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STRONG EXTRAPOLATION SPACES AND INTERPOLATION

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Abstract: We introduce a new class of rearrangement invariant spaces on the segment $[0,1]$ which contains the most common extrapolation spaces with respect to the L_p-scale as $p \to \infty$. We characterize the class and demonstrate under certain conditions that the Peetre $\mathscr{K}$-functional has extrapolatory description in the couple (E, L_∞) if and only if E belongs to the class. By way of application, we establish a new extrapolation theorem for the bounded operators in L_p.

Keywords: rearrangement invariant space, Lorentz space, Marcinkiewicz space, Orlicz space, operator extrapolation, Peetre $\mathscr{K}$-functional

Introduction

The L_p spaces serve as the most common and important examples of rearrangement invariant or symmetric spaces. At the same time, the norms in many other rearrangement invariant spaces to within equivalence are expressed with the help of L_p-norms. In this sense a conventional example is the Zygmund space $\operatorname{Exp} L^\beta$ ($\beta > 0$) of the measurable functions $x(t)$ on $[0,1]$ with the finite quasinorm

$$\|x\|_{\operatorname{Exp} L^\beta} = \sup_{0<t\le 1} \log_2^{-1/\beta}(2/t) x^*(t) \tag{1}$$

(here and in the sequel $x^*(t)$ stands for a nonincreasing rearrangement of $|x(t)|$). As is well known (for instance, see [1, pp. 22–23] or [2]),

$$\|x\|_{\operatorname{Exp} L^\beta} \asymp \sup_{1<p<\infty} (p^{-1/\beta}\|x\|_{L_p}). \tag{2}$$

These relations are of interest in their own right as regards applications since they allow us to describe the limit spaces in the extrapolation Yano-type theorems [3] applicable when studying some various important operators that are met in analysis [1, 4, 5].

More general spaces and constructions are presented in [6, 7]. In particular, these articles contain an extrapolatory description of the exponential Orlicz spaces $\operatorname{Exp} L^\Phi$ (Φ is an increasing convex function on $[0,\infty)$ and $\Phi(0) = 0$). These spaces consist of the functions $x = x(t)$, $t \in [0,1]$, satisfying

$$\|x\|_{\operatorname{Exp} L^\Phi} = \inf\left\{u > 0 : \int_0^1 \left(\exp \Phi\left(\frac{|x(t)|}{u}\right) - 1\right) dt \le 1\right\} < \infty.$$

Namely, we have the following refinement of (2) [7, Corollary 1]:

$$\|x\|_{\operatorname{Exp} L^\Phi} \asymp \sup_{1<p<\infty} \frac{\|x\|_{\Phi(p)}}{p}. \tag{3}$$

Inspection of these and other examples prompted the authors to introduce the following general definition (see our previous article [8]). A rearrangement invariant space E is called an *extrapolation space*

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(as $p \to \infty$) whenever there exists a Banach ideal space F on $[1,\infty)$ such that $\|x\|_E \asymp \|\|x\|_{L_p}\|_F$. Using this definition, in [8] we find necessary and sufficient conditions for the Marcinkiewicz and Lorentz spaces, which are important in applications, to be extrapolation spaces and study the bounds of the extrapolatory description of rearrangement invariant spaces. Simultaneously, we elucidate a very interesting fact that the norms in many (but not all) extrapolation spaces E admit the above description with a parameter $F = \widetilde{E}$ consisting of the measurable functions $x(u)$ on $[1,\infty)$ such that $\tilde{x}(t) := x(\log_2(2/t))$ $(0 < t \le 1)$ belongs to E; in this case $\|x\|_{\widetilde{E}} = \|\tilde{x}\|_E$. The examples of similar relations are (2) and (3). The main aim of this article is to study the properties of these spaces (we call them *strong extrapolation spaces*).

The article is organized as follows: In Section 1 we exhibit basic definitions and some preliminaries of the theory of rearrangement invariant spaces. The main results are exposed in Section 2. Here we characterize strong extrapolation spaces and prove that a rearrangement invariant space E possesses the above property if and only if $Sx(t) = x(t^2)$ is a bounded operator in E. This criterion can be simplified in one important case that includes the Marcinkiewicz and Lorentz spaces and reduces to the fact that the fundamental function of the space under study satisfies the so-called Δ^2-condition. At the same time, at the end of the section we exhibit an example of an extrapolation space whose fundamental function satisfies this condition and it fails to be a strong extrapolation space. In Section 3 we study a possibility of extrapolatory description of the $\mathscr{K}$-functional $\mathscr{K}(t,f;E,L_\infty)$ under the condition that E is an extrapolation space. First of all, as it is demonstrated here, this description is guaranteed by the fact that E is a strong extrapolation space, which refines the corresponding results in the case of the Orlicz spaces (see [2, 7]). Moreover (what is unexpected), under certain conditions the converse statement is also true and thus E is a strong extrapolation space provided that the $\mathscr{K}$-functional admits extrapolatory description. Finally, in Section 4, we prove an extrapolation theorem that is sharp in a certain sense and generalizes the Yano theorem.

§ 1. The Basic Notation and Preliminaries

In what follows, by an embedding of one Banach space into the other we mean a continuous embedding; i.e., $X_1 \subset X_0$ means that if $x \in X_1$ then $x \in X_0$ and $\|x\|_{X_0} \le C\|x\|_{X_1}$ for some $C > 0$. Writing $X_1 = X_0$ for some Banach spaces, we mean that X_0 and X_1 coincide and the norms of X_0 and X_1 are equivalent.

We consider real functions on the segment $[0,1]$ or the half-axis $[1,\infty)$ with the usual Lebesgue measure μ and assume these functions measurable and finite almost everywhere. As usual, we identify the functions that coincide almost everywhere.

Recall that a function space X is called *ideal* whenever the conditions $y = y(t) \in X$ and $|x(t)| \le |y(t)|$ imply that $x = x(t) \in X$ and $\|x\| \le \|y\|$.

The distribution function $n_x(\tau)$ of a function $x(t)$ is defined as

$$n_x(\tau) = \mu\{t : |x(t)| > \tau\}, \quad \tau > 0.$$

Two functions $x(t)$ and $y(t)$ are called *equimeasurable* if their distribution functions coincide. The *rearrangement* of $x(t)$ is the nonnegative function $x^*(t)$ defined on $[0,\infty)$, equimeasurable with $x(t)$, decreasing, and right continuous. This rearrangement is unique. Moreover, it always exists and can be defined by the formula [9, p. 83]

$$x^*(t) = \inf\{\tau : n_x(\tau) < t\}.$$

In the article we speak of rearrangement invariant function spaces. A detailed exposition of the theory of these spaces can be found in [9, 10]. A Banach ideal space E of measurable functions on $[0,1]$ is called a *rearrangement invariant space* (r.i.s.) if the relations $y = y(t) \in X$ and $n_x(\tau) = n_y(\tau)$ $(\tau > 0)$ imply that $x \in X$ and $\|x\|_E = \|y\|_E$.

The simplest and important example of an r.i.s. is given by the L_p space $(1 \le p \le \infty)$ with the usual norm

$$\|x\|_p = \Bigl(\int_0^1 |x(t)|^p\,dt\Bigr)^{1/p} \quad (1 \le p < \infty) \quad \text{and} \quad \|x\|_\infty = \operatorname*{ess\,sup}_{0\le t\le 1} |x(t)|.$$

If $p > q$ then $L_p \subset L_q$ (with the constant 1). Moreover, L_∞ is the smallest space among all rearrangement invariant spaces on $[0,1]$ and L_1 is the largest [9, Theorem 2.4.1]. The L_p spaces are a particular case of the broad class of the $L_{p,q}$ spaces, in which the following quasinorm is finite

$$\|x\|_{p,q} := \left(\frac{q}{p} \int\limits_0^1 (t^{\frac{1}{p}} x^*(t))^q \, \frac{dt}{t} \right)^{\frac{1}{q}} \ (1 \le q < \infty) \quad \text{and} \quad \|x\|_{p,\infty} = \sup_{0 \le t \le 1} t^{\frac{1}{p}} x^*(t).$$

The other examples of rearrangement invariant spaces are Lorentz and Marcinkiewicz spaces. Let $\varphi(t)$ be an increasing concave function on $[0,1]$ with $\varphi(0) = 0$. The Marcinkiewicz space $M(\varphi)$ comprises all measurable functions $x(t)$ (on $[0,1]$) with the finite norm

$$\|x\|_{M(\varphi)} = \sup_{0 < s \le 1} \frac{\varphi(s)}{s} \int\limits_0^s x^*(t) \, dt.$$

The Lorentz space $\Lambda(\varphi)$ is defined by the norm

$$\|x\|_{\Lambda(\varphi)} = \int\limits_0^1 x^*(t) \, d\varphi(t).$$

An important characteristic of an r.i.s. E is its fundamental function $\phi_E(t) = \|\chi_{(0,t)}\|_E$, where $\chi_{(0,t)}$ designates the characteristic function (indicator) of $(0,t)$. In particular,

$$\phi_{M(\varphi)}(t) = \varphi(t), \ \phi_{\Lambda(\varphi)}(t) = \varphi(t), \text{ and } \phi_{L_{p,q}}(t) = t^{1/p} \quad (1 < p < \infty, \ 1 \le q \le \infty).$$

The space $\Lambda(\varphi)$ is the smallest and $M(\varphi)$, the largest among all rearrangement invariant spaces with the same fundamental function $\varphi(t)$ [9, Theorems 4.5.5 and 4.5.7]. A fundamental function is always quasiconcave (recall that a function $\psi(t)$ on $[0,1]$ is *quasiconcave* if $\psi(0) = 0$, $\psi(t)$ is positive and increases, while $\psi(t)/t$ decreases). A quasiconcave function $\psi(t)$ is equivalent to its least concave majorant $\widetilde{\psi}(t)$; more exactly, $1/2\widetilde{\psi}(t) \le \psi(t) \le \widetilde{\psi}(t)$ $(0 \le t \le 1)$ [9, Theorem 4.1.1].

The dilation function of a positive function $\varphi(t)$, $t \in (0,1]$, is defined as

$$\mathscr{M}_\varphi(s) = \sup_{0 < t \le \min(1, 1/s)} \frac{\varphi(st)}{\varphi(t)}, \quad 0 < s < \infty.$$

The numbers

$$\gamma_\varphi = \lim_{s \to 0+} \frac{\log \mathscr{M}_\varphi(s)}{\log s}, \quad \delta_\varphi = \lim_{s \to \infty} \frac{\log \mathscr{M}_\varphi(s)}{\log s}$$

are called *lower* and *upper dilation indices* of $\varphi(t)$. If φ is quasiconcave then $0 \le \gamma_\varphi \le \delta_\varphi \le 1$ [9, p. 76].

If (X_0, X_1) is a Banach couple (i.e., the Banach spaces X_0 and X_1 are linearly and continuously embedded into some Hausdorff topological vector space) then we can naturally define the intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ with the following norms

$$\|x\|_{X_0 \cap X_1} = \max_{i=0,1} \|x\|_{X_i}$$

and

$$\|x\|_{X_0 + X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i, \, i = 0,1\}.$$

A Banach space X such that $X_0 \cap X_1 \subset X \subset X_0 + X_1$ is called an *interpolation space with respect to* (X_0, X_1) if every linear operator T defined on $X_0 + X_1$ and bounded in X_0 and X_1 is bounded in X. An important role in different questions of the theory of operator interpolation belongs to the Peetre $\mathscr{K}$-functional [11, Section 3.1]:

$$\mathscr{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

For a fixed $t > 0$, it is a norm on the sum $X_0 + tX_1$; and for a fixed $x \in X_0 + X_1$, it is an increasing concave function in t.

If F is a Banach ideal space of functions on $[1, \infty)$ and $w = w(p)$ is a positive function (weight) then the symbol $F(w)$ stands for an ideal space of functions $x = x(p)$ with the norm $\|x\|_{F(w)} = \|x \cdot w\|_F$. At last, the expression $f \asymp g$ means that $cf \le g \le Cf$ for some $c > 0$ and $C > 0$; moreover, these constants are independent of all or some part of arguments of the functions (quasinorms) f and g.

§ 2. Strong Extrapolation Spaces and Their Characterization

Let F be a Banach ideal space of functions on $[1, \infty)$. Denote by $\mathscr{L}_F$ the space of measurable (on $[0,1]$) functions $x(t)$ such that $\xi_x = \xi_x(p) := \|x\|_p \in F$ and endow this space with the norm

$$\|x\|_{\mathscr{L}_F} := \|\xi_x\|_F.$$

It is immediate that $\mathscr{L}_F$ is an r.i.s. on $[0,1]$ and $L_p \supset \mathscr{L}_F \supset L_\infty$ for $1 \le p < \infty$.

Definition 2.1. An r.i.s. E is called an *extrapolation space* (as $p \to \infty$) if there exists a Banach ideal space F on $[1, \infty)$ such that $E = \mathscr{L}_F$ (to within equivalence of norms). In this case we write $E \in \mathscr{E}$.

Define a subclass of $\mathscr{E}$ (see [12]). If E is an r.i.s. on $[0,1]$ then by $\widetilde{E}$ we denote the Banach ideal space of functions on $[1, \infty)$ with the norm

$$\|f\|_{\widetilde{E}} := \|f(\log_2(2/t))\|_E.$$

Definition 2.2. An r.i.s. E is called a *strong extrapolation space* ($E \in \mathscr{S}\mathscr{E}$) if $E = \mathscr{L}_{\widetilde{E}}$ (to within equivalence of norms).

In particular, (2) and (3) show that the Zygmund space $\operatorname{Exp} L^\beta$ ($\beta > 0$) and the exponential Orlicz space $\operatorname{Exp} L^\Phi$ are strong extrapolation spaces.

One of the main results of this article is the following characterization of strong extrapolation spaces.

Theorem 2.3. *For every r.i.s. E on $[0,1]$, the following are equivalent:*
(a) $E \in \mathscr{S}\mathscr{E}$;
(b) $Sx(t) = x(t^2)$ *is a bounded operator in* E.

To prove Theorem 1 and some other statements, we need the following lemma.

Lemma 2.4. *The operator $Sx(t) = x(t^2)$ is bounded in an r.i.s. E if and only if $\|Sx^*\|_E \le C\|x\|_E$ for some $C > 0$ and all $x \in E$.*

Proof. Since necessity is obvious, we prove sufficiency. Let $x \in E$ and $\tau > 0$. If

$$A = \{t \in [0,1] : |x(t)| > \tau\} \text{ and } B = \{t \in [0,1] : |x(t^2)| > \tau\}$$

then $B = \kappa(A)$, with $\kappa(u) = \sqrt{u}$. Hence (for instance, see [13, Lemma 9.5.1]),

$$\mu\{t : |Sx(t)| > \tau\} = \mu(B) = \int_A \frac{du}{2\sqrt{u}} \le \int_0^{\mu\{t : x^*(t) > \tau\}} \frac{du}{2\sqrt{u}} = \mu\{t : Sx^*(t) > \tau\}.$$

Thereby $(Sx)^*(t) \le Sx^*(t)$ and the condition of the lemma implies

$$Sx \in E \text{ and } \|Sx\|_E \le \|Sx^*\|_E \le C\|x\|_E. \quad \square$$

Proof of Theorem 1. Let $x = x(t)$ be measurable on $[0,1]$. Consider the function

$$\tilde{x}(t) := \|x\|_{\log_2(2/t)} \quad (0 < t \le 1).$$

Since

$$\|x\|_p \ge \|x^* \chi_{[0, 2^{-p+1}]}\|_p \ge \frac{1}{2} x^*(2^{-p+1}) \quad (p \ge 1);$$

therefore, $\|\tilde{x}\|_E \ge \frac{1}{2}\|x\|_E$. In view of the definition of strong extrapolation space, the condition (a) of the theorem is equivalent to the inequality

$$\|\tilde{x}\|_E \le C\|x\|_E \tag{4}$$

valid for all $x \in E$ and some $C > 0$.

First, we assume S to be bounded in E. Since $\|S^n x\|_E \le \|S\|^n \|x\|_E$ $(n = 1, 2, \dots)$, we infer

$$\tilde{x}(2^{-p+1}) = \|x\|_p \le 2\|x^* \chi_{[0,2^{-p})}\|_p$$
$$\le 2\sum_{n=0}^{\infty} \|x^* \chi_{[2^{-2^{n+1}p}, 2^{-2^n p})}\|_p \le 2\sum_{n=0}^{\infty} 2^{-2^n} x^*(2^{-2^{n+1}p})$$

for all $p \ge 1$ or equivalently

$$\tilde{x}(2t) \le 2\sum_{n=0}^{\infty} 2^{-2^n} S^{n+1} x^*(t) \quad (0 < t \le 1/2).$$

Hence,

$$\|\tilde{x}(2t)\|_E \le 2\sum_{n=0}^{\infty} 2^{-2^n} \|S\|^{n+1} \|x\|_E \le C_1 \|x\|_E.$$

Since the norm of the dilation operator $\sigma x(t) = x(t/2)$ is bounded in every r.i.s. [9, Theorem 2.4.4], the last inequality implies (4) with $C = 2C_1$.

Let now $E = \mathscr{L}_{\widetilde{E}}$, i.e., the condition $x \in E$ implies $\tilde{x} \in E$ and (4). If $t = 2^{-p+1}$ $(p \ge 1)$ then

$$x^*(t^2) \le x^*(2^{-2p}) \le 4\|x^* \chi_{[0,2^{-2p}]}\|_p \le 4\|x^*\|_p = 4\tilde{x}(t).$$

Therefore, $\|Sx^*\|_E \le 4\|\tilde{x}\|_E \le 4C\|x\|_E$. Applying Lemma 2.4, we conclude that S is bounded in E. $\square$

Corollary 2.5. *An r.i.s. E is a strong extrapolation space if and only if there exists a constant $C > 0$ such that*

$$\left\| \sum_{n=1}^{\infty} \|x^* \chi_{[2^{-2n}, 2^{-2n+1})}\|_n \chi_{[2^{-n}, 2^{-n+1})} \right\|_E \le C\|x\|_E$$

for all $x \in E$.

Proof. First of all, since the dilation operator is bounded in the r.i.s. E [9, Theorem 2.4.4], we obtain

$$\frac{1}{2}\left\| \sum_{k=1}^{\infty} x^*(2^{-k}) \chi_{[2^{-k}, 2^{-k+1}]} \right\|_E \le \|x\|_E \le 2\left\| \sum_{k=1}^{\infty} x^*(2^{-k+1}) \chi_{[2^{-k}, 2^{-k+1}]} \right\|_E.$$

This inequality and the elementary inequality

$$x^*(2^{-k+1}) \le 2\|x^* \chi_{[2^{-k}, 2^{-k+1}]}\|_k \le x^*(2^{-k}) \quad (1 \le k < \infty)$$

imply that

$$\left\| \sum_{k=1}^{\infty} \|x^* \chi_{[2^{-k}, 2^{-k+1}]}\|_k \chi_{[2^{-k}, 2^{-k+1}]} \right\|_E \le \|x\|_E \le 4\left\| \sum_{k=1}^{\infty} \|x^* \chi_{[2^{-k}, 2^{-k+1}]}\|_k \chi_{[2^{-k}, 2^{-k+1}]} \right\|_E. \tag{5}$$

The claim results from the last relation and Theorem 2.3. $\square$

Remark 2.6. While discussing (2) in [14], the attention is payed to the fact that (2) remains valid after the replacement of the L_p-norm of $x(t)$ with the "cut off" L_p-norms $\|x^* \chi_{[2^{-k}, 2^{-k+1}]}\|_k$ $(k = 1, 2, \dots)$. Not diminishing the role of this phenomenon that was used by the authors in the study of some questions in [14, 15], we observe that it has no extrapolation sense since similar relations are valid in any r.i.s. (see (5)). Theorem 2.3 shows that the problem is in the inverse change of the "cut off" L_p-norms by the "complete" (or, as in Corollary 2.5, "shifted" L_p-norms $\|x^* \chi_{[2^{-2k}, 2^{-2k+1})}\|_k$, $k = 1, 2, \dots$).

REMARK 2.7. As is easily seen, we can use the seminorm

$$\|f\|_{\widetilde{E},p_0} = \|f\chi_{[p_0,\infty)}\|_{\widetilde{E}}$$

($p_0 > 1$ is a fixed number) rather than the norm in $\widetilde{E}$ in Theorem 2.3.

REMARK 2.8. Instead of the L_p-scale we can also examine the scale of the $L_{p,q}$ spaces ($1 < p < \infty$) for a fixed $q \in [1,\infty]$ defining the $\mathscr{L}_{F,q}$ space as comprising all functions $x = x(t)$ for which $\xi_{x,q} = \xi_{x,q}(p) := \|x\|_{p,q} \in F$. Applying the arguments of Theorem 2.3, we can demonstrate that the conditions (a) and (b) in the statement of this theorem are equivalent to the following:

(c) $E = \mathscr{L}_{\widetilde{E},q}$ for every $q \in [1,\infty]$ (i.e., $\|x\|_E \asymp \| \|x\|_{\log_2(2/t),q}\|_E$ ($x \in E$));

(d) there exists $q \in [1,\infty]$ such that $E = \mathscr{L}_{\widetilde{E},q}$.

In some important cases the condition (b) ensuring the property of an r.i.s. to be extrapolation space is equivalent to a much simpler condition. Recall [8] that a nonnegative function $\varphi(t)$ on $[0,1]$ meets the Δ^2-condition ($\varphi \in \Delta^2$) whenever there exists $C > 0$ such that

$$\varphi(t) \le C\varphi(t^2), \quad 0 \le t \le 1, \tag{6}$$

for all $0 \le t \le 1$.

Proposition 2.9. *Given an r.i.s. E on $[0,1]$, we consider the following conditions:*

(i) *$Sx(t) = x(t^2)$ is a bounded operator in E,*

(ii) *the fundamental function $\varphi(t)$ of E meets the Δ^2-condition.*

Then (ii) *results from* (i). *Conversely, if E is an interpolation space with respect to the couple* $(\Lambda(\varphi), M(\varphi))$ *of Lorentz and Marcinkiewicz spaces* (*in particular, coincides with $\Lambda(\varphi)$ or $M(\varphi)$*) *then* (i) *results from* (ii).

PROOF. To prove the implication (i)⇒(ii) it suffices to write the condition of boundedness of S for the characteristic functions $\chi_{[0,t]}$ ($0 < t \le 1$).

(ii)⇒(i) For $E = \Lambda(\varphi)$, the boundedness of a linear operator continuous in the space of measurable functions with respect to convergence in measure is equivalent to its boundedness on the collection of characteristic functions [9, Lemma 2.5.2]. For the operator S this is equivalent to (6).

Examine the case of a Marcinkiewicz space. First of all, if $\varphi \in \Delta^2$ then

$$\varphi(t)Sx^*(t) = \varphi(t)x^*(t^2) \le C\varphi(t^2)x^*(t^2) \le C \sup_{0<s\le 1} \varphi(s)x^*(s) \le C\|x\|_{M(\varphi)}$$

for every $x \in M(\varphi)$.

It is immediate that the condition $\varphi \in \Delta^2$ implies that $\delta_\varphi = 0$. Hence, using the last relation and [9, Theorem 2.5.3], we infer

$$\|Sx^*\|_{M(\varphi)} \le C' \sup_{0<t\le 1} Sx^*(t)\varphi(t) \le C'C\|x\|_{M(\varphi)}.$$

Thereby it remains to refer to Lemma 2.4.

At last, if E is an interpolation space with respect to $(\Lambda(\varphi), M(\varphi))$ ($\varphi \in \Delta^2$) then by the previous arguments S is bounded in $\Lambda(\varphi)$ and $M(\varphi)$. Hence, S is bounded in E. □

Theorem 2.3 and the last proposition imply the following result which strengthens Theorem 2.3 in [12].

Theorem 2.10. *Every r.i.s. E which is an interpolation space with respect to the couple* $(\Lambda(\varphi)$, $M(\varphi))$ (*in particular, $E = \Lambda(\varphi)$ or $E = M(\varphi)$*) *is a strong extrapolation space if and only if $\varphi \in \Delta^2$.*

The following statement resulting from the last theorem was established for the first time in [8] (see Theorems 2 and 3).

Corollary 2.11. *If $\varphi \in \Delta^2$ then*

$$\|x\|_{M(\varphi)} \asymp \sup_{1\le p<\infty} \varphi(2^{-p})\|x\|_p \quad \text{and} \quad \|x\|_{\Lambda(\varphi)} \asymp \int_1^\infty \|x\|_p 2^{-p}\varphi'(2^{-p})\,dp.$$

In the concluding part of the section we demonstrate that there exists an r.i.s. failing to be a strong extrapolation space and its fundamental function meets the Δ^2-condition. More exactly, we construct a closed subspace of the Marcinkiewicz space $M(\varphi)$, with $\varphi \in \Delta^2$, where the operator S does not act.

Example 2.12. Let φ be an increasing concave function on $[0,1]$ such that $\varphi(+0) = \varphi(0) = 0$ and $\varphi \in \Delta^2$. Define the sequence $t_n \in [0,1]$ by induction as follows: $t_0 = 1$ and the points t_n are chosen arbitrarily so that

$$\frac{\varphi\big(t_{n-1}^2\big)}{\varphi(t_n)} > n.$$

Clearly, $t_n \to 0$. We put

$$\bar{x}(t) = \sum_{n=1}^\infty \frac{1}{\varphi\big(t_{n-1}^2\big)}\chi_{(t_n^2, t_{n-1}^2]}(t)$$

and $E_0 = \{y \in M(\varphi) :$ there exist $B > 0$ and $\beta > 0$ such that $y^*(t) \le B\bar{x}(\beta t)$ $(0 < t \le 1)\}$. It is easy to see that E_0 is a linear subset of $M(\varphi)$. Define E as the closure of E_0 in $M(\varphi)$. Then E is an r.i.s. endowed with the norm of $M(\varphi)$. In particular, the fundamental function of this space is $\varphi_E(t) = \varphi(t) \in \Delta^2$.

Show now that $x_1(t) := \bar{x}(t^2) \notin E$. It suffices to check that

$$\inf_{y\in E_0} \|x_1 - y\|_{M(\varphi)} > 0. \tag{7}$$

Let $y \in E_0$. Applying the well-known inequality [9, Theorem 2.3.1]:

$$\|a-b\|_{M(\varphi)} \ge \|a^* - b^*\|_{M(\varphi)},$$

we obtain

$$\|x_1 - y\|_{M(\varphi)} \ge \|x_1 - y^*\|_{M(\varphi)} \ge \|(x_1 - y^*)\chi_{(t_n^2, t_n]}\|_{M(\varphi)}. \tag{8}$$

Choose a sufficiently large $n \in \mathbb{N}$ so that

$$\frac{\varphi\big(t_{n-1}^2\big)}{\varphi(t_n)} > 2B \quad \text{and} \quad \Big(1 + \frac{1}{\beta}\Big)t_n^2 \le t_n,$$

where $B > 0$ and $\beta \in (0,1)$ are such that $y^*(t) \le B\bar{x}(\beta t)$. In this case

$$\begin{aligned}
\|(x_1 - y^*)\chi_{(t_n^2,t_n]}\|_{M(\varphi)} &\ge \|(x_1(t_n) - B\bar{x}(\beta t))\chi_{(t_n^2/\beta, t_n]}\|_{M(\varphi)} \\
&= \big(x_1(t_n) - B\bar{x}\big(t_{n-1}^2\big)\big)\varphi(t_n - t_n^2/\beta) \ge \big(\bar{x}\big(t_n^2\big) - B\bar{x}\big(t_{n-1}^2\big)\big)\varphi\big(t_n^2\big) \\
&= \bigg(\frac{1}{\varphi\big(t_n^2\big)} - \frac{B}{\varphi\big(t_{n-1}^2\big)}\bigg)\varphi\big(t_n^2\big) \ge \bigg(\frac{1}{\varphi(t_n)} - \frac{B}{\varphi\big(t_{n-1}^2\big)}\bigg)\varphi\big(t_n^2\big) \\
&= \frac{1}{\varphi(t_n)}\bigg(1 - \frac{B\varphi(t_n)}{\varphi\big(t_{n-1}^2\big)}\bigg)\varphi\big(t_n^2\big) > \frac{\varphi\big(t_n^2\big)}{\varphi(t_n)} \cdot \frac{1}{2} \ge \frac{1}{2C}.
\end{aligned}$$

Hence, (8) implies the inequality

$$\|x_1 - y\|_{M(\varphi)} \ge \frac{1}{2C},$$

where $y \in E_0$, and thus (7) is proven. Now, Theorem 2.3 guarantees that the r.i.s. E is not a strong extrapolation space.

§ 3. Extrapolatory Description of the $\mathscr{K}$-functional for the Banach Couple $(\mathscr{L}_F, L_\infty)$

Theorem 2.3 implies that every r.i.s. presenting an interpolation space with respect to a couple of strong extrapolation spaces (E_0, E_1) is a strong extrapolation space. Here we demonstrate that the $\mathscr{K}$-functional $\mathscr{K}(t, f; E, L_\infty)$, with $E \in \mathscr{S}\mathscr{E}$, has extrapolatory description, thereby generalizing the corresponding earlier results for the case of the exponential Orlicz spaces from [6, 7]. Furthermore, it will be proven that the converse is also true under certain conditions; i.e., if the $\mathscr{K}$-functional admits extrapolatory description then $E \in \mathscr{S}\mathscr{E}$.

Theorem 3.1. *Let E be an r.i.s. on $[0, 1]$.*

(i) *If $E \in \mathscr{S}\mathscr{E}$ then*

$$\mathscr{K}(t, f; E, L_\infty) \asymp \mathscr{K}(t, \|f\|_{\log_2 2/u}; E, L_\infty), \tag{9}$$

with the constants independent of $f \in E$ and $t > 0$.

(ii) *If for some Banach ideal space F on $[1, \infty)$, $F \supset L_\infty$, we have*

$$\mathscr{K}(t, f; E, L_\infty) \asymp \mathscr{K}(t, \|f\|_p; F, L_\infty[1, \infty)), \tag{10}$$

with constants independent of $f \in E$ and $t > 0$, then the fundamental function φ of E satisfies the Δ^2-condition.

To prove the claim, we need several auxiliary assertions. The first of them is actually known; however, we give its proof for the sake of completeness.

Let E be an r.i.s. on $[0, 1]$, $E \neq L_\infty$. Then its fundamental function $\varphi(t)$ satisfies the condition $\lim_{t\to 0+} \varphi(t) = 0$. Without loss of generality, we may assume that $\varphi(t)$ is strictly increasing on $[0, 1]$ and $\varphi(1) = 1$.

Lemma 3.2. *If $f \in E$ and $0 < t \leq 1$ then*

$$\|f^* \chi_{[0,\varphi^{-1}(t)]}\|_E \leq \mathscr{K}(t, f; E, L_\infty) \leq 2\|f^* \chi_{[0,\varphi^{-1}(t)]}\|_E.$$

Proof. Since $\mathscr{K}(t, f; E, L_\infty) = \mathscr{K}(t, f^*; E, L_\infty)$, it suffices to consider the case of $f = f^*$. On the one hand, $f = g + h$, where $g := f\chi_{[0,\varphi^{-1}(t)]}$ and $h := f - g$. Then

$$\mathscr{K}(t, f; E, L_\infty) \leq \|g\|_E + t\|h\|_\infty \leq \|f\chi_{[0,\varphi^{-1}(t)]}\|_E$$
$$+ f(\varphi^{-1}(t))\|\chi_{[0,\varphi^{-1}(t)]}\|_E \leq 2\|f\chi_{[0,\varphi^{-1}(t)]}\|_E,$$

and the right-hand side of the inequality is proven.

Conversely, if $f = f_0 + f_1$, where $f_0 \in E$ and $f_1 \in L_\infty$, then

$$\|f_0\|_E + t\|f_1\|_\infty \geq \|f_0\chi_{[0,\varphi^{-1}(t)]}\|_E + \|f_1\|_\infty\|\chi_{[0,\varphi^{-1}(t)]}\|_E$$
$$\geq \|(f_0 + f_1)\chi_{[0,\varphi^{-1}(t)]}\|_E = \|f\chi_{[0,\varphi^{-1}(t)]}\|_E.$$

Taking the infimum over all representations, we obtain the left-hand side of the inequality. □

Let now $E \in \mathscr{E}$, i.e., $E = \mathscr{L}_F$, with F a Banach ideal space on $[1, \infty)$. If $E \neq L_\infty$ then there exists $f \in E$ such that $\lim_{p\to+\infty} \|f\|_p = \infty$. Hence, the function

$$\psi(u) := \|\chi_{[u,\infty)}\|_F \quad (u \geq 1) \tag{11}$$

enjoys the relation $\lim_{u\to+\infty} \psi(u) = 0$. Without loss of generality, we may assume that ψ is strictly decreasing on $[1, \infty)$ and $\psi(1) = 1$.

Lemma 3.3. *If $a \in F$ is nonnegative monotonically increasing on* $[1, \infty)$ *and* $0 < t \le 1$ *then*

$$\|a\chi_{[\psi^{-1}(t),\infty)}\|_F \le \mathscr{K}(t, a; F, L_\infty[1,\infty)) \le 2\|a\chi_{[\psi^{-1}(t),\infty)}\|_F.$$

Proof. Consider the representation

$$a = b + c, \quad \text{where } b = a\chi_{[\psi^{-1}(t),\infty)},\ c = a - b.$$

Since $a(s)$ is nonnegative and increasing, by the definition of $\mathscr{K}$-functional we see that

$$\begin{aligned}\mathscr{K}(t, a; F, L_\infty[1,\infty)) &\le \|b\|_F + t\|c\|_{L_\infty[1,\infty)} \le \|a\chi_{[\psi^{-1}(t),\infty)}\|_F + ta(\psi^{-1}(t))\\ &\le \|a\chi_{[\psi^{-1}(t),\infty)}\|_F + a(\psi^{-1}(t))\|\chi_{[\psi^{-1}(t),\infty)}\|_F \le 2\|a\chi_{[\psi^{-1}(t),\infty)}\|_F.\end{aligned}$$

Conversely, if $a = a_0 + a_1$, with $a_0 \in F$ and $a_1 \in L_\infty[1,\infty)$, then

$$\begin{aligned}\|a_0\|_F + t\|a_1\|_{L_\infty[1,\infty)} &\ge \|a_0\chi_{[\psi^{-1}(t),\infty)}\|_F + \|a_1\|_{L_\infty[1,\infty)}\|\chi_{[\psi^{-1}(t),\infty)}\|_F\\ &\ge \|(a_0 + a_1)\chi_{[\psi^{-1}(t),\infty)}\|_F = \|a\chi_{[\psi^{-1}(t),\infty)}\|_F.\end{aligned}$$

Taking the infimum over all representations on the left, we arrive to the needed inequality. $\square$

In what follows, a connection between the fundamental function φ of the extrapolation space $E = \mathscr{L}_F$ and the function ψ defined by (11) plays an important role. In particular, if $E \in \mathscr{S}\mathscr{E}$ then

$$\psi(u) = \|\chi_{[u,\infty)}\|_{\widetilde{E}} = \|\chi_{[u,\infty)}(\log_2(2/s))\|_E = \varphi(2^{1-u}) \quad (u \ge 1);$$

hence,

$$\varphi(t) = \psi(\log_2(2/t)) \quad (0 < t \le 1). \tag{12}$$

In the general case the following claim is valid: for every $h \in (0, 1)$, there exists $A = A(h) > 0$ such that

$$\psi(h\log_2(2/t)) \le A\varphi(t), \quad \text{if } 0 < t \le 2^{1-1/h}. \tag{13}$$

Indeed, if $p \ge h\log_2(2/t)$ then $t^{1/p} \ge t^{\frac{1}{h\log_2(2/t)}} \ge 2^{-1/h}$ and the definitions of φ and ψ imply that $\varphi(t) \ge C^{-1}2^{-1/h}\psi(h\log_2(2/t))$, where C is the constant of equivalence of the norms in E and $\mathscr{L}_F$. Thereby the relation (13) with $A = 2^{1/h}C$ is true.

To prove the second part of Theorem 3.1, we will use the next statement.

Lemma 3.4. *Let $E = \mathscr{L}_F$, where F is a Banach ideal space on* $[1, \infty)$, *and let φ be the fundamental function of E. Then the following are equivalent:*

(a) *there exists $C_1 > 0$ such that*

$$\varphi(t) \le C_1\|t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),\infty)}\|_F \tag{14}$$

for all $0 < t \le 1$*;*

(b) *there exist $C_2 > 0$ and $0 < h < 1$ such that*

$$\varphi(t) \le C_2\psi(h\log_2(2/t)) \tag{15}$$

for all $0 < t \le 2^{1-1/h}$.

Proof. First, we assume that (b) holds. If $0 < t \le 2^{1-1/h}$ then the definition of ψ yields

$$\begin{aligned}\varphi(t) &\le C_2\|\chi_{[h\log_2(2/t),\infty)}\|_F \le 2^{1/h}C_2\|t^{\frac{1}{h\log_2(2/t)}}\chi_{[h\log_2(2/t),\infty)}\|_F\\ &\le 2^{1/h}C_2\|t^{1/p}\chi_{[h\log_2(2/t),\infty)}\|_F.\end{aligned}$$

Moreover, ψ^{-1} is decreasing and (13) implies that

$$\psi^{-1}(A\varphi(t)) \le h\log_2\frac{2}{t} \quad \text{if } 0<t\le 2^{1-1/h}.$$

From here and the last relation we infer

$$\varphi(t) \le 2^{1/h}C_2\|t^{1/p}\chi_{[\psi^{-1}(A\varphi(t)),\infty)}\|_F.$$

At the same time, Lemma 3.3 and the concavity of the $\mathscr{K}$-functional yield

$$\begin{aligned}&\|t^{1/p}\chi_{[\psi^{-1}(A\varphi(t)),\infty)}\|_F \le \mathscr{K}(A\varphi(t),t^{1/p};F,L_\infty[1,\infty))\\ &\le A\mathscr{K}(\varphi(t),t^{1/p};F,L_\infty[1,\infty)) \le 2A\|t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),\infty)}\|_F.\end{aligned}$$

Thus, (14) is proven for $0<t\le 2^{1-1/h}$ and thus for all $0<t\le 1$.

Examine the converse implication (a)$\Rightarrow$(b). Let h be an arbitrary number from $(0,1/2)$. In this case if $0<t\le 2^{1-1/h}$ then $t<1/2$. Thereby $\frac{\log_2 t}{1-\log_2 t}\le -\frac{1}{2}$ and $t^{h^{-1}\log_2^{-1}(2/t)}\le 2^{-(2h)^{-1}}$. Hence, assuming that $h\log_2(2/t)>\psi^{-1}(\varphi(t))$, we obtain

$$\|t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),h\log_2(2/t))}\|_F \le t^{h^{-1}\log_2^{-1}(2/t)}\|\chi_{[\psi^{-1}(\varphi(t)),\infty)}\|_F \le 2^{-(2h)^{-1}}\varphi(t).$$

Thus, if $h\in(0,1/2)$ is so that $2C_1 2^{-(2h)^{-1}}<1$ then (14) yields

$$\begin{aligned}&\varphi(t)\le C_1\|t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),\infty)}\|_F\\ &\le C_1\|t^{1/p}\chi_{[\psi^{-1}(\varphi(t)),h\log_2(2/t))}\|_F + C_1\|t^{1/p}\chi_{[h\log_2(2/t),\infty)}\|_F\\ &\le C_1 2^{-\frac{1}{2h}}\varphi(t)+C_1\psi(h\log_2(2/t)) \le (1/2)\varphi(t)+C_1\psi(h\log_2(2/t));\end{aligned}$$

hence,

$$\varphi(t)\le 2C_1\psi(h\log_2(2/t)) \quad (0<t\le 2^{1-1/h}).$$

Since the last inequality is a direct consequence of (14) in the case of $h\log_2(2/t)\le\psi^{-1}(\varphi(t))$, the lemma is proven. $\square$

Proof of Theorem 3.1. Since for $E=L_\infty$ both statements of the theorem are obvious, we confine exposition to the case of $E\ne L_\infty$. Thereby we may assume that the fundamental function φ of E satisfies the conditions stated before Lemma 3.2. Since $\varphi(1)=1$, we have $\|f\|_E\le\|f\|_\infty$ $(f\in L_\infty)$ and thus

$$\mathscr{K}(t,f;E,L_\infty)=\|f\|_E \quad (t\ge 1). \tag{16}$$

Moreover, $\|a\|_{\widetilde{E}}\le\|a\|_{L_\infty[1,\infty)}$ $(a\in L_\infty[1,\infty))$ and, therefore, if $a\in\widetilde{E}$ then

$$\mathscr{K}(t,a;\widetilde{E},L_\infty[1,\infty))=\|a\|_{\widetilde{E}} \quad (t\ge 1).$$

This relation and (16) imply that under the condition $E=\mathscr{L}_{\widetilde{E}}$ we have

$$\mathscr{K}(t,f;E,L_\infty)\asymp\mathscr{K}(t,\|f\|_p;\widetilde{E},L_\infty[1,\infty)) \quad (t\ge 1).$$

Thus, in view of Lemmas 3.2 and 3.3 combined with (12), the statement (i) results from the relation

$$\left\|\Bigg(\int\limits_0^{\varphi^{-1}(t)} f^*(s)^p\,ds\Bigg)^{1/p}\right\|_{\widetilde{E}} \asymp \|\|f\|_p\chi_{[V(t),\infty)}\|_{\widetilde{E}} \quad (0<t\le 1), \tag{17}$$

where $V(t):=\log_2(2/\varphi^{-1}(t))$.

First of all, if $p \ge V(t)$ then

$$\left(\int_0^{\varphi^{-1}(t)} f^*(s)^p\,ds\right)^{1/p} \ge \varphi^{-1}(t)^{1/p}\|f\|_p \ge \varphi^{-1}(t)^{1/V(t)}\|f\|_p.$$

Since $\frac{\log_2 z}{1-\log_2 z} \ge -1$ $(0 < z \le 1)$, the definition of $V(t)$ and the previous relation yield

$$\left(\int_0^{\varphi^{-1}(t)} f^*(s)^p\,ds\right)^{1/p} \ge \frac{1}{2}\|f\|_p \quad (p \ge V(t));$$

hence,

$$\left\|\left(\int_0^{\varphi^{-1}(t)} f^*(s)^p\,ds\right)^{1/p}\right\|_{\widetilde{E}} \ge \frac{1}{2}\|\|f\|_p\chi_{[V(t),\infty)}\|_{\widetilde{E}} \quad (0 < t \le 1). \tag{18}$$

To prove the reverse inequality, we may assume that

$$\{s \in [0,1] : f^*(s) > 0\} \subset [0, \varphi^{-1}(t)]. \tag{19}$$

If $2^{1-p} \le \varphi^{-1}(t)$ (which is equivalent to $p \ge V(t)$) then

$$f^*(2^{1-p}) \le 2\left(\int_0^{2^{1-p}} f^*(s)^p\,ds\right)^{1/p} \le 2\|f\|_p\chi_{[V(t),\infty)}(p).$$

In view of (19),

$$\|f\|_E = \|f^*(2^{1-p})\|_{\widetilde{E}} \le 2\|\|f\|_p\chi_{[V(t),\infty)}\|_{\widetilde{E}}.$$

On the other hand, since E is a strong extrapolation space,

$$\left\|\left(\int_0^{\varphi^{-1}(t)} f^*(s)^p\,ds\right)^{1/p}\right\|_{\widetilde{E}} \le C\|f\|_E.$$

The last two relations yield

$$\left\|\left(\int_0^{\varphi^{-1}(t)} f^*(s)^p\,ds\right)^{1/p}\right\|_{\widetilde{E}} \le 2C\|\|f\|_p\chi_{[V(t),\infty)}\|_{\widetilde{E}}$$

which along with (18) implies (17). Thus, (i) is proven.

Proceed with the proof of (ii). First of all, since $F \supset L_\infty[1,\infty)$, we have

$$\mathscr{K}(t, a; F, L_\infty[1,\infty)) \asymp \|a\|_F \quad (t \ge 1)$$

for $a \in F$. This relation, (16), and (10) imply that $E = \mathscr{L}_F$. Hence, in view of Lemmas 3.2 and 3.3 (we may assume that the function ψ defined by (11) satisfies the conditions before the last of them) and the relations (10), we infer

$$\left\|\left(\int_0^{\varphi^{-1}(t)} f^*(s)^p\,ds\right)^{1/p}\right\|_F \asymp \|\|f\|_p\chi_{[\psi^{-1}(t),\infty)}\|_F \quad (0 < t \le 1).$$

Choosing $f = \chi_{[0,u)}$ and $t = \varphi(u)$, where $0 < u \le 1$, we conclude that

$$\varphi(u) \asymp \|u^{1/p}\|_F \le C\|u^{1/p}\chi_{[\psi^{-1}(\varphi(u)),\infty)}\|_F.$$

Thus, Lemma 3.4 yields (15). This fact and (13) imply that

$$\begin{aligned}\varphi(u) &\le C_2\psi(h\log_2(2/u)) = C_2\psi((h/2)\log_2(2/u)^2)\\ &\le C_2\psi((h/2)\log_2(2/u^2)) \le C_2A(h/2)\varphi(u^2)\end{aligned}$$

for some $h \in (0,1)$ and all $0 < u \le 2^{1/2-1/h}$, i.e., $\varphi \in \Delta^2$. The theorem is proven. $\square$

As is known, there exist Marcinkiewicz spaces $M(\varphi) \in \mathscr{E}$ such that $\varphi \notin \Delta^2$ [8, Example 1]. At the same time, it follows from Theorems 2.10 and 3.1 that

Corollary 3.5. *If an r.i.s. E is an interpolation space with respect to a Banach couple $(\Lambda(\varphi), M(\varphi))$ (in particular, if $E = \Lambda(\varphi)$ or $E = M(\varphi)$) then the following are equivalent:*

(i) *there exists a Banach ideal space F on $[1,\infty)$ such that $F \supset L_\infty[1,\infty)$ and*

$$\mathscr{K}(t, f; E, L_\infty) \asymp \mathscr{K}(t, \|f\|_p; F, L_\infty[1,\infty)),$$

with constants independent of $f \in E$ and $t > 0$;

(ii) $\varphi \in \Delta^2$.

In both cases the relation from the condition (i) *is fulfilled for $F = \widetilde{E}$.*

Example 3.6. Let E be an exponential Orlicz space $\operatorname{Exp} L^\Phi$ constructed on using $e^{\Phi(u)} - 1$, with Φ an increasing convex function on the half-axis $(0,\infty)$ and $\Phi(0) = 0$. As is well known [16, 17], $\operatorname{Exp} L^\Phi = M(\varphi)$ (to within equivalence of the norms), where $\varphi(u) = 1/\Phi^{-1}(\log_2 2/u)$. Since the function Φ^{-1} is concave then

$$\Phi^{-1}(\log_2(2/u^2)) \le \Phi^{-1}(2\log_2(2/u)) \le 2\Phi^{-1}(\log_2(2/u)),$$

$\varphi(u) \le 2\varphi(u^2)$, and thus $\varphi \in \Delta^2$. Therefore, $\operatorname{Exp} L^\Phi \in \mathscr{S}\mathscr{E}$ and

$$\mathscr{K}(t, f; \operatorname{Exp} L^\Phi, L_\infty) \asymp \|\|f\|_p\chi_{[\psi^{-1}(t),\infty)}\|_F \quad (0 < t \le 1).$$

In this case $\psi^{-1}(t) = \Phi(1/t)$ $(0 < t \le 1)$ and, as it was demonstrated in [8], $F = L_\infty(\varphi(2^{-u}))$. Thereby

$$\mathscr{K}(t, f; \operatorname{Exp} L^\Phi, L_\infty) \asymp \sup_{p \ge \Phi(1/t)} \frac{\|f\|_p}{\Phi^{-1}(p)} \quad (0 < t \le 1)$$

or

$$\mathscr{K}(t, f; L_\infty, \operatorname{Exp} L^\Phi) \asymp t\mathscr{K}(1/t, f; \operatorname{Exp} L^\Phi, L_\infty) \asymp t \sup_{p \ge \Phi(t)} \frac{\|f\|_p}{\Phi^{-1}(p)} \quad (t \ge 1).$$

The last relation coincides with the statement of Proposition 1 in [7] (also see Theorem 2 in [6]) to within change of the variable.

In the power case, i.e. $\Phi(t) = t^\beta$, we arrive at the classical Zygmund space $\operatorname{Exp} L^\beta$ for which a similar relation

$$\mathscr{K}(t, f; L_\infty, \operatorname{Exp} L^\beta) \asymp t \sup_{p \ge t^\beta} \frac{\|f\|_p}{p^{1/\beta}} = t \sup_{p \ge t} \frac{\|f\|_{p^\beta}}{p} \quad (t \ge 1)$$

is valid for all $\beta > 0$ [2, Theorem 2].

§ 4. A New Extrapolation Theorem

As usual, let $x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds$. Given an r.i.s. E, define $E(\log^{-1})$ as follows: this space consists of the functions $x(t)$ with $x^{**}(t)\log_2^{-1} 2/t \in E$ and is endowed with the norm $\|x\|_{E(\log^{-1})} = \|x^{**}(t)\log_2^{-1}(2/t)\|_E$. These spaces arise in the theory of interpolation of weak type operators [9, Chapter II, Section 6] as well as in studying various questions of the geometry of rearrangement invariant spaces (for instance, see [18, Theorem 2.17]).

First, we demonstrate that if E is a strong extrapolation space then the replacement of two "stars" with one "star" in the norm of $E(\log^{-1})$ leads to an equivalent functional.

Proposition 4.1. *Let $E \in \mathscr{S}\mathscr{E}$. Then $E(\log^{-1})$ consists of all functions $x(t)$ such that*

$$x^*(t)\log_2^{-1}(2/t) \in E$$

and

$$\|x\|_{E(\log^{-1})} \asymp \|x^* \log_2^{-1}(2/t)\|_E. \tag{20}$$

Proof. Since $x^{**}(t)$ decreases, we obtain the inequality

$$\begin{aligned}\big\|x^{**}(t)\log_2^{-1}(2/t)\cdot\chi_{[1/2,1]}\big\|_E &\le 2\|x^{**}(t)\cdot\chi_{[1/2,3/4]}\|_E \\ &\le 6\big\|x^{**}(t)\log_2^{-1}(2/t)\cdot\chi_{[1/4,1/2]}\big\|_E.\end{aligned}$$

Hence,

$$\begin{aligned}\|x\|_{E(\log^{-1})} &\le \big\|x^{**}(t)\log_2^{-1}(2/t)\cdot\chi_{[0,1/2]}\big\|_E + \big\|x^{**}(t)\log_2^{-1}(2/t)\cdot\chi_{[1/2,1]}\big\|_E \\ &\le 7\big\|x^{**}(t)\log_2^{-1}(2/t)\cdot\chi_{[0,1/2]}\big\|_E.\end{aligned}$$

Assume further that $t \in \big[0, \frac{1}{2}\big]$. In this case

$$x^{**}(t) = \frac{1}{t}\sum_{n=0}^{\infty}\int_{t^{2^{n+1}}}^{t^{2^n}} x^*(s)\,ds \le \frac{1}{t}\sum_{n=0}^{\infty} x^*(t^{2^{n+1}})t^{2^n} = \sum_{n=0}^{\infty} x^*(t^{2^{n+1}})t^{2^n-1}.$$

Hence,

$$\begin{aligned}x^{**}(t)\log_2^{-1}(2/t) &\le \sum_{n=0}^{\infty} x^*(t^{2^{n+1}})\log_2^{-1}\left(\frac{2}{t^{2^{n+1}}}\right)\frac{\log_2(2/t^{2^{n+1}})}{\log_2(2/t)}t^{2^n-1} \\ &\le \sum_{n=0}^{\infty} S^{n+1}\left(x^*(t)\log_2^{-1}\left(\frac{2}{t}\right)\right)2^{n+1}\left(\frac{1}{2}\right)^{2^n-1},\end{aligned}$$

where as before $Sx(t) = x(t^2)$. By Theorem 2.3, S is bounded in E and so

$$\begin{aligned}\|x\|_{E(\log^{-1})} &= \big\|x^{**}(t)\log_2^{-1}(2/t)\big\|_E \\ &\le 7\left(\sum_{n=0}^{\infty}\|S\|^{n+1}2^{n+2-2^n}\right)\big\|x^*(t)\log_2^{-1}(2/t)\big\|_E = C\big\|x^*(t)\log_2^{-1}(2/t)\big\|_E.\end{aligned}$$

In view of the inequality $x^*(t) \le x^{**}(t)$, the reverse inequality is obvious and the claim is proven. □

Corollary 4.2. *Under the conditions of the previous proposition, $E(\log^{-1})$ is a strong extrapolation space. Moreover, we have $E(\log^{-1}) = \mathscr{L}_{\widetilde{E}(1/p)}$, where $\|f\|_{\widetilde{E}(1/p)} := \|f/p\|_{\widetilde{E}}$.*

Proof. The relation (20) indicates that the boundedness of S in E implies its boundedness in $E(\log^{-1})$. Therefore, by Theorem 2.3 this space is also a strong extrapolation space. The last assertion follows form the equality $\widetilde{E(\log^{-1})} = \widetilde{E}(1/p)$ which can be checked directly. □

Theorem 4.3. *Let T be a bounded linear operator in L_p for all $p \geq p_0 \geq 1$ such that for some $C > 0$*

$$\|T\|_{L_p \to L_p} \leq Cp \quad (p \geq p_0). \tag{21}$$

Then T is bounded from E into $E(\log^{-1})$ for all $E \in \mathscr{S}\mathscr{E}$.

Furthermore, this statement is sharp in the following sense: there exists a linear operator T_0 satisfying (21) and such that its boundedness from $E \in \mathscr{S}\mathscr{E}$ into an r.i.s. F implies the inclusion

$$E(\log^{-1}) \subset F. \tag{22}$$

Proof. The estimate (21), Corollary 4.2, and Remark 2.7 yield

$$\|Tx\|_{E(\log^{-1})} \leq C_1 \|\|Tx\|_p/p\|_{\widetilde{E},p_0} \leq C_2 \|\|x\|_p\|_{\widetilde{E},p_0} \leq C_3 \|x\|_E.$$

As T_0 we take the adjoint of the Hardy–Littlewood operator, i.e.,

$$T_0 x(t) = \int_t^1 \frac{x(s)ds}{s}.$$

This operator is bounded in L_p for $1 \leq p < \infty$ and its norm equals $\|T_0\|_{L_p \to L_p} = p$. In accord with Theorem 4.3 T_0 is bounded from E into $E(\log^{-1})$ for all $E \in \mathscr{S}\mathscr{E}$. We now prove (22) assuming that T_0 is bounded as an operator from E into an r.i.s. F.

Let $x \in E(\log^{-1})$. For some $y \in E$, we have

$$x^*(t) \leq y(t) \log_2(2/t) \quad (0 < t \leq 1)$$

and the properties of rearrangements [9, p. 93] yield

$$x^*(t) \leq (y(t) \log_2(2/t))^* \leq y^*(t/2) \log_2(4/t) \leq z^*(t) \log_2(2/t),$$

where $z \in E$. Put $z_1(t) = 8z^*(t^2)$. Since E is a strong extrapolation space, by Theorem 2.3 $z_1 \in E$ and

$$T_0 z_1(t) \geq \int_t^{\sqrt{t}} \frac{z_1(s)ds}{s} \geq z_1(\sqrt{t}) \int_t^{\sqrt{t}} \frac{ds}{s} = 4z^*(t) \log \frac{1}{t} \geq z^*(t) \log_2(2/t) \cdot \chi_{[0,1/2]}(t) \geq x^*(t)\chi_{[0,1/2]}(t).$$

If T_0 acts from E into F then $T_0 z_1 \in F$ and $x \in F$ due to the previous inequality, i.e., (22) is fulfilled. □

Remark 4.4. If $E = L_\infty$ then (see (1)) the space $E(\log^{-1})$ coincides with the Zygmund space $\operatorname{Exp} L^1$. In this case Theorem 4.3 implies the claim of the classical Yano theorem; i.e., if (21) holds then T is bounded from L_∞ into $\operatorname{Exp} L^1$.

Remark 4.5. In one particular case the statement of Theorem 4.3 is a consequence of the following interpolation theorem (see [19]). Let $1 < p < \infty$. Assume that an r.i.s. E is an interpolation space between L_1 and L_∞ and $\log_2(2/t) \in E$. Then every linear operator A of weak type $(1,1)$ bounded in L_p is bounded from $E(\log)$ with the norm $\|x\|_{E(\log)} = \|x^*(t) \log_2(2/t)\|_E$ into E.

If now $T = A^*$ (i.e., T is the adjoint of A) then under the conditions of this theorem T is bounded from E' into $E'(\log^{-1})$, where E' is the r.i.s. associated with E [9, p. 65]. Observe that by the Marcinkiewicz interpolation theorem [11, Theorem 1.3.1] the norm of this operator satisfies the estimate (21). At the same time, the conditions of this theorem are much more restrictive and it does not "cover" the statement of Theorem 4.3. To justify this fact, we may use the construction that is briefly described in [20, Section 5.9]. Let $\psi(p)$ $(1 < p < 3/2)$ be an arbitrary function such that $\lim_{p\to 1} \psi(p) = \infty$. Then there exists a convolution operator A bounded from L_p into L_2 for all $p \in (1, 3/2)$, $\|A\|_{L_p \to L_p} \leq \|A\|_{L_p \to L_2} \leq \psi(p)$ but not of the weak type $(1,1)$. In this case, if $\psi(p) \leq C(p-1)^{-1}$ for some $C > 0$, then $T = A^*$ satisfies the conditions of Theorem 4.3 but fails to satisfy the conditions of the corresponding theorem in [19].

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