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Independent functions in rearrangement invariant spaces and the Kruglov property

S. V. Astashkin

Abstract. Let X be a separable or maximal rearrangement invariant space on $[0,1]$. It is shown that the inequality

$$\Bigl\|\sum_{k=1}^{\infty} f_k\Bigr\|_X \leqslant C\Bigl\|\Bigl(\sum_{k=1}^{\infty} f_k^2\Bigr)^{1/2}\Bigr\|_X$$

holds for an arbitrary sequence of independent functions $\{f_k\}_{k=1}^{\infty} \subset X$, $\int_0^1 f_k(t)\,dt = 0$, $k=1,2,\dots$, if and only if X has the Kruglov property. As a consequence, it is proved that the same property is necessary and sufficient for a version of Maurey's well-known inequality for vector-valued Rademacher series with independent coefficients to hold in X.

Bibliography: 24 titles.

§ 1. Introduction

Let $r_k(t) = \operatorname{sign}\sin 2^k\pi t$, $k=1,2,\dots$, be the Rademacher functions on the interval $[0,1]$. By Khintchine's classical inequality (see [1]), for each $p>0$ there exists a positive constant C_p such that for arbitrary real a_k, $k=1,2,\dots$, we have

$$\Bigl\|\sum_{k=1}^{\infty} a_k r_k\Bigr\|_{L_p} \leqslant C_p\Bigl(\sum_{k=1}^{\infty} a_k^2\Bigr)^{1/2}.$$

This result prompted a great number of studies and generalizations and found many applications in various areas of analysis and function theory (see, for instance, [2]–[4]). In particular, in 1975, Rodin and Semenov proved that the inequality

$$\Bigl\|\sum_{k=1}^{\infty} a_k r_k\Bigr\|_{X} \leqslant C\Bigl(\sum_{k=1}^{\infty} a_k^2\Bigr)^{1/2} \tag{1}$$

holds in a rearrangement invariant space X on $[0,1]$ if and only if X contains the space G, the closure of L_∞ in the Orlicz space L_{N_2} defined by the function $N_2(u) = e^{u^2}-1$ (see [5]). In this paper we solve a similar problem in the case

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when arbitrary independent mean-zero functions are considered in place of scalar multiples of Rademacher functions. More precisely, we find a description of the rearrangement invariant spaces X such that the following inequality holds with some $C>0$ for all sequences of independent functions $\{f_k\}_{k=1}^{\infty}\subset X$, with $\int_0^1 f_k(t)\,dt=0$, $k=1,2,\dots$:

$$\Bigl\|\sum_{k=1}^{\infty} f_k\Bigr\|_X \leqslant C\Bigl\|\Bigl(\sum_{k=1}^{\infty} f_k^2\Bigr)^{1/2}\Bigr\|_X. \tag{2}$$

We point out that if the class of sequences of independent mean-zero functions is replaced by the wider class of martingale differences, then the answer to this question is known: a necessary and sufficient condition on X is that the lower Boyd index of this space be positive: $\alpha_X>0$ (see [6]).

In the case when, in place of the entire class of martingale differences, we consider only independent functions with mean value zero we shall see that another property comes to the forefront. It was introduced and actively studied by Braverman [7], who called it the Kruglov property because it was inspired by some probabilistic constructions due to Kruglov [8]. The Kruglov property proved to be very useful in the study of geometric properties of rearrangement invariant spaces [7], for instance, Rosenthal's inequality [9] and its generalizations [10], [11]. Simplifying matters slightly we can say that the Kruglov property is a property of spaces positioned sufficiently 'far away' from L_∞, the minimal rearrangement invariant space on $[0,1]$. For instance, a rearrangement invariant space X has the Kruglov property ($X\in\mathbb{K}$) if $X\supset L_p$ for some $p<\infty$ (and, in particular, if $\alpha_X>0$). More interestingly, however, the same holds for some spaces that are 'much closer' to L_∞: for instance, for the Orlicz exponential spaces L_{N_γ} with $N_\gamma(u)=e^{u^\gamma}-1$, provided that $0<\gamma\leqslant 1$.

The central result of this paper (Theorem 1) shows that the condition $X\in\mathbb{K}$ is necessary and sufficient in order that we have Khintchine's generalized inequality (2) in a separable or maximal rearrangement invariant space X. Thus, in view of the previous observation, this inequality also holds for some spaces containing no L_p with $p<\infty$. The condition $X\in\mathbb{K}$, which is less restrictive than the positiveness of the lower Boyd index, on the one hand demonstrates the difference between properties of independent mean-zero functions and general martingale differences. On the other hand it is stronger than the condition $X\supset G$, which ensures that inequality (1) holds (a special case of (2) for scalar Rademacher series).

Finally we give a brief description of the structure and the contents of this paper. In § 2 we introduce the definitions and notation used in what follows. In § 3 we obtain necessary and sufficient conditions ensuring inequality (2) holds in a rearrangement invariant space X (we even have a slightly more general inequality, in which the norm l_2 on the right-hand side is replaced by the norm of a symmetric sequence space $E\neq l_1$). In § 4 we show that the Kruglov property is also useful in the investigation of vector-valued Rademacher series with independent coefficients (Theorem 2). Finally, in § 5 we prove Theorem 3, which, in particular, yields necessary and sufficient conditions for a certain weaker inequality of type (2). These conditions use the notion of Kruglov operator, introduced and analyzed in [12] (see also [13]).

§ 2. Definitions and notation

2.1. Rearrangement invariant spaces. A Banach space $(X,\|\cdot\|_X)$ of real-valued Lebesgue-measurable functions on an interval $[0,\alpha)$, $0<\alpha\leqslant\infty$, is called *rearrangement invariant* (symmetric) if whenever $y\in X$ and $x^*(t)\leqslant y^*(t)$ ($t\in[0,\alpha)$), it follows that $x\in X$ and $\|x\|_X\leqslant\|y\|_X$. Here and throughout λ is the Lebesgue measure and $x^*(t)$ the right-continuous non-increasing *rearrangement* of $|x(s)|$:

$$x^*(t)=\inf\bigl\{\tau\geqslant 0:\lambda(\{s\in[0,\alpha):|x(s)|>\tau\})\leqslant t\bigr\},\qquad t>0.$$

If X is a rearrangement invariant (r.i.) space on an interval $[0,\alpha)$, then the *dual* (or *associated*) space X' consists of all measurable functions y such that

$$\|y\|_{X'}:=\sup\biggl\{\int_0^\alpha|x(t)y(t)|\,dt:x\in X,\ \|x\|_X\leqslant 1\biggr\}<\infty.$$

If X^* is the dual of X, then $X'\subset X^*$, and $X'=X^*$ if and only if X is separable. The natural embedding of X in its second dual X'' is an isometric surjection if and only if X is *maximal* (or has the *Fatou property*): if $\{f_n\}_{n=1}^\infty\subseteq X$, $f\in S[0,\alpha)$, $f_n\to f$ a.e. on $[0,\alpha)$ and $\sup_n\|f_n\|_X<\infty$, then $f\in X$ and $\|f\|_X\leqslant\liminf_{n\to\infty}\|f_n\|_X$. In the case when an r.i. space X is separable or maximal, the natural embedding of X in its second dual is an isometry.

For each r.i. space X on $[0,1]$ we have the continuous embeddings

$$L_\infty\subseteq X\subseteq L_1.$$

In what follows we denote by X_0 the closure of L_∞ in X. If $X\neq L_\infty$, then X_0 is separable. Moreover, $(X'')_0=X_0$.

If X is an r.i. space, then the function $\varphi_X(t):=\|\chi_A\|_X$, where $A\subset(0,\infty)$, $\lambda(A)=t$, and χ_A is the indicator of A, is called the *fundamental function* of X. Other important characteristics of an r.i. space are its Boyd indices. Recall their definition for spaces on $[0,1]$. If $\tau>0$, then the dilation operator $\sigma_\tau x(t):=x(t/\tau)\chi_{(0,\min(1,\tau))}(t)$ is bounded in each r.i. space X (see [14], Theorem 2.4.4). The quantities

$$\alpha_X:=\lim_{\tau\to 0}\frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln\tau}\,,\qquad \beta_X:=\lim_{\tau\to\infty}\frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln\tau}$$

are called the *Boyd indices* of the space X; we always have $0\leqslant\alpha_X\leqslant\beta_X\leqslant 1$.

The L_p-spaces, $1\leqslant p\leqslant\infty$, are important examples of rearrangement invariant spaces as are their generalization, the Orlicz spaces. Let Φ be an Orlicz function on $[0,\infty)$, that is, a continuous convex increasing function on $[0,\infty)$ such that $\Phi(0)=0$ and $\Phi(\infty)=\infty$. Then the *Orlicz space* $L_\Phi=L_\Phi[0,1]$ consists of all measurable functions f on $[0,1]$ with finite norm

$$\|f\|_{L_\Phi}=\inf\biggl\{\rho>0:\int_0^1\Phi\biggl(\frac{|f(t)|}{\rho}\biggr)\,dt\leqslant 1\biggr\}.$$

Other examples of rearrangement invariant spaces are Lorentz and Marcinkiewicz spaces. Let ψ be an increasing concave function on $[0,1]$, with $\psi(0)=0$. Then the *Marcinkiewicz space* $M(\psi)=M(\psi)[0,1]$ (or the *Lorentz space* $\Lambda(\psi)=\Lambda(\psi)[0,1]$) consists of all measurable functions $x(t)$ on $[0,1]$ that have a finite norm

$$\|x\|_{M(\psi)}=\sup_{0<s\leqslant 1}\frac{\psi(s)}{s}\cdot\int_0^s x^*(t)\,dt\quad\biggl(\text{correspondingly,}\ \ \|x\|_{\Lambda(\psi)}=\int_0^1 x^*(t)\,d\psi(t)\biggr).$$

It can easily be verified that $\varphi_{\Lambda(\psi)}(t)=\varphi_{M(\psi)}(t)=\psi(t)$. Furthermore, the space $\Lambda(\psi)$ is the narrowest, while $M(\psi)$ is the broadest rearrangement invariant space with fundamental function $\psi(t)$ (see [14], Theorems 2.5.5 and 2.5.7).

Following [15], § 2f, for each r.i. space X on $[0,1]$ we set

$$Z_X^2:=\bigl\{f\in(L_1+L_\infty)(0,\infty):\|f\|_{Z_X^2}:=\|f^*\chi_{[0,1]}\|_X+\|f^*\chi_{[1,\infty)}\|_{L_2[1,\infty)}<\infty\bigr\}.$$

It can easily be shown that the quasinorm $\|\cdot\|_{Z_X^2}$ is equivalent to a norm, so that Z_X^2 is an r.i. space on $[0,\infty)$. If $\{f_k\}_{k=1}^\infty$ is a sequence of functions on $[0,1]$, then throughout we denote by $\{\bar f_k\}_{k=1}^\infty$ a sequence of disjoint functions on $[0,\infty)$ that are equidustributed with the f_k; for instance,

$$\bar f_k(t)=f_k(t-k+1)\chi_{[k-1,k)}(t),\qquad k=1,2,\dots .$$

For greater detail on rearrangement invariant spaces see [14]–[16].

2.2. The Kruglov property and several definitions from probability theory. Let f be a measurable function on $[0,1]$ (a random variable). Then we denote by $\pi(f)$ the random variable $\sum_{i=1}^N f_i$, where the f_i are independent copies of f (that is, independent random variables equidistributed with f) and N is a random variable independent of the sequence $\{f_i\}$ and having the Poisson distribution with parameter 1, that is,

$$\mathsf{P}\{N=k\}=\frac{1}{e\cdot k!}\,,\qquad k=0,1,2,\dots .$$

We can give another (equivalent) definition of the random variable $\pi(f)$ by expressing its characteristic function $\theta_{\pi(f)}$ in terms of the characteristic function θ_f of f:

$$\theta_{\pi(f)}(t)=\exp(\theta_f(t)-1),\qquad t\in\mathbb{R}.$$

The following property has its origin in Kruglov's paper [8] and was actively studied and used by Braverman (see [7]).

Definition. We say that an r.i. space X on $[0,1]$ has the *Kruglov property* ($X\in\mathbb{K}$) if it follows from the relation $f\in X$ that $\pi(f)\in X$.

As already mentioned in the introduction, this is a property of spaces sufficiently 'remote' from L_∞, first of all, of spaces containing some L_p with $p<\infty$. However, this is not necessary: for instance, the Orlicz exponential space L_{N_γ} belongs to $\mathbb{K}$ if $0<\gamma\leqslant 1$.

Recall the definition of the operator $\mathscr{K}$ introduced in [12] (see also [13]) and closely connected with the Kruglov property (it is called the Kruglov operator in [12] for this reason). Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint measurable subsets of $[0,1]$ and let $\lambda(B_n) = 1/(e\cdot n!)$. If $f \in L_1[0,1]$, then let

$$\mathscr{K}f(\omega_0,\omega_1,\dots) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} f(\omega_k)\chi_{B_n}(\omega_0).$$

Then $\mathscr{K}$ is a positive linear operator, $\mathscr{K}\colon L_1[0,1] \to L_1(\Omega,P)$, where $(\Omega,P) = \prod_{n=0}^{\infty}([0,1],\lambda_n)$ and λ_n is the Lebesgue measure on $[0,1]$. Since the random variables $\mathscr{K}f$ and $\pi(f)$ are equidustributed (see [12]), we can regard $\mathscr{K}f$ as an explicit representation of $\pi(f)$. In particular, an r.i. space X belongs to $\mathbb{K}$ if and only if $\mathscr{K}$ is bounded as an operator from X into $X(\Omega,P)$ (see [12], Lemma 3.3). Another (equivalent) description of the operator $\mathscr{K}$ (see [12]) is sometimes more convenient. If $f \in L_1[0,1]$, $\{B_n\}$ is the same sequence of subsets of $[0,1]$ and if the functions in the system $f_{n,1}, f_{n,2},\dots,f_{n,n}$, χ_{B_n} are independent for each $n\in\mathbb{N}$ and satisfy the equalities $f_{n,k}^* = f^*$ for all $n\in\mathbb{N}$ and $k=1,2,\dots,n$, then $\mathscr{K}f(t)$ can be defined as a certain rearrangement of the function

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{n,k}(t)\chi_{B_n}(t), \qquad 0\leqslant t\leqslant 1.$$

Note also that the condition that $\mathscr{K}$ is a bounded operator from an r.i. space X into an r.i. space Y is sufficient (and also necessary under certain fairly broad assumptions) for the equivalence of the sums of positive independent variables in Y and disjoint functions in X equidistributed with them (see [12], Theorem 3.5).

We shall require in what follows also a definition of the weak convergence of random variables. We say that a sequence of random variables $\{f_k\}_{k=1}^{\infty}$ is *weakly convergent* to a random variable f if for each bounded continuous function g on $\mathbb{R}$ we have $\lim_{n\to\infty}\mathsf{E}g(f_n) = \mathsf{E}g(f)$, where we set

$$\mathsf{E}h = \int_0^1 h(t)\,dt.$$

As is well known (see [17], Ch. 6, § 4, Theorem 3), the weak convergence of $\{f_k\}$ to f is equivalent to the pointwise convergence of the corresponding characteristic functions, that is, to the relation $\theta_{f_n}(t)\to\theta_f(t)$, $t\in\mathbb{R}$. Finally, as usual, we set $\operatorname{supp} f = \{t : f(t)\neq 0\}$.

§ 3. Khintchine's generalized inequality in rearrangement invariant spaces

We start with the central result of this paper: necessary and sufficient conditions for an r.i. space X so that inequality (2) holds for each sequence of independent functions $\{f_k\}_{k=1}^{\infty}\subset X$, $\int_0^1 f_k(t)\,dt = 0$, $k=1,2,\dots$. In fact, we shall even consider a slightly more general situation, when the l_2-norm on the right-hand side of (2) is replaced by the norm of an arbitrary symmetric sequence space $E\neq l_1$.

Theorem 1. *Let X be an arbitrary separable or maximal r.i. space on* $[0,1]$. *Then the following conditions are equivalent*:

(a) $X \in \mathbb{K}$;

(b) *there exists* $C > 0$ *such that for each sequence of independent functions* $\{f_k\}_{k=1}^{\infty} \subset X$, $\int_0^1 f_k(t)\,dt = 0$, $k = 1, 2, \dots$,

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_X \leqslant C \Big\| \sum_{k=1}^{\infty} \bar{f}_k \Big\|_{Z_X^2}; \tag{3}$$

(c) *there exists* $C > 0$ *such that for each sequence of independent functions* $\{f_k\}_{k=1}^{\infty} \subset X$, $\int_0^1 f_k(t)\,dt = 0$, $k = 1, 2, \dots$,

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_X \leqslant C \Big\| \Big(\sum_{k=1}^{\infty} f_k^2 \Big)^{1/2} \Big\|_X; \tag{4}$$

(d) *there exist a symmetric sequence space* $E \neq l_1$ *and a positive constant* C *such that for each sequence of independent functions* $\{f_k\}_{k=1}^{\infty} \subset X$, $\int_0^1 f_k(t)\,dt = 0$, $k = 1, 2, \dots$,

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_X \leqslant C \big\| \|(f_k)\|_E \big\|_X; \tag{5}$$

(e) *there exist a symmetric sequence space* $E \neq l_1$ *and a positive constant* C *such that* (5) *holds for each sequence of independent functions* $\{f_k\}_{k=1}^{\infty} \subset X$, $\int_0^1 f_k(t)\,dt = 0$, $k = 1, 2, \dots$, *such that*

$$\sum_{k=1}^{\infty} \lambda(\operatorname{supp} f_k) \leqslant 1; \tag{6}$$

(f) *there exists* $C > 0$ *such that relation* (3) *or equivalently*,

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_X \leqslant C \Big\| \sum_{k=1}^{\infty} \bar{f}_k \Big\|_X \tag{7}$$

holds for each sequence of independent functions $\{f_k\}_{k=1}^{\infty} \subset X$ *satisfying* (6).

We require two lemmas for the proof, the first of which we prove in a slightly more general form, so that we can use it again in the last section of this paper.

Lemma 1. *Let X and Y be r.i. spaces on* $[0,1]$ *and let E be a symmetric sequence space*, $E \neq l_1$. *Assume that there exists* $C > 0$ *such that*

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_Y \leqslant C \big\| \|(f_k)\|_E \big\|_X \tag{8}$$

for each sequence of symmetrically distributed independent functions $\{f_k\}_{k=1}^{\infty} \subset X$ *satisfying* (6).

Then a similar inequality (*not necessarily with the same constant*) *holds also for an arbitrary sequence of independent functions satisfying condition* (6).

Proof. We observe first that $Y \neq L_\infty$. In fact, let $\{a_k\}$ and $\{b_k\}$ be disjoint subsets of $[0,1]$ and let $\lambda(a_k) = \lambda(b_k) = 2^{-k}$, $k = 1, 2, \dots$. Consider the sequence of independent functions

$$f_k(t) = \begin{cases} 1 & \text{if } t \in a_k, \\ -1 & \text{if } t \in b_k, \\ 0 & \text{if } t \in [0,1] \setminus (a_k \cup b_k), \end{cases} \qquad k = 1, 2, \dots .$$

On the one hand, for each $n \in \mathbb{N}$ we have $\left\| \sum_{k=1}^n f_k \right\|_{L_\infty} = n$. On the other hand, if e_k are the standard basis vectors in the space of sequences and

$$\varphi(n) = \left\| \sum_{k=1}^{n} e_k \right\|_E, \qquad n \in \mathbb{N}, \tag{9}$$

then in each r.i. space X we have

$$\big\| \|(f_k)_{k=1}^n\|_E \big\|_X \leqslant \big\| \|(f_k)_{k=1}^n\|_E \big\|_{L_\infty} = \varphi(n).$$

Since E is symmetric and $E \neq l_1$, it follows that

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = 0, \tag{10}$$

so that relation (8) fails for this sequence $\{f_k\}$ if $Y = L_\infty$.

We see that $Y \neq L_\infty$ and therefore $\lim_{t\to 0+} \varphi_Y(t) = 0$, where φ_Y is the fundamental function of the space Y. Thus, there exists $a \in (0, 1/2]$ such that

$$\frac{\|\chi_{[0,a)}\|_Y}{\|\chi_{[0,1)}\|_Y} \leqslant \frac{1}{2}\,. \tag{11}$$

Assume first that the sequence $\{f_n\}_{n=1}^\infty$ satisfies the condition

$$\sum_{n=1}^{\infty} \lambda(\operatorname{supp} f_n) \leqslant a. \tag{12}$$

Suppose that f_n' has the same distribution as f_n, $n = 1, 2, \dots$, and, in addition, the family of functions $\{f_n, f_m'\}_{n,m=1}^\infty$ is jointly independent. Then the functions $h_n := f_n - f_n'$ are independent, symmetrically distributed and, as $a \leqslant 1/2$, it follows from (12) that

$$\sum_{k=1}^{\infty} \lambda(\operatorname{supp} h_n) \leqslant 2 \sum_{n=1}^{\infty} \lambda(\operatorname{supp} f_n) \leqslant 2a \leqslant 1,$$

that is, the sequence $\{h_n\}$ satisfies (6). Hence

$$\left\| \sum_{n=1}^{\infty} h_n \right\|_Y \leqslant C \big\| \|(h_n)\|_E \big\|_X \tag{13}$$

by assumption.

Now using Proposition 1.11 from [7], the equality $h_n = f_n - \mathsf{E}f_n - (f'_n - \mathsf{E}f_n)$ and relation (12) we obtain

$$\Big\|\sum_{n=1}^{\infty} h_n\Big\|_Y \geqslant c\Big\|\sum_{n=1}^{\infty} f_n - \mathsf{E}\Big(\sum_{n=1}^{\infty} f_n\Big)\Big\|_Y \geqslant c\Big\|\sum_{n=1}^{\infty} f_n - \mathsf{E}\Big(\sum_{n=1}^{\infty} f_n\Big)\chi_{[0,a]}\Big\|_Y$$
$$\geqslant c\bigg(\Big\|\sum_{n=1}^{\infty} f_n\Big\|_Y - \mathsf{E}\Big|\sum_{n=1}^{\infty} f_n\Big| \cdot \|\chi_{[0,a]}\|_Y\bigg).$$

Since in each r.i. space Z on $[0,1]$ we have

$$\|g\|_Z \geqslant \|\chi_{[0,1]}\|_Z \cdot \int_0^1 |g(t)|\,dt, \qquad g \in Z \tag{14}$$

(see [14], Theorem 2.4.1), it follows from (11) that

$$\Big\|\sum_{n=1}^{\infty} h_n\Big\|_Y \geqslant c\Big\|\sum_{n=1}^{\infty} f_n\Big\|_Y \cdot \bigg(1 - \frac{\|\chi_{[0,a]}\|_Y}{\|\chi_{[0,1]}\|_Y}\bigg) \geqslant \frac{c}{2}\Big\|\sum_{n=1}^{\infty} f_n\Big\|_Y.$$

Thus, by (13) we obtain

$$\Big\|\sum_{n=1}^{\infty} f_n\Big\|_Y \leqslant \frac{2}{c}\Big\|\sum_{n=1}^{\infty} h_n\Big\|_Y \leqslant \frac{2C}{c}\big\|\|(h_n)\|_E\big\|_X \leqslant \frac{4C}{c}\big\|\|(f_n)\|_E\big\|_X,$$

and the proof in the case (12) is complete.

Now let $\{f_n\}_{n=1}^{\infty} \subset X$ be an arbitrary sequence of independent functions such that (6) holds. We can assume without loss of generality that $\lambda(\{t : f_n(t) = b\}) = 0$ for each $b \in \mathbb{R}$ and $n = 1, 2, \dots$. Then for $l = [1/a] + 1$ there exist quantities $-\infty = b_0 < b_1 < \dots < b_l = \infty$ such that

$$\sum_{i=1}^{\infty} \lambda\big(\{t \in [0,1] : b_{k-1} < f_i(t) < b_k\}\big) \leqslant a, \qquad k = 1, 2, \dots, l.$$

Let

$$f_{i,k}(t) := f_i(t)\chi_{\{b_{k-1} < f_i < b_k\}}(t), \qquad i = 1, 2, \dots, \qquad k = 1, 2, \dots, l.$$

Clearly, $f_i = \sum_{k=1}^{l} f_{i,k}$ and for each $k = 1, 2, \dots, l$ the sequence of independent functions $\{f_{i,k}\}_{i=1}^{\infty} \subset X$ satisfies condition (12). Hence, using the case discussed before and the inequalities $|f_{i,k}| \leqslant |f_i|$, $k = 1, 2, \dots, l$, we obtain

$$\Big\|\sum_{i=1}^{\infty} f_i\Big\|_Y \leqslant \sum_{k=1}^{l}\Big\|\sum_{i=1}^{\infty} f_{i,k}\Big\|_Y \leqslant \frac{4C}{c}\sum_{k=1}^{l}\big\|\|(f_{i,k})_{i=1}^{\infty}\|_E\big\|_X$$
$$\leqslant \frac{4C}{c}\bigg(\bigg[\frac{1}{a}\bigg] + 1\bigg)\big\|\|(f_i)\|_E\big\|_X,$$

which proves Lemma 1.

The next result can be deduced by means of general arguments relating to interpolation theory for linear operators (see, for instance, [18], § 12); however, we give a direct proof for completeness.

Lemma 2. *Let E be a symmetric sequence space, $E \neq l_1$. Then there exists a positive homogeneous function $\psi(u,v)$ of degree one, increasing in each variable and satisfying the relation $\lim_{u\to 0+}\psi(u,v)=0$ such that for all $a=(a_k)_{k=1}^\infty \in l_1$,*

$$\|a\|_E \leqslant \psi(\|a\|_{l_\infty}, \|a\|_{l_1}). \tag{15}$$

Proof. Let $\varphi(n)$ be the fundamental function of the space E defined by (9). Consider the piecewise linear function on $[0,\infty)$ that is equal to 0 for $t=0$ and to $\varphi(n)$ for $t=n$; we shall also denote it by $\varphi(t)$. Since $\varphi(n)$ is increasing and $\varphi(n)/n$ is decreasing, $\varphi(t)$ has similar properties: it is an increasing function, but $\varphi(t)/t$ decreases for $t\geqslant 0$. Hence if $\tilde\varphi(t)$ is the smallest concave majorant of $\varphi(t)$, then by Theorem 2.1.1 in [14],

$$\varphi(t) \leqslant \tilde\varphi(t) \leqslant 2\varphi(t), \qquad t>0. \tag{16}$$

Finally, let $\psi(u,v) := u\tilde\varphi(u)$, $u,v>0$. It can easily be verified that $\psi(u,v)$ is a positive homogeneous function of degree one, which is increasing with respect to each variable. Moreover, by (16) and (10),

$$\lim_{u\to 0+}\psi(u,v) = v\lim_{t\to\infty}\frac{\tilde\varphi(t)}{t} = \lim_{n\to\infty}\frac{\varphi(n)}{n} = 0.$$

We conclude by proving (15). Let $a=(a_k)_{k=1}^\infty \in l_1$. We can assume without loss of generality that $a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$. For each $n\in\mathbb{N}$ let $a^n=\sum_{k=1}^n a_k e_k$. Since

$$a^n = \sum_{k=1}^n (a_k - a_{k+1})\sum_{i=1}^k e_i, \qquad a_{n+1}=0$$

and $\tilde\varphi(t)$ is concave, using relation (16) and the definition and properties of the function ψ, therefore

$$\begin{aligned}\|a^n\|_E &\leqslant \sum_{k=1}^n (a_k-a_{k+1})\varphi(k) \leqslant a_1\sum_{k=1}^n \frac{a_k-a_{k+1}}{a_1}\,\tilde\varphi(k)\\ &\leqslant a_1\tilde\varphi\biggl(\sum_{k=1}^n \frac{a_k-a_{k+1}}{a_1}k\biggr) = \psi\biggl(a_1, \sum_{k=1}^n (a_k-a_{k+1})k\biggr)\\ &\leqslant \psi(\|a\|_{l_\infty}, \|a\|_{l_1}).\end{aligned}$$

Since $\|a^n-a\|_{l_1}\to 0$, it follows that $\|a^n-a\|_E\to 0$, therefore passing to the limit as $n\to\infty$ in the last inequality we obtain (15). The proof of Lemma 2 is complete.

Proof of Theorem 1. (a) $\Rightarrow$ (b). This is proved in [19], Theorem 1.

(b) $\Rightarrow$ (c). Recall that $\bar f_k(t) = f_k(t-k+1)\chi_{[k-1,k)}(t)$, $t>0$, and Z_X^2 is an r.i. space on $(0,\infty)$ with quasinorm

$$\|g\|_{Z_X^2} := \|g^*\chi_{[0,1]}\|_X + \|g^*\chi_{[1,\infty)}\|_{L_2[1,\infty)}. \tag{17}$$

If $\bar f = \sum_{k=1}^\infty \bar f_k$, then it follows from [20], Theorem 1 that

$$\|\bar f^*\chi_{[0,1]}\|_X + \biggl(\sum_{k=1}^\infty \bar f^*(k)^2\biggr)^{1/2} \leqslant C\biggl\|\biggl(\sum_{k=1}^\infty f_k^2\biggr)^{1/2}\biggr\|_X. \tag{18}$$

For each measurable function g on the half-axis we have

$$\|g^*\chi_{[1,\infty)}\|_{L_2[1,\infty)} \leqslant \left(\sum_{k=1}^{\infty} g^*(k)^2\right)^{1/2},$$

therefore it follows from (17) and (18) that

$$\left\|\sum_{k=1}^{\infty} \bar{f}_k\right\|_{Z_X^2} \leqslant C\left\|\left(\sum_{k=1}^{\infty} f_k^2\right)^{1/2}\right\|_X.$$

Hence (3) immediately yields (4) and the proof of (c) is complete.

The implications (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (e) are obvious.

(e) $\Rightarrow$ (f). By Lemma 1 inequality (5) holds for an arbitrary sequence of independent functions satisfying (6). Applying this inequality to the functions $|f_k|$, $k = 1, 2, \ldots$, while taking (15) into account we obtain

$$\left\|\sum_{k=1}^{\infty} |f_k|\right\|_X \leqslant C\big\|\|(f_k)\|_E\big\|_X \leqslant C\big\|\psi\big(\|(f_k)\|_{l_\infty}, \|(f_k)\|_{l_1}\big)\big\|_X. \tag{19}$$

By properties of ψ, for arbitrary $x, y \in X$ and $s, t > 0$ we have

$$\psi(x,y) \leqslant \max\left(\frac{x}{s}, \frac{y}{t}\right)\psi(s,t) \leqslant \left(\frac{x}{s} + \frac{y}{t}\right)\psi(s,t),$$

therefore taking s and t such that $\|x\|/s = \|y\|/t$ we obtain

$$\|\psi(x,y)\| \leqslant \left(\frac{\|x\|}{s} + \frac{\|y\|}{t}\right)\psi(s,t) \leqslant 2\max\left(\frac{\|x\|}{s}, \frac{\|y\|}{t}\right)\psi(s,t) = 2\psi(\|x\|, \|y\|).$$

Thus, it follows from (19) that

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} |f_k|\right\|_X &\leqslant 2C\psi\big(\big\|\|(f_k)\|_{l_\infty}\big\|_X, \big\|\|(f_k)\|_{l_1}\big\|_X\big) \\ &= 2C\left\|\sum_{k=1}^{\infty} |f_k|\right\|_X \psi\left(\frac{\big\|\|(f_k)\|_{l_\infty}\big\|_X}{\big\|\|(f_k)\|_{l_1}\big\|_X}, 1\right),\end{aligned}$$

and therefore

$$\psi\left(\frac{\big\|\|(f_k)\|_{l_\infty}\big\|_X}{\big\|\|(f_k)\|_{l_1}\big\|_X}, 1\right) \geqslant \frac{1}{2C}.$$

By Lemma 2, $\lim_{u\to 0+}\psi(u,v) = 0$, so it follows from the last inequality that

$$\left\|\sum_{k=1}^{\infty} |f_k|\right\|_X \leqslant C_1\left\|\sup_{k=1,2,\ldots} |f_k|\right\|_X$$

for some $C_1 > 0$.

Now, it is well known (see, for instance, [21], Proposition 1) that if (6) holds, then in an arbitrary r.i. space X we have

$$\Big\| \sup_{k=1,2,\dots} |f_k| \Big\|_X \leqslant \Big\| \sum_{k=1}^{\infty} \bar{f}_k \Big\|_X \leqslant 2 \Big\| \sup_{k=1,2,\dots} |f_k| \Big\|_X .$$

Hence it follows from the preceding inequality that

$$\Big\| \sum_{k=1}^{\infty} |f_k| \Big\|_X \leqslant C_1 \Big\| \sum_{k=1}^{\infty} \bar{f}_k \Big\|_X ,$$

which proves (f).

(f) $\Rightarrow$ (a). First of all, by Theorem 3.5 in [12] the Kruglov operator $\mathscr{K}$ is bounded from X to its second dual X''. We claim that in this case we have

$$\mathscr{K} : X'' \to X'' \tag{20}$$

(that is, $X'' \in \mathbb{K}$). For an arbitrary function $f \in X''$ we set

$$f_n := f \cdot \chi_{\{|f| \leqslant n\}}, \qquad n \in \mathbb{N}.$$

Then $f_n \to f$ a.e. on $[0,1]$ and $f_n \in L_\infty \subset X$. Since $\mathscr{K} : X \to X''$, $|f_n| \leqslant |f|$ and X is isometrically embedded in X'' by assumption, it follows that

$$\|\mathscr{K} f_n\|_{X''} \leqslant C \|f_n\|_X \leqslant C \|f\|_{X''}. \tag{21}$$

By the definition of $\mathscr{K}$, $\mathscr{K} f_n \to \mathscr{K} f$ a.e. Hence it follows from (21) and the properties of X'' that $\mathscr{K} f \in X''$ and $\|\mathscr{K} f\|_{X''} \leqslant C \|f\|_{X''}$, which proves (20).

Now if X is maximal (that is, $X = X''$), then the proof is complete. Assume that X is separable. By (20) and Theorem 7.2 of [12],

$$h_1(u) := \frac{\ln(e/u)}{\ln(\ln(\ln \alpha/u))} \in X'', \quad \text{where } \ln\ln\ln \alpha > e,$$

that is, the Marcinkiewicz space $M(\psi)$ generated by

$$\psi(t) = \frac{\ln(\ln(\ln \alpha/t))}{\ln(e/t)}$$

is embedded in X''. Since X is separable, it follows that

$$(M(\psi))_0 \subset (X'')_0 = X_0 = X.$$

Obviously,

$$h_2(u) := \frac{\ln(e/u)}{(\ln(\ln \alpha/u))}$$

is a function in $(M(\psi))_0$, therefore $h_2 \in X$ by this embedding. Hence it follows from Theorem 4.4 in [12] that

$$\mathscr{K} : L_\infty \to X. \tag{22}$$

Let $f \in X$. Since X is separable, there exists $\{g_n\} \subset L_\infty$ such that $\|g_n - f\|_X \to 0$. Thus, $\|\mathscr{K} g_n - \mathscr{K} f\|_{X''} \to 0$ by (20). On the other hand we see from (22) that $\{\mathscr{K} g_n\} \subset X$, and since X is isometrically embedded in X'', it follows that $\|\mathscr{K} g_n - \mathscr{K} f\|_X \to 0$ and $\mathscr{K} f \in X$. The proof of Theorem 1 is complete.

Remark 1. As seen from the proof of Theorem 1 and also from Theorem 3.5 in [12], (a)–(f) are equivalent to either of the following conditions:

(g) there exists $C > 0$ such that (4) holds for each sequence of symmetrically distributed independent functions $\{f_k\}_{k=1}^{\infty} \subset X$ satisfying condition (6);

(h) there exists $C > 0$ such that (7) holds for each sequence of equimeasurable independent functions $\{f_k\}_{k=1}^{\infty} \subset X$ satisfying condition (6).

At the same time, the simple example of $f_k(t) = 1$, $k = 1, 2, \dots$, demonstrates that the assumption that there exists $C > 0$ such that (4) holds for each sequence of equimeasurable independent functions $\{f_k\}_{k=1}^{\infty} \subset X$ is never fulfilled in a rearrangement invariant space.

Remark 2. If we ask for which r.i. space of functions X and symmetric sequence space E (5) holds only in the case when the f_k, $k = 1, 2, \dots$, are scalar multiples of Rademacher functions, then the answer is quite different. Namely, it was shown in [22] that the two-sided estimate

$$C^{-1}\|(a_k)\|_E \leqslant \left\|\sum_{k=1}^{\infty} a_k r_k\right\|_X \leqslant C\|(a_k)\|_E$$

holds in an r.i. space X on $[0,1]$ if and only if the sequence space E is an interpolation space for the couple (l_1, l_2).

§ 4. Vector-valued Rademacher series with independent coefficients

Here we apply Theorem 1 to series of the form

$$F(s,t) = \sum_{k=1}^{\infty} f_k(s) r_k(t), \qquad s, t \in [0,1],$$

where $\{f_k\}$ is some sequence of independent functions and the r_k are Rademacher functions. If X is an r.i. space on $I = [0,1]$, then $X(I \times I)$ is the corresponding r.i. space on the square $I \times I$ with norm $\|f\|_{X(I\times I)} = \|f^*\|_X$.

Theorem 2. *If an r.i. space X is separable or maximal, then the following conditions are equivalent*:

(α) $X \in \mathbb{K}$;

(β) *there exists $C > 0$ such that for each sequence of independent functions $\{f_k\}_{k=1}^{\infty} \subset X$*,

$$C^{-1}\left\|\left(\sum_{k=1}^{\infty} f_k^2\right)^{1/2}\right\|_X \leqslant \left\|\sum_{k=1}^{\infty} f_k(s) r_k(t)\right\|_{X(I\times I)} \leqslant C\left\|\left(\sum_{k=1}^{\infty} f_k^2\right)^{1/2}\right\|_X; \tag{23}$$

(γ) *there exists $C > 0$ such that for each sequence of independent functions $\{f_k\}_{k=1}^{\infty} \subset X$*,

$$C^{-1}\left\|\left(\sum_{k=1}^{\infty} f_k^2\right)^{1/2}\right\|_X \leqslant \int_0^1 \left\|\sum_{k=1}^{\infty} f_k(s) r_k(t)\right\|_X dt \leqslant C\left\|\left(\sum_{k=1}^{\infty} f_k^2\right)^{1/2}\right\|_X. \tag{24}$$

Proof. First of all we observe that the left-hand inequalities in (23) and (24) hold in each r.i. space. For example, we shall verify this for (23). Since $\{f_k(s)r_k(t)\}_{k=1}^\infty$ is an unconditional sequence with constant 1 in $X(I\times I)$ (see [7], Proposition 1.14) and in view of Khintchine's inequality for L_1 with the optimal constant from [23], it follows that

$$\begin{aligned}\Big\|\sum_{k=1}^\infty f_k(s)r_k(t)\Big\|_{X(I\times I)} &= \int_0^1\Big\|\sum_{k=1}^\infty f_k(s)r_k(t)r_k(u)\Big\|_{X(I\times I)}\,du\\ &\geqslant \Big\|\int_0^1\Big|\sum_{k=1}^\infty f_k(s)r_k(t)r_k(u)\Big|\,du\Big\|_{X(I\times I)} \geqslant \frac{1}{\sqrt{2}}\Big\|\Big(\sum_{k=1}^\infty f_k^2\Big)^{1/2}\Big\|_X.\end{aligned}$$

The proof of the left-hand side of (24) is completely similar.

Now since the $f_k(s)r_k(t)$, $k=1,2,\dots$, are independent functions and

$$\int_0^1\int_0^1 f_k(s)r_k(t)\,ds\,dt=0,\qquad k=1,2,\dots,$$

it follows from Theorem 1 that the right-hand inequality in (23) is an immediate consequence of (α). We now prove the right-hand inequality in (24). Let $\{f_k\}_{k=1}^\infty\subset X$ be a sequence of independent functions. Then the functions

$$g_k(s)=f_k(s)-\int_0^1 f_k(t)\,dt,\qquad k=1,2,\dots,$$

are also independent. Furthermore,

$$\int_0^1 g_k(t)\,dt=0,\qquad k=1,2,\dots.$$

Applying Theorem 1 to the sequence $\{g_k(\,\cdot\,)r_k(t)\}_{k=1}^\infty$ for each $t\in[0,1]$ and using relation (14) we therefore obtain

$$\begin{aligned}\int_0^1\Big\|\sum_{k=1}^\infty f_k(s)r_k(t)\Big\|_X\,dt &= \int_0^1\Big\|\sum_{k=1}^\infty\Big(g_k(s)+\int_0^1 f_k(s)\,ds\Big)\cdot r_k(t)\Big\|_X\,dt\\ &\leqslant \int_0^1\Big\|\sum_{k=1}^\infty g_k(s)r_k(t)\Big\|_X\,dt+\int_0^1\Big|\sum_{k=1}^\infty\int_0^1 f_k(s)\,ds\cdot r_k(t)\Big|\,dt\cdot\|\chi_{[0,1]}\|_X\\ &\leqslant C\Big\|\Big(\sum_{k=1}^\infty g_k^2\Big)^{1/2}\Big\|_X+\Big(\sum_{k=1}^\infty\Big(\int_0^1 f_k(s)\,ds\Big)^2\Big)^{1/2}\cdot\|\chi_{[0,1]}\|_X\\ &\leqslant C\Big\|\Big(\sum_{k=1}^\infty f_k^2\Big)^{1/2}\Big\|_X+(C+1)\|\chi_{[0,1]}\|_X\cdot\Big(\sum_{k=1}^\infty\Big(\int_0^1 f_k(s)\,ds\Big)^2\Big)^{1/2}\\ &\leqslant C\Big\|\Big(\sum_{k=1}^\infty f_k^2\Big)^{1/2}\Big\|_X+(C+1)\|\chi_{[0,1]}\|_X\cdot\int_0^1\Big(\sum_{k=1}^\infty f_k^2(s)\Big)^{1/2}\,ds\\ &\leqslant (2C+1)\Big\|\Big(\sum_{k=1}^\infty f_k^2\Big)^{1/2}\Big\|_X.\end{aligned}$$

Conversely, assume that (β) or (γ) (or, more precisely, the right-hand inequality in (23) or (24)) holds. In view of Theorem 1 and Remark 1, to prove (α) it is sufficient to show that the space X has property (g) from Remark 1. Let $\{f_k\}_{k=1}^\infty \subset X$ be a sequence of symmetrically distributed independent functions satisfying condition (6). Using Proposition 1.14 in [7] again we conclude that $\{f_k\}$ is an unconditional sequence with constant 1 in X. Hence if the right-hand inequality in (24) holds, then

$$\Big\|\sum_{k=1}^\infty f_k\Big\|_X = \int_0^1 \Big\|\sum_{k=1}^\infty f_k(s)r_k(t)\Big\|_X\,dt \leqslant C\Big\|\Big(\sum_{k=1}^\infty f_k^2\Big)^{1/2}\Big\|_X,$$

which proves (g). Moreover, the distribution of $f_k(s)$ on I coincides with the distribution of $f_k(s)r_k(t)$ on $I\times I$, $k=1,2,\dots$. Indeed, since the f_k are symmetrically distributed, for each $\tau\in\mathbb{R}$,

$$\begin{aligned}&\lambda_2\big(\{(s,t)\in I\times I\colon f_k(s)r_k(t)>\tau\}\big)\\ &\qquad=\frac{1}{2}\big[\lambda(\{s\in I: f_k(s)>\tau\})+\lambda(\{s\in I: -f_k(s)>\tau\})\big]\\ &\qquad=\lambda(\{s\in I: f_k(s)>\tau\}).\end{aligned}$$

Since the functions f_k, $k=1,2,\dots$, and $f_k(s)r_k(t)$, $k=1,2,\dots$, are independent, it therefore follows that

$$\Big\|\sum_{k=1}^\infty f_k\Big\|_X = \Big\|\sum_{k=1}^\infty f_k(s)r_k(t)\Big\|_{X(I\times I)},$$

so (g) is also a consequence of the right-hand inequality in (23). The proof of Theorem 2 is complete.

Remark 3. It is convenient to compare the result of Theorem 2 with two well-known results.

The first of them states that the following conditions are equivalent for a separable or maximal r.i. space X: (i) there exists $C>0$ such that (23) holds for an arbitrary sequence of functions $\{f_k\}_{k=1}^\infty \subset X$; (ii) the lower Boyd index α_X is positive. (The implication (ii) $\Rightarrow$ (i) is proved in [15], Proposition 2.d.1, and the implication (i) $\Rightarrow$ (ii) is proved in [24].)

The second (Maurey's inequality) says that if X is a q-concave Banach lattice (see [15], Definition 1.d.3) for some $q<\infty$, then there exists $C>0$ such that (24) holds for an arbitrary sequence of functions $\{f_k\}_{k=1}^\infty \subset X$ (see [15], Theorem 1.d.6(i)).

Note that, on the one hand, the coefficients of the series in both statements are arbitrary functions; on the other hand the conditions ensuring inequalities (23) and (24) in these statements are much more restrictive than (α).

§ 5. Khintchine's weak inequality and the Kruglov operator

Let $\varphi(t)$ be the fundamental function of an r.i. space X on $[0,1]$. Then as we already mentioned, we have the continuous embeddings

$$\Lambda(\varphi)\subset X\subset M(\varphi),$$

where $\Lambda(\varphi)$ and $M(\varphi)$ are the Lorentz and the Marcinkiewicz spaces, respectively (see [14], Theorems 2.5.5 and 2.5.7). Consequently, if inequality (2) holds in X for all sequences of independent functions $\{f_k\}_{k=1}^\infty \subset X$, $\int_0^1 f_k(t)\,dt = 0$, $k = 1, 2, \dots$, then the following inequality must also hold for such sequences $\{f_k\}_{k=1}^\infty \subset \Lambda(\varphi)$:

$$\Bigl\|\sum_{k=1}^\infty f_k\Bigr\|_{M(\varphi)} \leqslant C\Bigl\|\Bigl(\sum_{k=1}^\infty f_k^2\Bigr)^{1/2}\Bigr\|_{\Lambda(\varphi)}. \tag{25}$$

We show below that the last property can be characterized similarly to (2); only this time we have to consider the Kruglov operator in place of the Kruglov property.

For convenience we recall the following definition from operator interpolation theory (see [14], Definition 2.6.1). Let φ be an increasing concave function on $[0,1]$, $\varphi(0)=0$. Then we say that a linear operator T acting on finite measurable functions has *weak type* φ if it is bounded from $\Lambda(\varphi)$ into $M(\varphi)$.

Theorem 3. *The following conditions are equivalent*:

(i) *the Kruglov operator* $\mathscr{K}$ *has weak type* φ;

(ii) *there exists* $C > 0$ *such that* (25) *holds for each sequence of independent functions* $\{f_k\}_{k=1}^\infty \subset \Lambda(\varphi)$, $\int_0^1 f_k(t)\,dt = 0$, $k = 1, 2, \dots$;

(iii) *there exists* $C > 0$ *such that* (25) *holds for each sequence of independent functions* $\{f_k\}_{k=1}^\infty \subset \Lambda(\varphi)$, $\int_0^1 f_k(t)\,dt = 0$, $k = 1, 2, \dots$, *that satisfies condition* (6).

Proof. If (i) holds, then it follows from Theorem 2.1 in [19] and Theorem 1 in [20] (see also the proof of Theorem 1) that for each sequence of independent functions $\{f_k\}_{k=1}^\infty \subset \Lambda(\varphi)$, $\int_0^1 f_k(t)\,dt = 0$, $k = 1, 2, \dots$, we have

$$\Bigl\|\sum_{k=1}^\infty f_k\Bigr\|_{M(\varphi)} \leqslant C_1\Bigl\|\sum_{k=1}^\infty \bar{f}_k\Bigr\|_{Z^2_{\Lambda(\varphi)}} \leqslant C_2\Bigl\|\Bigl(\sum_{k=1}^\infty f_k^2\Bigr)^{1/2}\Bigr\|_{\Lambda(\varphi)}.$$

This proves (ii).

The implication (ii) $\Rightarrow$ (iii) is obvious, therefore it remains to show that (iii) implies (i).

Let $f = \chi_{[0,u]}$, $0 < u \leqslant 1$, $n \in \mathbb{N}$ and let $\{f_{n,k}\}_{k=1}^n$ be a system of independent functions equidistributed with $\sigma_{1/n} f(t) = \chi_{[0,u/n]}(t)$. Now if $f_n := \sum_{k=1}^n f_{n,k}$, then it is easy to see that

$$\lambda\bigl(\{t \in [0,1] : f_n(t) = k\}\bigr) = \frac{n!}{k!\,(n-k)!}\Bigl(\frac{u}{n}\Bigr)^k\Bigl(1-\frac{u}{n}\Bigr)^{n-k}, \qquad k = 1, 2, \dots, n.$$

Hence

$$f_n^*(t) = \sum_{k=1}^n \chi_{[0,\tau_k^n]}, \quad \text{where } \tau_k^n = \sum_{k=1}^n \frac{n!}{k!\,(n-k)!}\Bigl(\frac{u}{n}\Bigr)^k\Bigl(1-\frac{u}{n}\Bigr)^{n-k}. \tag{26}$$

We also observe that by condition (iii) and Lemma 1 relation (25) holds for each system $\{f_{n,k}\}_{k=1}^n$, that is,

$$\Big\|\sum_{k=1}^n f_{n,k}\Big\|_{M(\varphi)} \leqslant C\Big\|\Big(\sum_{k=1}^n f_{n,k}^2\Big)^{1/2}\Big\|_{\Lambda(\varphi)}, \qquad n=1,2,\dots. \tag{27}$$

It follows from the definition of the norm in the Marcinkiewicz space (see also [14], Ch. 2, § 5, inequality (5.15)) that

$$\|f_n\|_{M(\varphi)} = \Big\|\sum_{k=1}^n f_{n,k}\Big\|_{M(\varphi)} \geqslant \max_{k=1,2,\dots,n}\big(k\varphi(\tau_k^n)\big).$$

Moreover,

$$\Big(\sum_{k=1}^n f_{n,k}^2\Big)^{1/2} = \Big(\sum_{k=1}^n f_{n,k}\Big)^{1/2} = f_n^{1/2}. \tag{28}$$

Hence, in view of (27), we obtain

$$\max_{k=1,2,\dots,n}\big(k\varphi(\tau_k^n)\big) \leqslant C\|f_n^{1/2}\|_{\Lambda(\varphi)}. \tag{29}$$

It is now easy to see that

$$\big(f_n^{1/2}\big)^*(s) = \sum_{k=1}^n(\sqrt{k}-\sqrt{k-1}\,)\chi_{[0,\tau_k^n]}(s) = \sum_{k=1}^n \frac{1}{\sqrt{k}+\sqrt{k-1}}\,\chi_{[0,\tau_k^n]}(s),$$

so that from the definition of the norm in the Lorentz space (see also [14], Ch. 2, § 5, equality (5.4)) we obtain

$$\|f_n^{1/2}\|_{\Lambda(\varphi)} \leqslant \sum_{k=1}^n \frac{1}{\sqrt{k}}\,\varphi(\tau_k^n). \tag{30}$$

Hence, by (29),

$$\max_{k=1,2,\dots,n}\big(k\varphi(\tau_k^n)\big) \leqslant C\sum_{k=1}^n \frac{1}{\sqrt{k}}\,\varphi(\tau_k^n). \tag{31}$$

Let

$$\alpha_n := \max_{k=1,2,\dots,n}\big(k\varphi(\tau_k^n)\big), \qquad n\in\mathbb{N}.$$

Then $\varphi(\tau_k^n) \leqslant \alpha_n/k$, $k=1,2,\dots,n$, and for arbitrary $n \geqslant N \geqslant 1$ we have

$$\sum_{k=N}^n \frac{1}{\sqrt{k}}\,\varphi(\tau_k^n) \leqslant \sum_{k=N}^n k^{-3/2}\cdot\alpha_n.$$

Hence taking N such that $(C+1)\sum_{k=N}^\infty k^{-3/2} \leqslant 1$, for all $n \geqslant N$ we obtain

$$(C+1)\sum_{k=N}^n \frac{1}{\sqrt{k}}\,\varphi(\tau_k^n) \leqslant \alpha_n.$$

Combining this with (31) yields

$$(C+1)\sum_{k=N}^{n}\frac{1}{\sqrt{k}}\,\varphi(\tau_k^n)\leqslant C\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\,\varphi(\tau_k^n).$$

Consequently,

$$\sum_{k=N}^{n}\frac{1}{\sqrt{k}}\,\varphi(\tau_k^n)\leqslant C\sum_{k=1}^{N}\frac{1}{\sqrt{k}}\,\varphi(\tau_k^n),$$

or

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\,\varphi(\tau_k^n)\leqslant (C+1)\sum_{k=1}^{N}\frac{1}{\sqrt{k}}\,\varphi(\tau_k^n)\leqslant C'\varphi(\tau_1^n)\leqslant C'\varphi(u),$$

where the last inequality is a consequence of $\tau_1^n=1-(1-u/n)^n\leqslant u$ (see (26)). Thus, taking (28) and (30) into account we arrive at the inequality

$$\left\|\left(\sum_{k=1}^{n}f_{n,k}^2\right)^{1/2}\right\|_{\Lambda(\varphi)}\leqslant C'\varphi(u).$$

Hence it follows from (27) that

$$\left\|\sum_{k=1}^{n}f_{n,k}\right\|_{M(\varphi)}\leqslant C''\varphi(u),\qquad n=1,2,\dots. \tag{32}$$

It is a straightforward computation that each variable $f_{n,k}$, $k=1,2,\dots,n$, has the characteristic function

$$\theta_{f_{n,k}}(t)=n^{-1}\theta_f(t)+(1-n^{-1}),\qquad t\in\mathbb{R},$$

where θ_f is the characteristic function of $f=\chi_{[0,u]}$. Hence the sum $f_n=\sum_{k=1}^{n}f_{n,k}$ has the characteristic function

$$\theta_{f_n}(t)=(n^{-1}(\theta_f(t)-1)+1)^n,\qquad t\in\mathbb{R}.$$

Since

$$\lim_{n\to\infty}\theta_{f_n}(t)=\exp(\theta_f(t)-1)=\theta_{\pi(f)}(t)$$

for all $t\in\mathbb{R}$, the f_n converge weakly to $\mathscr{K}f$ (see § 2). Thus, in view of relation (32), Proposition 1.5 in [7] and the equality $M(\varphi)''=M(\varphi)$ (see [14], Ch. 2, § 5), we obtain

$$\|\mathscr{K}\chi_{[0,u]}\|_{M(\varphi)}\leqslant C''\varphi(u),\qquad 0<u\leqslant 1.$$

In view of Lemma 2.5.2 in [14], this means that the operator $\mathscr{K}$ is bounded from $\Lambda(\varphi)$ into $M(\varphi)$, which completes the proof of Theorem 3.

Remark 4. As shown in [12], Remark 5.2, the operator $\mathscr{K}$ has weak type φ if and only if

$$\sup_{u\in(0,1],\,k\in\mathbb{N}}\frac{k\varphi(u^k/k!)}{\varphi(u)}<\infty.$$

It can be easily shown that, in fact,

$$\max_{k=1,2,\dots,n}\bigl(k\varphi(\tau_k^n)\bigr)\leqslant\biggl\|\sum_{k=1}^{n}f_{n,k}\biggr\|_{M(\varphi)}\leqslant C\max_{k=1,2,\dots,n}\bigl(k\varphi(\tau_k^n)\bigr),$$

so that this result can be also deduced by passing to the limit in (32).

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S. V. Astashkin
Samara State University
E-mail: astashkn@ssu.samara.ru

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