ISSN 0001-4346, Mathematical Notes, 2008, Vol. 83, No. 2, pp. 145–151. © Pleiades Publishing, Ltd., 2008.
Original Russian Text © S. V. Astashkin, 2008, published in Matematicheskie Zametki, 2008, Vol. 83, No. 2, pp. 163–169.

Lieb–Thirring Inequality for L_p Norms

S. V. Astashkin*

Samara State University

Received May 31, 2007; in final form, August 28, 2007

Abstract—In this paper, we obtain the Lieb–Thirring inequality for L_p-norms. The proof uses only the standard apparatus of the theory of orthogonal series.

DOI: 10.1134/S0001434608010173

Key words: *Lieb–Thirring inequality, L_p-norm, orthonormal system, orthogonal series, Marcinkiewicz theorem, Fourier multiplier, Rademacher function.*

In 1976, Lieb and Thirring obtained the following result [1].

Theorem A. *If $\Phi = \{\phi_j\}_{j=1}^N \subset L^2(\mathbb{R}^d)$ is an arbitrary orthonormal system,*

$$\max\left(1, \frac{d}{2}\right) < p \le 1 + \frac{d}{2},$$

and

$$\rho_\Phi := \sum_{j=1}^N \phi_j^2, \tag{1}$$

then

$$\left(\int_{\mathbb{R}^d} \rho_\Phi^{p/(p-1)}\,dx\right)^{2(p-1)/d} \le C_{p,d} \sum_{j=1}^N \|\nabla\phi_j\|^2_{L^2(\mathbb{R}^d)}, \tag{2}$$

where, as is customary, $\nabla\phi = (\partial\phi/\partial x_1, \dots, \partial\phi/\partial x_d)$.

More recently, inequalities of type (2) for finite orthonormal systems were established by different authors (they were of interest mainly because of their applications to the theory of partial differential equations). The well-known methods of proof based on nontrivial results from spectral theory used by these authors were quite complicated (for details, see [2] and [3]).

Recently, Kashin [4] proposed a new approach to the proof of inequalities of Lieb–Thirring type, in which he used only the standard apparatus of the theory of orthogonal series: L_p-inequalities for vector-valued Rademacher series and the classical Littlewood–Paley theorem. Let us recall some definitions and notation from [4]. Suppose that

$$\Phi = \{\phi_j\}_{j=1}^N \subset L^2(S^1), \qquad \text{where} \quad \phi_j \perp 1, \ 1 \le j \le N. \tag{3}$$

Let us define the operator $P_\Phi \colon l_2^N \to L^2(S^1)$ as follows:

$$P_\Phi(\{c_j\}_{j=1}^N) = \sum_{j=1}^N c_j\phi_j.$$

*E-mail: astashkn@ssu.samara.ru .

Besides, suppose that $\pi_m\colon L^2(S^1) \to T_{2^m}$ is the orthogonal projection onto the space T_{2^m}, $m = 0, 1, 2, \dots$, of trigonometric polynomials $t(z)$ of the form

$$t(z) = \sum_{2^m \le |k| < 2^{m+1}} a_k z^k$$

and, finally,

$$\lambda_m(\Phi) = \|\pi_m \cdot P_\Phi\|_{l_2^N \to T_{2^m}}. \tag{4}$$

Note that, for the special case in which Φ is an orthonormal (or suborthonormal) system, we have

$$\lambda_m(\Phi) \le 1, \qquad m = 0, 1, \dots . \tag{5}$$

Recall that a system $\{\phi_j\}_{j=1}^N$ is said to be *suborthonormal* if it satisfies the inequality

$$\sum_{i,j=1}^N c_i c_j \int_{S^1} \phi_i \phi_j \, d\mu \le \sum_{j=1}^N c_j^2$$

for all real $c_1, c_2, \dots, c_N$.

In [4], the following theorem was proved.

Theorem B. *There exists an absolute constant $C > 0$ such that, for an arbitrary system Φ of the form* (3)*, the following inequality holds*:

$$\int_{S^1} \rho_\Phi^2 \, d\mu \le C \sum_{m=0}^\infty \gamma_m(\Phi) \sum_{j=1}^N \sum_{2^m \le |i| < 2^{m+1}} |\widehat{\phi}_j(i)|^2, \tag{6}$$

where the function ρ_Φ is defined by (1)*, and*

$$\gamma_m(\Phi) := \sum_{k=0}^m 2^k \lambda_k^2(\Phi), \qquad m = 0, 1, 2, \dots .$$

In particular, if Φ is an orthonormal (or suborthonormal) system, then, in view of (5), inequality (6) can be rewritten as

$$\int_{S^1} \rho_\Phi^2 \, d\mu \le C_1 \sum_{j=1}^N \|\phi_j^{(1/2)}\|_{L^2(S^1)}^2, \tag{7}$$

where $f^{(1/2)}$ denotes the "one-half" derivative of a function f. Recall that, for an arbitrary $\alpha > 0$ and a function

$$f(z) = \sum_{j \ne 0} \widehat{f}(j) z^j \in L^2(S^1),$$

the derivative of order α is defined by the relation

$$f^{(\alpha)}(z) := \sum_{j \ne 0} |j|^\alpha \widehat{f}(j) z^j.$$

In this paper, we show that, following the approach used in [4], one can generalize inequalities (6) and (7) to the case of L_p-norms. In the proof, we shall use the Marcinkiewicz theorem on Fourier multipliers.

We begin by stating the results and making a few remarks.

Theorem 1. *For any $k = 2, 3, \dots,$ there exists a constant $C = C(k)$ such that, for an arbitrary system Φ of the form* (3)*, the following inequality holds*:

$$\int_{S^1} \rho_\Phi^k \, d\mu \le C \sum_{m=0}^{\infty} \gamma'_m(\Phi) \sum_{j=1}^{N} \sum_{2^m \le |i| < 2^{m+1}} |\widehat{\phi}_j(i)|^2, \tag{8}$$

where, for $m = 0, 1, 2, \dots,$

$$\gamma'_m(\Phi) := \sum_{m_{k-1}=0}^{m} 2^{m_{k-1}} \lambda^2_{m_{k-1}}(\Phi) \sum_{m_{k-2}=0}^{m_{k-1}} 2^{m_{k-2}} \lambda^2_{m_{k-2}}(\Phi) \cdots \sum_{m_1=0}^{m_2} 2^{m_1} \lambda^2_{m_1}(\Phi),$$

while the numbers $\lambda_m(\Phi)$ are defined by relation (4)*.*

Remark 1. For $k = 1$, in view of Parseval's equality, we can write

$$\int_{S^1} \rho_\Phi \, d\mu = \sum_{j=1}^{N} \int_{S^1} \phi_j^2 = \sum_{m=0}^{\infty} \sum_{j=1}^{N} \sum_{2^m \le |i| < 2^{m+1}} |\widehat{\phi}_j(i)|^2.$$

Remark 2. Inequality (8) generalizes relation (6). If, additionally, Φ is an orthonormal (or suborthonormal) system, then, in view of (5),

$$\gamma'_m(\Phi) \le \sum_{m_{k-1}=0}^{m} 2^{m_{k-1}} \sum_{m_{k-2}=0}^{m_{k-1}} 2^{m_{k-2}} \cdots \sum_{m_1=0}^{m_2} 2^{m_1} \le 2^{k(k-1)/2} 2^{(k-1)m},$$

and we obtain the inequality

$$\int_{S^1} \rho_\Phi^k \, d\mu \le C_1 \sum_{j=1}^{N} \|\phi_j^{((k-1)/2)}\|^2_{L^2(S^1)}. \tag{9}$$

Note that, in the case $k = 2m + 1$, $m \in \mathbb{N}$, inequality (9) was obtained by another method in [5] (for a simpler proof, see [6]).

A more general relation than (7) and (9) is established in the following theorem.

Theorem 2. *Suppose that $k, l \in \mathbb{N}$, where $k \ge 2$ and k is divisible by l. Then there exists a constant $C = C(k)$ such that, for an arbitrary orthonormal* (*or suborthonormal*) *system Φ of the form* (3)*, the following inequality holds*:

$$\int_{S^1} \rho_\Phi^k \, d\mu \le C \biggl(\sum_{j=1}^{N} \|\phi_j^{((l-1)/2)}\|^2_{2k/l} \biggr)^{k/l}. \tag{10}$$

Let us now pass to the proof of these results.

Proof of Theorem 1. Following [4], we can express the function f as

$$f = \sum_{m=0}^{\infty} \delta_m(f), \qquad \text{where} \quad \delta_m(f) := \sum_{2^m \le |i| < 2^{m+1}} \widehat{f}(i) z^i.$$

Denote

$$\delta_m(t, z) := \delta_m \biggl(\sum_{j=1}^{N} r_j(t) \phi_j(z) \biggr) = \sum_{j=1}^{N} r_j(t) \delta_m(\phi_j)(z),$$

where the $r_j(t)$ are Rademacher functions on $[0,1]$. Then, in view of the vector-valued version of Khinchine's inequality [7, p. 52] and the Littlewood–Paley inequality [8, p. 63, inequalities (9)], we find

$$Q := \int_{S^1} \rho_\Phi^k \, d\mu = \int_{S^1} \biggl(\sum_{j=1}^N \phi_j^2(z) \biggr)^k d\mu$$
$$\le C_1 \int_0^1 \biggl\| \sum_{j=1}^N r_j(t)\phi_j \biggr\|_{L^{2k}}^{2k} dt \le C_2 \int_0^1 \int_{S^1} \biggl(\sum_{m=0}^\infty \delta_m^2(t,z) \biggr)^k d\mu \, dt,$$

where the constants depend only on k. Hence

$$\frac{Q}{C_2} \le \sum_{m_1=0}^\infty \cdots \sum_{m_k=0}^\infty \int_0^1 \int_{S^1} \delta_{m_1}^2(t,z)\delta_{m_2}^2(t,z)\cdots\delta_{m_k}^2(t,z)\, d\mu\, dt. \qquad (11)$$

For each $i = 1, 2, \dots, k$, we have

$$\delta_{m_i}^2(t,z) = \biggl(\sum_{j=1}^N r_j(t)\delta_{m_i}(\phi_j)(z) \biggr)^2 = \sum_{j,q=1}^N r_j(t) r_q(t) \delta_{m_i}(\phi_j)(z)\delta_{m_i}(\phi_q)(z)$$

and, therefore, for all $t \in [0,1]$,

$$\int_{S^1} \delta_{m_1}^2(t,z)\delta_{m_2}^2(t,z)\cdots\delta_{m_k}^2(t,z)\, d\mu = \sum_{j_1=1}^N \sum_{q_1=1}^N \cdots \sum_{j_k=1}^N \sum_{q_k=1}^N r_{j_1}(t) r_{q_1}(t) r_{j_2}(t) r_{q_2}(t) \cdots r_{j_k}(t) r_{q_k}(t)$$
$$\times \int_{S^1} \delta_{m_1}(\phi_{j_1})\delta_{m_1}(\phi_{q_1})\cdots\delta_{m_k}(\phi_{j_k})\delta_{m_k}(\phi_{q_k})\, d\mu.$$

Note that the integral

$$\int_0^1 r_{j_1}(t) r_{q_1}(t) r_{j_2}(t) r_{q_2}(t) \cdots r_{j_k}(t) r_{q_k}(t)\, dt$$

is nonzero (is equal to 1) if and only if, in the collection of the natural numbers $(j_1, q_1, \dots, j_k, q_k)$, each number occurs an even number of times. Therefore, the desired quantity

$$I_{m_1,m_2,\dots,m_k} := \int_0^1 \int_{S^1} \delta_{m_1}^2(t,z)\cdots\delta_{m_k}^2(t,z)\, d\mu\, dt = \int_{S^1}\int_0^1 \delta_{m_1}^2(t,z)\dots\delta_{m_k}^2(t,z)\, dt\, d\mu$$

is the sum of integrals of the form

$$\int_{S^1} \delta_{m_1}(\phi_{j_1})\delta_{m_{\pi(1)}}(\phi_{j_1})\delta_{m_2}(\phi_{j_2})\delta_{m_{\pi(2)}}(\phi_{j_2})\cdots\delta_{m_k}(\phi_{j_k})\delta_{m_{\pi(k)}}(\phi_{j_k})\, d\mu, \qquad (12)$$

where π is a permutation of the set $\{1, 2, \dots, k\}$. Conversely, expressing any such permutation π as the product of cycles, we can readily verify that the corresponding integral (12) is one of the summands making up $I_{m_1,m_2,\dots,m_k}$. Thus,

$$I_{m_1,m_2,\dots,m_k} = \sum_{j_1=1}^N \sum_{j_2=1}^N \cdots \sum_{j_k=1}^N \sum_\pi \int_{S^1} \delta_{m_1}(\phi_{j_1})\delta_{m_{\pi(1)}}(\phi_{j_1})\delta_{m_2}(\phi_{j_2})$$
$$\times \delta_{m_{\pi(2)}}(\phi_{j_2})\cdots\delta_{m_k}(\phi_{j_k})\delta_{m_{\pi(k)}}(\phi_{j_k})\, d\mu$$
$$= \sum_\pi \int_{S^1} \sum_{j_1=1}^N \delta_{m_1}(\phi_{j_1})\delta_{m_{\pi(1)}}(\phi_{j_1})$$
$$\times \sum_{j_2=1}^N \delta_{m_2}(\phi_{j_2})\delta_{m_{\pi(2)}}(\phi_{j_2}) \cdots \sum_{j_k=1}^N \delta_{m_k}(\phi_{j_k})\delta_{m_{\pi(k)}}(\phi_{j_k})\, d\mu.$$

Hence, by Cauchy's inequality, we have

$$\begin{aligned} I_{m_1,m_2,\dots,m_k} &\le \sum_{\pi} \int_{S^1} \biggl(\sum_{j_1=1}^{N} \delta_{m_1}^2(\phi_{j_1})\biggr)^{1/2} \biggl(\sum_{j_1=1}^{N} \delta_{m_{\pi(1)}}^2(\phi_{j_1})\biggr)^{1/2} \cdots \biggl(\sum_{j_k=1}^{N} \delta_{m_k}^2(\phi_{j_k})\biggr)^{1/2} \\ &\qquad \times \biggl(\sum_{j_k=1}^{N} \delta_{m_{\pi(k)}}^2(\phi_{j_k})\biggr)^{1/2} d\mu \\ &= \sum_{\pi} \int_{S^1} \sum_{j=1}^{N} \delta_{m_1}^2(\phi_j) \cdots \sum_{j=1}^{N} \delta_{m_k}^2(\phi_j)\, d\mu \\ &= k! \int_{S^1} \sum_{j=1}^{N} \delta_{m_1}^2(\phi_j) \cdots \sum_{j=1}^{N} \delta_{m_k}^2(\phi_j)\, d\mu. \end{aligned}$$

Thus,

$$I_{m_1,m_2,\dots,m_k} \le k!\, J_{m_1,m_2,\dots,m_k},$$

where

$$J_{m_1,m_2,\dots,m_k} := \int_{S^1} \sum_{j=1}^{N} \delta_{m_1}^2(\phi_j) \cdots \sum_{j=1}^{N} \delta_{m_k}^2(\phi_j)\, d\mu.$$

Since, for any permutation $\pi\colon \{1,2,\dots,k\} \to \{1,2,\dots,k\}$,

$$J_{m_1,m_2,\dots,m_k} = J_{m_{\pi(1)},m_{\pi(2)},\dots,m_{\pi(k)}},$$

in view of (11), we can write

$$\begin{aligned} \frac{Q}{C_2} &\le k! \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} J_{m_1,m_2,\dots,m_k} = (k!)^2 \sum_{0\le m_1\le\cdots\le m_k} J_{m_1,m_2,\dots,m_k} \\ &\le (k!)^2 \sum_{0\le m_1\le\cdots\le m_k} \biggl\|\sum_{j=1}^{N} \delta_{m_1}^2(\phi_j)\biggr\|_{L^\infty(S^1)} \cdots \biggl\|\sum_{j=1}^{N} \delta_{m_{k-1}}^2(\phi_j)\biggr\|_{L^\infty(S^1)} \\ &\qquad \times \int_{S^1} \sum_{j=1}^{N} \delta_{m_k}^2(\phi_j)\, d\mu. \end{aligned}$$

By a lemma from [4], we have

$$\biggl\|\sum_{j=1}^{N} \delta_i^2(\phi_j)\biggr\|_{L^\infty(S^1)} \le 2^{i+1} \lambda_i^2(\Phi) \tag{13}$$

for each $i = 0,1,2,\dots$. Therefore,

$$\begin{aligned} \frac{Q}{C_2} &\le (k!)^2 \sum_{m_k=0}^{\infty} \sum_{m_{k-1}=0}^{m_k} 2^{m_{k-1}+1} \lambda_{m_{k-1}}^2(\Phi) \\ &\qquad \times \sum_{m_{k-2}=0}^{m_{k-1}} 2^{m_{k-2}+1} \lambda_{m_{k-2}}^2(\Phi) \cdots \sum_{m_1=0}^{m_2} 2^{m_1+1} \lambda_{m_1}^2(\Phi) \int_{S^1} \sum_{j=1}^{N} \delta_{m_k}^2(\phi_j)\, d\mu \\ &= (k!)^2 2^{k-1} \sum_{m=0}^{\infty} \gamma'_m(\Phi) \sum_{j=1}^{N} \sum_{2^m \le |i| < 2^{m+1}} |\widehat{\phi}_j(i)|^2, \end{aligned}$$

and inequality (8) holds with constant $C = C_2 2^{k-1} (k!)^2$. □

Proof of Theorem 2. Preserving the notation from the proof of Theorem 1, we can write

$$\frac{Q}{C_2} \le k! \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} J_{m_1,m_2,\dots,m_k},$$

where, as before,

$$J_{m_1,m_2,\dots,m_k} := \int_{S^1} \sum_{j=1}^{N} \delta_{m_1}^2(\phi_j) \cdots \sum_{j=1}^{N} \delta_{m_k}^2(\phi_j)\, d\mu.$$

Suppose that $n = k/l$. Then again, in view of (13) and (5) (see also Remark 2), we have

$$\begin{aligned}\frac{Q}{C_2} &\le k!(l!)^n \sum_{0\le m_1\le\cdots\le m_l} \prod_{i=1}^{l-1} \Big\| \sum_{j=1}^{N} \delta_{m_i}^2(\phi_j) \Big\|_{L^\infty(S^1)} \sum_{0\le m_{l+1}\le\cdots\le m_{2l}} \prod_{i=1}^{l-1} \Big\| \sum_{j=1}^{N} \delta_{m_{l+i}}^2(\phi_j) \Big\|_{L^\infty(S^1)} \cdots \\ &\qquad \times \sum_{0\le m_{l(n-1)+1}\le\cdots\le m_k} \prod_{i=1}^{l-1} \Big\| \sum_{j=1}^{N} \delta_{m_{(n-1)l+i}}^2(\phi_j) \Big\|_{L^\infty(S^1)} \\ &\qquad \times \int_{S^1} \sum_{j=1}^{N} \delta_{m_l}^2(\phi_j) \sum_{j=1}^{N} \delta_{m_{2l}}^2(\phi_j) \cdots \sum_{j=1}^{N} \delta_{m_k}^2(\phi_j)\, d\mu \\ &\le k!(l!)^n 2^{(l-1)(n+k/2)} \sum_{m_l=0}^{\infty} 2^{(l-1)m_l} \sum_{m_{2l}=0}^{\infty} 2^{(l-1)m_{2l}} \cdots \sum_{m_k=0}^{\infty} 2^{(l-1)m_k} \\ &\qquad \times \int_{S^1} \sum_{j=1}^{N} \delta_{m_l}^2(\phi_j) \sum_{j=1}^{N} \delta_{m_{2l}}^2(\phi_j) \cdots \sum_{j=1}^{N} \delta_{m_k}^2(\phi_j)\, d\mu \\ &= k!(l!)^n 2^{(l-1)(n+k/2)} \int_{S^1} \bigg(\sum_{j=1}^{N} \sum_{m=0}^{\infty} 2^{(l-1)m} \delta_m^2(\phi_j) \bigg)^n d\mu.\end{aligned}$$

Hence applying the Minkowski and Littlewood–Paley inequalities, we find

$$\begin{aligned}\frac{Q}{C_2} &\le k!(l!)^n 2^{(l-1)(n+k/2)} \Bigg(\sum_{j=1}^{N} \bigg(\int_{S^1} \bigg(\sum_{m=0}^{\infty} 2^{(l-1)m} \delta_m^2(\phi_j) \bigg)^n \bigg)^{1/n} d\mu \Bigg)^n \\ &\le k!(l!)^n 2^{(l-1)(n+k/2)} C_3 \bigg(\sum_{j=1}^{N} \|\psi_j\|_{2n}^2 \bigg)^n, \end{aligned} \tag{14}$$

where $C_3 > 0$ is a constant depending only on k and

$$\psi_j(z) = \sum_{m=0}^{\infty} 2^{m(l-1)/2} \sum_{2^m\le|i|<2^{m+1}} \widehat{\phi}_j(i) z^i, \qquad j = 1, 2, \dots, N.$$

Define the sequence $(\alpha_i)_{i\in\mathbb{Z}}$:

$$\alpha_0 = 0 \quad \text{and} \quad \alpha_i = 2^{(l-1)m/2}|i|^{(1-l)/2} \quad \text{if } 2^m \le |i| < 2^{m+1},\ m = 0, 1, 2, \dots.$$

Since $|\alpha_i| \le 1$, $i \in \mathbb{Z}$, and the "variation" of this sequence on each binary interval, i.e., on the interval $[2^m, 2^{m+1}]$ or $[-2^{m+1}, -2^m]$, $m = 0, 1, 2, \dots$, is at most $1 - 2^{(1-l)/2}$, it follows by the Marcinkiewicz theorem (see [8, pp. 67–69] or [9, Chap. 4, Theorem 6]), that the sequence $(\alpha_i)_{i\in\mathbb{Z}}$ defines a Fourier multiplier in the space $L_p(S^1)$ for each $p \in (1, \infty)$. Thus, for all $f \in L_p(S^1)$,

$$\bigg(\int_{S^1} \bigg(\sum_{m=0}^{\infty} 2^{m(l-1)/2} \sum_{2^m\le|i|<2^{m+1}} \widehat{f}(i) z^i \bigg)^p d\mu \bigg)^{1/p} \le C_4 \|f^{((l-1)/2)}\|_{L_p(S^1)},$$

where $C_4 > 0$ is a constant depending only on p. Combining this with inequality (14), we obtain (10). The theorem is proved. □

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (grant no. 07-01-96603).

REFERENCES

1. E. Lieb and W. Thirring, "Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities," in *Studies in Mathematical Physics* (Princeton Univ. Press, Princeton, NJ, 1976), pp. 269–303.
2. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, in *Applied Mathematical Sciences* (Springer-Verlag, New York, 1997), Vol. 68.
3. A. A. Il'in, "Lieb-Thirring integral inequalities and their applications to attractors of Navier-Stokes equations," Mat. Sb. **196** (1), 33–66 (2005) [Russian Acad. Sci. Sb. Math. **196** (1), 29–61 (2005)].
4. B. S. Kashin, "On a class of inequalities for orthonormal systems," Mat. Zametki **80** (2), 204–208 (2006) [Math. Notes **80** (1–2), 199–203 (2006)].
5. J.-M. Ghidaglia, M. Marion, and R. Temam, "Generalization of the Sobolev–Lieb–Thirring inequalities and applications to the dimension of attractors," Differential Integral Equations **1** (1), 1–21 (1988).
6. A. Eden and C. Foias, "A simple proof of the generalized Lieb–Thirring inequalities in one-space dimension," J. Math. Anal. Appl. **162** (1), 250–254 (1991).
7. B. S. Kashin and A. A. Saakyan, *Orthogonal Series* (AFTs, Moscow, 1999) [in Russian].
8. S. M. Nikol'skii, *Approximation of Functions of Several Variables and Embedding Theorems* (Nauka, Moscow, 1969) [in Russian].
9. I. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, Princeton, NJ, 1970; Mir, Moscow, 1973).