum_{k=1}^{s} \sum_{U\in\mathscr{A}_k,\, S(U)>\tau/2} \frac{(n-k)!}{(k-1)!\,n!} \geqslant \frac{1}{12} \operatorname{mes}\{t : T_n a(t) > \tau\}.
\end{aligned}$$

This proves the lemma for $n \geqslant 4$.

If $1 \leqslant n < 4$, then one can readily show (see the argument preceding Eq. (20)) that

$$\operatorname{mes}\{t : T_n a(t) > \tau\} \leqslant 5 \operatorname{mes}\{t : 2H_m a(t) > \tau\}$$

for all sufficiently large $m \in \mathbb{N}$ and every $\tau > 0$. □

Remark 9. *At the same time, the estimate*

$$\operatorname{mes}\{t : T_n a(t) > \tau\} \leqslant C \operatorname{mes}\{t : H_n a(t) > \tau\} \qquad (\tau > 0)$$

fails for any constant $C > 0$ independent of $n \in \mathbb{N}$. Indeed, if $a_1 = \cdots = a_n = 1$, then

$$\operatorname{mes}\{t : T_n a(t) = n\} = \frac{1}{n!},$$

while

$$\operatorname{mes}\{t : H_n a(t) = n\} = \frac{1}{n^n}\,.$$

5. Kruglov Property and Random Permutations

Theorem 10. *Let E be an r. i. space. The operator K is bounded in E if and only if the sequence of operators T_n is uniformly bounded in E.*

Proof. We shall use notation (3), (12), and (13).

Necessity. It follows from Lemma 8 that

$$\operatorname{mes}\{t : T_n a(t) > \tau\} \leqslant 12 \operatorname{mes}\{t : 2H_m a(t) > \tau\}$$

for arbitrary $n \in \mathbb{N}$, $a = (a_1, \dots, a_n) \geqslant 0$, and $\tau > 0$ and for all sufficiently large $m \in \mathbb{N}$. As was already indicated at the beginning of the preceding section, $H_m a \Rightarrow K f_a$ as $m \to \infty$. Therefore [14, Sec. 6.2],

$$\operatorname{mes}\{t : H_m a(t) > \tau\} \to \operatorname{mes}\{t : K f_a(t) > \tau\} \qquad (m \to \infty)$$

for all $\tau > 0$ such that the right-hand side of the last relation is continuous (i.e., for τ outside an at most countable set). Hence it follows from the preceding inequality that

$$\operatorname{mes}\{t : T_n a(t) > \tau\} \leqslant 12 \operatorname{mes}\{t : 2K f_a(t) > \tau\}$$

for all such τ. Since both functions in this inequality are monotone and right continuous, it follows that it holds for all $\tau > 0$.

It is well known [10, Sec. 2.4.3] that, for every r. i. space E, if $y \in E$ and

$$\operatorname{mes}\{t : |x(t)| > \tau\} \leqslant C \operatorname{mes}\{t : |y(t)| > \tau\} \qquad (\tau > 0),$$

then $x \in E$ and $\|x\|_E \leqslant \max(C, 1)\|y\|_E$. Therefore, by the preceding inequality,

$$\|T_n f_a\|_E \leqslant 24 \cdot \|K f_a\|_E, \quad \text{or} \quad \sup\{\|T_n f_a\|_E : \|f_a\| \leqslant 1\} \leqslant 24 \cdot \|K\|_E.$$

By the definition of T_n, we have $T_n x = T_n f_{a_n(x)}$, where $a_n(x) = (a_{n,k}(x))_{k=1}^n$ and $a_{n,k}(x) = n \int_{(k-1)/n}^{k/n} x(s)\, ds$. Since $\|f_{a_n(x)}\|_E \leqslant \|x\|_E$ [10, Sec. 2.3.2] and E is either separable or coincides with its second Köthe dual, we obtain

$$\sup_n \|T_n\|_E \leqslant 24 \cdot \|K\|_E.$$

Sufficiency. Assume that $\sup_n \|T_n\|_E = C < \infty$. It follows from Lemma 6 and [10, Sec. 2.4.3] that

$$\|H_m f_a\|_E \leqslant 3\|T_{nm}\|_E \|f_a\|_E \leqslant 3C\|f_a\|_E.$$

Since $H_m f_a \Rightarrow K f_a$ as $m \to \infty$, it follows from [1, Proposition 1.5] that

$$\|K f_a\|_{E''} \leqslant 3C\|f_a\|_E. \tag{21}$$

Now let $f = f^* \in E$ be arbitrary. If

$$f_n(t) = \sum_{k=1}^{2^n} f(k2^{-n})\chi_{((k-1)2^{-n}, k2^{-n})}(t) \qquad (0 \leqslant t \leqslant 1,\ n \in \mathbb{N}),$$

then $f_n(t) \uparrow f(t)$ a.e. and hence $f_n \Rightarrow f$ [14, Sec. 6.2]. It follows that $\varphi_n(t) \to \varphi(t)$ $(t \in \mathbb{R})$, where φ_n and φ are the characteristic functions of f_n and f, respectively [14, Sec. 6.4]. Since

$\varphi_{K\xi}(t) = \exp(\varphi_\xi(t) - 1)$ for every random variable ξ by [1, 1.6], we have $\varphi_{Kf_n}(t) \to \varphi_{Kf}(t)$ $(t \in \mathbb{R})$, i.e., $Kf_n \Rightarrow Kf$. Furthermore,

$$\|Kf_n\|_{E''} \leqslant 3C\|f_n\|_E \leqslant 3C\|f\|_E \qquad (n \in \mathbb{N})$$

by (21). Thus, using [1, Proposition 1.5] once more, we obtain $\|Kf\|_{E''} \leqslant 3C\|f\|_E$. Since the distribution function of Kf depends only on that of f, it follows from the last inequality that K is a bounded operator from E to E''. If $E = E''$, then everything is done. It remains to consider the case in which $E \neq E''$. In this case, the space E is separable.

First, by using the fact that every function $f \in E''$, $f \geqslant 0$, is the a.e. limit of its truncations $\tilde{f}_n := f\chi_{\{f_n \leqslant n\}}$ $(n \in \mathbb{N})$ and by arguing in the same way as above, one can prove the boundedness of K in E''. Therefore, by [3, Theorem 7.2], the function

$$g(t) := \frac{\ln(e/t)}{\ln(\ln(\ln(a/t)))},$$

where $a > 0$ is sufficiently large, belongs to E''. Now if

$$\psi(u) := \frac{u\ln(e/u)}{\ln(\ln(\ln(a/u)))} \qquad (0 < u \leqslant 1),$$

then $M_\psi \subset E''$. Since E is separable, we have $(M_\psi)_0 \subset (E'')_0 = E_0 = E$. One can readily verify that

$$h(t) := \frac{\ln(e/t)}{\ln(\ln(a/t))} \in (M_\psi)_0$$

and hence $h \in E$. This, together with [3, Th. 4.4], implies that

$$K \colon L_\infty \to E. \tag{22}$$

Now let $f \in E$. Since E is separable, it follows that there exists a sequence $\{f_n\} \subset L_\infty$ such that $\|f_n - f\|_E \to 0$. Since $K \colon E \to E''$, we have $\|Kf_n - Kf\|_{E''} \to 0$. On the other hand, by (22), since the embedding $E \subset E''$ is isometric, we have $\{Kf_n\} \subset E$ and hence $Kf \in E$. □

Remark 11. *It follows from the proof of the theorem that the following estimates hold in every r. i. space E:*

$$\frac{1}{24}\sup_n \|T_n\|_E \leqslant \|K\|_E \leqslant 3\sup_n \|T_n\|_E. \qquad \square$$

Let us give some corollaries of Theorem 10. Let $n \in \mathbb{N}$, let S_n be the set of all permutations of the set $\{1, \dots, n\}$, and let $l = l_n$ be an arbitrary one-to-one mapping of S_n onto $\{1, \dots, n!\}$. Now we extend the operator A_n earlier defined on $\mathbb{R}^n$ by formula (3) to the space of matrices $x = (x_{i,j})_{1 \leqslant i,j \leqslant n}$ as follows:

$$A_n x(t) = \sum_{i=1}^n x_{i,\pi(i)}, \qquad t \in \left(\frac{l(\pi) - 1}{n!}, \frac{l(\pi)}{n!}\right).$$

One major result in [8] (see Corollary 8 there) is that if the sequence $\{A_n\}_{n \geqslant 1}$ of operators is uniformly bounded on the set of diagonal matrices, then it is uniformly bounded on the set of all matrices. By applying Theorem 10, we obtain

Corollary 12. *If an r. i. space $E \in \mathbb{K}$, then*

$$\|A_n x\|_E \leqslant C\left(\left\|\sum_{k=1}^n x_k^* \chi_{(\frac{k-1}{n}, \frac{k}{n})}\right\|_E + \frac{1}{n}\sum_{k=n+1}^{n^2} x_k^*\right)$$

for every $n \in \mathbb{N}$ and every $x = (x_{i,j})_{1 \leqslant i,j \leqslant n}$, where $(x_k^*)_{k=1}^{n^2}$ is the nonincreasing permutation of the sequence $(|x_{i,j}|)_{i,j=1}^n$ and $C > 0$ depends on neither n nor x.

Corollary 13. *The operators T_n, $n \geqslant 1$, are uniformly bounded in the Orlicz space* $\exp L_p$ *if and only if $p \leqslant 1$.*

This assertion follows from Theorem 10 and the fact that the condition $p \leqslant 1$ is equivalent to the boundedness of K in the space $\exp L_p$ [1, 2.4, p. 42].

Theorem 10 and Corollary 3 imply the following assertion.

Corollary 14. *If E is an r. i. space and $\sup_n \|T_n\|_E < \infty$, then there exists an r. i. space $F \subsetneqq E$ such that $\sup_n \|T_n\|_F < \infty$.*

If E is an r. i. space and $p \geqslant 1$, then by $E(p)$ we denote the space of all measurable functions $x = x(t)$ on the interval $[0,1]$ such that $|x|^p \in E$. We equip $E(p)$ with the norm

$$\|x\|_{E(p)} = \| \, |x|^p \, \|_E^{1/p}.$$

It is well known that $E(p) \subset E$ and $\|x\|_E \leqslant \|x\|_{E(p)}$ for all $x \in E(p)$ [9, 1.d].

Let E and F be r. i. spaces such that $E \subset F$ and K is bounded in E. This does not necessarily imply that K is bounded in F [3, Corollaries 5.6 and 5.7]. However, we have the following assertion.

Corollary 15. *If the operator K is bounded in $E(p)$, then it is bounded in E.*

Proof. In view of Theorem 10, it suffices to prove that the uniform boundedness of the operators T_n in $E(p)$ implies their uniform boundedness in E.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x \geqslant 0$, and $\|T_n x\|_{E(p)} \leqslant C\|x\|_{E(p)}$ $(n \in \mathbb{N})$. This means that

$$\|(T_n x)^p\|_E^{1/p} \leqslant C\|x^p\|_E^{1/p}.$$

By setting $x^p = y$, we obtain

$$\|(T_n y^{1/p})^p\|_E \leqslant C^p\|y\|_E.$$

It follows from the definition of T_n that $(T_n y^{1/p})^p \geqslant T_n y$. Hence $\|T_n y\|_E \leqslant C^p\|y\|_E$. Thus, the operators T_n are uniformly bounded in E. $\square$

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