Functional Analysis and Its Applications, Vol. 42, No. 2, pp. 144–147, 2008
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 42, No. 2, pp. 78–81, 2008

A Generalized Khintchine Inequality in Rearrangement Invariant Spaces

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Received October 25, 2006

Abstract. Let X be a separable or maximal rearrangement invariant space on $[0,1]$. Necessary and sufficient conditions are found under which the generalized Khintchine inequality

$$\Big\|\sum_{k=1}^{\infty} f_k\Big\|_X \leqslant C\Big\|\Big(\sum_{k=1}^{\infty} f_k^2\Big)^{1/2}\Big\|_X$$

holds for an arbitrary sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of mean zero independent variables. Moreover, the subspace spanned in a rearrangement invariant space by the Rademacher system with independent vector coefficients is studied.

Key words: Khintchine inequality, rearrangement invariant space, Rademacher system, independent functions, Kruglov property, Boyd indices.

1. Introduction. Let $r_k(t) = \operatorname{sign}\sin 2^k\pi t$ $(k = 1, 2, \dots)$ be the Rademacher functions on the interval $[0,1]$. By the classical Khintchine inequality [1], for every $p > 0$ there exists a constant $C_p > 0$ such that $\|\sum_{k=1}^{\infty} a_k r_k\|_{L_p} \leqslant C_p(\sum_{k=1}^{\infty} a_k^2)^{1/2}$ for arbitrary real a_k, $k = 1, 2, \dots$.

This relation has led to a number of studies and generalizations and found numerous applications in various fields of analysis [2]. In particular, Rodin and Semenov [3] proved in 1975 that the inequality

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_X \leqslant C\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \tag{1}$$

holds for a rearrangement invariant space X on $[0,1]$ if and only if X contains the separable part G of the Orlicz space L_{N_2} generated by the function $N_2(u) = e^{u^2} - 1$. In this note, we solve a similar problem in which scalar multiples of Rademacher functions are replaced by any mean zero independent functions. More precisely, we characterize rearrangement invariant spaces X for which there exists a constant $C > 0$ such that

$$\Big\|\sum_{k=1}^{\infty} f_k\Big\|_X \leqslant C\Big\|\Big(\sum_{k=1}^{\infty} f_k^2\Big)^{1/2}\Big\|_X \tag{2}$$

for all sequences $\{f_k\}_{k=1}^{\infty} \subset X$ of independent functions satisfying the conditions $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$ and $(\sum_{k=1}^{\infty} f_k^2)^{1/2} \in X$. Note that such a characterization is known if, instead of the class of sequences of mean zero independent functions, the larger class of all martingale differences is considered: inequality (2) holds in X if and only if the lower Boyd index of X is positive, $\alpha_X > 0$ [4].

In our case, the central role is played by the so-called Kruglov property, which was introduced and intensively studied by Braverman [5]. The main result of this note (Theorem 1) provides a description, in terms of this property, of separable or maximal rearrangement invariant spaces for which the generalized Khintchine inequality (2) holds for arbitrary sequences of mean zero independent functions. The second result (Theorem 2) shows the same property to be useful when studying the subspace spanned in a rearrangement invariant space by the Rademacher system with independent vector coefficients.

 0016–2663/08/4202–0144

2. Definitions and notation. A Banach space X of real-valued Lebesgue measurable functions on the interval $[0,\alpha)$ $(0<\alpha\leqslant\infty)$ is called a *rearrangement invariant* space if the relations $y\in X$ and $x^*(t)\leqslant y^*(t)$ $(t\in[0,\alpha))$ imply that $x\in X$ and $\|x\|_X\leqslant\|y\|_X$. Here $x^*(t)$ is the nonincreasing right-continuous *rearrangement* of the function $|x(s)|$; i.e., $x^*(t)=\inf\{\tau\geqslant 0:\lambda(\{s\in[0,\alpha):|x(s)|>\tau\})\leqslant t\}$, $t>0$ (where λ stands for the Lebesgue measure).

Given a rearrangement invariant space X on $[0,\alpha)$, the *Köthe dual* (or *associate*) space X' consists of all measurable functions y such that $\|y\|_{X'}:=\sup\{\int_0^\alpha\|x(t)y(t)\|\,dt : x\in X,\ \|x\|_X\leqslant 1\}<\infty$. A rearrangement invariant space X is said to be *maximal* (or have the *Fatou property*) if the natural embedding of X in the Köthe bidual X'' is a surjective isometry. This is equivalent to the following property of X: if $x_n\in X$ $(n=1,2,\dots)$, $x_n\to x$ a.e. on $[0,\alpha)$, and $\sup_{n=1,2,\dots}\|x_n\|_X<\infty$, then $x\in X$ and $\|x\|\leqslant\liminf_{n\to\infty}\|x_n\|$. Note that all rearrangement invariant spaces important in applications (Orlicz, Lorentz, and Marcinkiewicz spaces etc.) are separable or maximal. The Boyd indices α_X and β_X of a rearrangement invariant space X are an important characteristic of X; they are related to the dilation operator and always satisfy $0\leqslant\alpha_X\leqslant\beta_X\leqslant 1$ [6, Theorem 2.4.4]. For any rearrangement invariant space X on $[0,1]$, one has the continuous embeddings $L_\infty[0,1]\subset X\subset L_1[0,1]$ [6, Theorem 2.4.1].

If X is a rearrangement invariant space on $[0,1]$, then the space Z_X^2 introduced in [7] consists of all functions $f\in(L_1+L_\infty)(0,\infty)$ such that

$$\|f\|_{Z_X^2}:=\|f^*\chi_{[0,1]}\|_X+\|f^*\chi_{[1,\infty)}\|_{L_2[1,\infty)}<\infty.$$

One can readily show that the quasi-norm $\|\cdot\|_{Z_X^2}$ is equivalent to some norm, and therefore, Z_X^2 is a rearrangement invariant space on $[0,\infty)$. If $\{f_k\}_{k=1}^\infty$ is a sequence of functions defined on $[0,1]$, then by $\{\bar f_k\}_{k=1}^\infty$ we denote a sequence of pairwise disjoint functions defined on $[0,\infty)$ and equidistributed with the respective f_k; for example, $\bar f_k(t)=f_k(t-k+1)\chi_{[k-1,k)}(t)$. For more detailed information on rearrangement invariant spaces, we refer the reader to the monographs [6], [8], and [9].

Let f be a measurable function (random variable) on $[0,1]$. By $\pi(f)$ denote the random variable $\sum_{i=1}^N f_i$, where the f_i are independent copies of f and N is a Poisson random variable with parameter 1 independent of the sequence $\{f_i\}$. The following property was introduced and intensively studied by Braverman [5], who used some probabilistic constructions due to Kruglov [10].

Definition 1. A rearrangement invariant space X on $[0,1]$ is said to have the *Kruglov property* $(X\in\mathbb{K})$ if $\pi(f)\in X$ whenever $f\in X$.

Roughly speaking, a rearrangement invariant space X belongs to the class $\mathbb{K}$ if X is “sufficiently far” from L_∞. In particular, this is true if $X\supset L_p$ for some $p<\infty$ [5, Theorem 1.2, p. 16]. Moreover, if L_{N_γ} is the exponential Orlicz space generated by the function $N_\gamma(u)=e^{u^\gamma}-1$, then $L_{N_\gamma}\in\mathbb{K}$ if and only if $0<\gamma\leqslant 1$ [5, p. 42].

3. Results.

Theorem 1. *Let X be a separable or maximal rearrangement invariant space on $[0,1]$. The following conditions are equivalent*:

(a) $X\in\mathbb{K}$.

(b) *There exists a constant $C>0$ such that*

$$\Big\|\sum_{k=1}^\infty f_k\Big\|_X\leqslant C\Big\|\sum_{k=1}^\infty \bar f_k\Big\|_{Z_X^2}\tag{3}$$

for any sequence $\{f_k\}_{k=1}^\infty\subset X$ of independent functions satisfying the conditions $\int_0^1 f_k(t)\,dt=0$, $k=1,2,\dots$, and $\sum_{k=1}^\infty\bar f_k\in Z_X^2$.

(c) *There exists a constant $C>0$ such that inequality* (2) *holds for any sequence $\{f_k\}_{k=1}^\infty\subset X$ of independent functions satisfying the conditions $\int_0^1 f_k(t)\,dt=0$, $k=1,2,\dots$, and $(\sum_{k=1}^\infty f_k^2)^{1/2}\in X$.*

Remark 1. The inequality opposite to (3) holds in any rearrangement invariant space X [7]. Moreover, it was proved in [7] that (3) is true if $X \supset L_p$ for some $p < \infty$. As was said above, the last condition is more restrictive than condition (a). In particular, Theorem 1 shows that relations (2) and (3) hold in the exponential Orlicz space L_{N_γ} for $0 < \gamma \leqslant 1$.

Remark 2. On the one hand, the condition $X \in \mathbb{K}$, which is weaker than the positivity of the lower Boyd index, shows the difference between the properties of mean zero independent functions and general martingale differences [4]. On the other hand, this condition is stronger than the embedding $X \supset G$; the latter guarantees inequality (1), which is the special case of inequality (2) for scalar multiples of the Rademacher system.

For an arbitrary rearrangement invariant space X on $I = [0,1]$, we define the corresponding rearrangement invariant space $X(I \times I)$ on the square $I \times I$ with the norm $\|f\|_{X(I\times I)} = \|f^*\|_X$.

Theorem 3. *If X is a separable or maximal rearrangement invariant space, then the following conditions are equivalent*:

(α) $X \in \mathbb{K}$.

(β) *There exists a constant $C > 0$ such that*

$$C^{-1}\Big\|\Big(\sum_{k=1}^{\infty} f_k^2\Big)^{1/2}\Big\|_X \leqslant \Big\|\sum_{k=1}^{\infty} f_k(s) r_k(t)\Big\|_{X(I\times I)} \leqslant C\Big\|\Big(\sum_{k=1}^{\infty} f_k^2\Big)^{1/2}\Big\|_X \tag{4}$$

for any sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of independent functions.

(γ) *There exists a constant $C > 0$ such that*

$$C^{-1}\Big\|\Big(\sum_{k=1}^{\infty} f_k^2\Big)^{1/2}\Big\|_X \leqslant \int_0^1 \Big\|\sum_{k=1}^{\infty} f_k(s) r_k(t)\Big\|_X dt \leqslant C\Big\|\Big(\sum_{k=1}^{\infty} f_k^2\Big)^{1/2}\Big\|_X \tag{5}$$

for any sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of independent functions.

Remark 3. It is instructive to compare Theorem 2 with the following two well-known results. The first one states that the following conditions are equivalent for any separable or maximal rearrangement invariant space X: (i) there exists a constant $C > 0$ such that inequality (4) holds for every sequence $\{f_k\}_{k=1}^{\infty} \subset X$; (ii) $\alpha_X > 0$. (The implication (ii)$\Rightarrow$(i) was proved in [8, Proposition 2.d.1], and the opposite implication, in [11].) The second result, known as Maurey's inequality, states that if X is a q-concave Banach lattice for some $q < \infty$ [8, Definition 1.d.3], then there exists a constant $C > 0$ such that inequality (5) holds for every sequence $\{f_k\}_{k=1}^{\infty} \subset X$ [8, Theorem 1.d.6(i)]. Note that, on the one hand, the coefficients of the series involved in both statements are arbitrary functions; on other hand, the conditions in these statements guaranteeing that inequalities (4) and (5) hold are much more restrictive than condition (α).

Remark 4. The proof of Theorems 1 and 2 uses the operator approach, developed in [12] and [13], to studying the Kruglov property.

References

[1] A. Khintchine, Math. Z., **18** (1923), 109–116.

[2] G. Peshkir and A. N. Shiryaev, Uspekhi Mat. Nauk, **50**:5 (1995), 3–62; English transl.: Russian Math. Surveys, **50**:5, 849–904.

[3] V. A. Rodin and E. M. Semenov, Anal. Math., **1**:3 (1975), 207–222.

[4] W. B. Johnson and G. Schechtman, Israel J. Math., **64**:3 (1988), 267–275.

[5] M. Sh. Braverman, Independent Random Variables and Rearrangement Invariant Spaces, Cambridge Univ. Press, Cambridge, 1994.

[6] S. G. Krein, Y. I. Petunin, and E. M. Semenov, Interpolation of Linear Operators, Transl. Math. Monographs, vol. 54, Amer. Math. Soc., Providence, RI, 1982.

[7] W. B. Johnson and G. Schechtman, Ann. Probab., **17**:2 (1989), 789–808.

[8] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces II. Function spaces, Springer-Verlag, Berlin–Heidelberg–New York, 1979.

[9] C. Bennett and R. Sharpley, Interpolation of Operators, Academic Press, Boston, 1988.
[10] V. M. Kruglov, Teor. Veroyatnost. Primenen., **15**:2 (1970), 331–336.
[11] S. V. Astashkin and M. Sh. Braverman, in: Operator Equations in Function Spaces, Voronezh Gos. Univ., Voronezh, 1986, 3–10.
[12] S. V. Astashkin and F. A. Sukochev, Mat. Zametki, **76**:4 (2004), 483–489; English transl.: Math. Notes, **76**:4 (2004), 449–454.
[13] S. V. Astashkin and F. A. Sukochev, Israel J. Math., **145** (2005), 125–156.

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Translated by S. V. Astashkin