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SERIES OF INDEPENDENT, MEAN ZERO RANDOM VARIABLES IN REARRANGEMENT-INVARIANT SPACES HAVING THE KRUGLOV PROPERTY

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This paper compares sequences of independent, mean zero random variables in a rearrangement-invariant space X on $[0,1]$ with sequences of disjoint copies of individual terms in the corresponding rearrangement-invariant space Z_X^2 on $[0,\infty)$. The principal results of the paper show that these sequences are equivalent in X and Z_X^2, respectively, if and only if X possesses the (so-called) Kruglov property. We also apply our technique to complement well-known results concerning the isomorphism between rearrangement-invariant spaces on $[0,1]$ and $[0,\infty)$. Bibliography: 20 titles.

1. Introduction

It follows from the classical Khintchine inequality that for any $p \in [1,\infty)$, the Rademacher system $\{r_n\}_{n=1}^\infty$ defined by $r_n(t) = \operatorname{sgn}\ \sin(2^n \pi t)$, $t \in [0,1)$, in the L_p-space on the interval $[0,1]$ is equivalent to the sequence of disjoint translates $\bar{r}_n(t) := r_n(t-n+1)$ in $L_2(0,\infty)$. If one treats the Rademacher system as a special example of sequences of independent, mean zero random variables, then a significant generalization of this inequality to the class of rearrangement-invariant (=r.i.) spaces X on $[0,1]$ is due to W. B. Johnson and G. Schechtman [1]. They introduced the r.i. space Z_X^2 on $[0,\infty)$ (our notation is different from that used in [1]) which is linked with a given r.i. space X on $[0,1]$ and showed that any sequence $\{f_k\}_{k=1}^\infty$ of independent, mean zero random variables in X is equivalent to the sequence of its disjoint translates $\{\bar{f}_k(\cdot) := f_k(\cdot - k + 1)\}_{k=1}^\infty$ in Z_X^2 provided that X contains an L_p-space for some $p < \infty$. In fact, the main difficulty is the proving of the inequality

$$\left\| \sum_{k=1}^n f_k \right\|_X \le C \left\| \sum_{k=1}^n \bar{f}_k \right\|_{Z_X^2} \tag{1.1}$$

(see the right-hand side of inequality (3) in [1, Theorem 1]).

The main tool used in the proof of (1.1) in [1] is a variant of the well-known Hoffman-Jørgensen inequality [2], which admits a reduction to the L_p-case. Our approach in this paper is based on a completely different reasoning which involves the theory of infinitely divisible distributions and study of a certain positive linear operator $\mathcal{K}$ on $L_1[0,1]$, which was introduced recently in [3] (see also [4]). Although the theory of infinitely divisible distributions is a well-known part of probability theory, its applications to the study of the geometry of an r.i. space X are less known. Such applications were pioneered by M. S. Braverman [5], who used earlier ideas and probabilistic constructions of V. M. Kruglov [6]. Our approach is also related to the method of stochastic integration with respect to the symmetrized Poisson process employed in [7] (see also [8]) to study the problem whether for a given r.i. space X on $[0,1]$ there exists an r.i. space on the semi-axis isomorphic to X. We discuss in detail these connections in Secs. 3 and 4 below, after introducing all the necessary definitions and some probabilistic constructions in Sec. 2. In Sec. 3, we prove our main result stating that for a wide class of r.i. spaces X, the operator $\mathcal{K}$ acts boundedly on X if and only if estimate (1.1) holds for any sequence $\{f_k\}_{k=1}^\infty \subseteq X$ of independent, mean zero random variables (see Theorems 3.1 and 3.6). Superficially, this assertion is analogous to [3, Theorem 6.1], where a similar statement is established for sequences of independent, nonnegative random variables (see the first part of Sec. 3). However, it should be pointed out that the proofs of these two results are completely different. This is explained by the difference in the settings (see also the difference between the proofs of inequalities (3) and (4) in [1, Theorem 1]). An advantage of our approach compared to that of [1] is seen in the fact that the former allows us to obtain inequality (1.1) in many r.i. spaces X (important in applications) that do not contain an L_p-space for any $p < \infty$. We exemplify this in Remark 3.5, by applying our results to the exponential Orlicz spaces, $\operatorname{Exp} L_p$, $0 < p \le 1$ (this class of spaces is not covered by [1, Theorem 1]). Theorem

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3.6 combined with results of [3] implies Corollary 3.7, which, in a certain sense, is a converse to the assertion given in the first part of [1, Theorem 1]. Simultaneously, this corollary shows that, unlike the situation with the Khintchine inequality, there is no "minimal" r.i. space E such that inequality (1.1) holds for every r.i. space $X \supseteq E$.

Finally, in Sec. 4, we apply our techniques to the following well-studied problem: does a given r.i. space X on $[0,1]$ admit an isomorphic representation as an r.i. space on the semi-axis (this problem was first formulated in [10] and then deeply studied in [7, 8]). First, it turns out that the assumption that the operator $\mathcal{K}$ acts boundedly on X implies the existence of a subspace in X that is isomorphic to the r.i. space Z_X^2 on $[0,\infty)$. Earlier, this assertion was established for a much smaller class of r.i. spaces with nontrivial lower Boyd index. Second, Theorem 3.1 can be used to provide a simple proof of the well-known result (see [7, §8] or [8, p. 203]) that an r.i. space X with nontrivial Boyd indices is isomorphic to an r.i. space on $[0,\infty)$. Finally, we expose in this section a deep connection between the operator $\mathcal{K}$ and the approach of [7] based on stochastic integration with respect to a symmetrized Poisson process.

We use the following abbreviations throughout this paper:

r.i. space is a rearrangement-invariant space;

r.v. is a random variable;

d.f. is a distribution function (of an r.v.);

ch.f. is a characteristic function (of an r.v.);

$f \overset{d}{=} g$ means that the d.f.'s of the r.v.'s f and g are equal.

2. Definitions and preliminaries

This section contains basic definitions and notation from the theory of rearrangement-invariant spaces (see, e.g., [11, 8]) and miscellaneous tools from probability theory (mainly taken from [5, 3]), which are used throughout this paper.

2.1. Rearrangement-invariant spaces.

A Banach space $(X, \|\cdot\|_X)$ of real-valued, Lebesgue measurable functions (with identification λ-a.e.) on an interval $J=[0,\alpha)$, $0<\alpha\le\infty$, is called *rearrangement-invariant* if

(i) X is an ideal lattice, i.e., if $y\in X$ and x is any measurable function on J such that $0\le|x|\le|y|$, then $x\in X$ and $\|x\|_X\le\|y\|_X$;

(ii) X is rearrangement-invariant in the sense that if $y\in X$ and x is any measurable function on J such that $x^*=y^*$, then $x\in X$ and $\|x\|_X=\|y\|_X$.

Here λ denotes Lebesgue measure and x^* denotes a nonincreasing, right-continuous rearrangement of x given by

$$x^*(t)=\inf\{\ \tau\ge 0 : n_{|x|}(\tau)\le t\ \},\quad t>0,$$

where $n_{|x|}(\tau):=\lambda\{s\ge 0:\ |x(s)|>\tau\}$.

We note that for any rearrangement-invariant (=r.i.) space $X=X[0,\infty)$ (respectively, $X=X[0,1]$),

$$L_1\cap L_\infty[0,\infty)\subseteq X\subseteq L_1+L_\infty[0,\infty)\quad (\text{respectively, } L_\infty[0,1]\subseteq X\subseteq L_1[0,1]),$$

with continuous embeddings [11] (see the definitions of a sum and intersection of Banach spaces which form a Banach couple below in this section).

Let X be an r.i. space on $[0,1]$. We also work with an r.i. space $X(\Omega,\mathcal{P})$ of measurable functions on a probability space $(\Omega,\mathcal{P})$ given by

$$X(\Omega,\mathcal{P}):=\{f\in L_1(\Omega,\mathcal{P}) : f^*\in X\},\quad \|f\|_{X(\Omega,\mathcal{P})}:=\|f^*\|_X.$$

Here, the decreasing rearrangement f^* is calculated with respect to the measure $\mathcal{P}$ on Ω. We denote by $S(\Omega)(=S(\Omega,\mu))$ the linear space of all measurable, finite a.e. functions on a given measure space (Ω,μ) equipped with the topology of convergence locally in measure.

The Köthe dual $X^\times$ of an r.i. space X on an interval $J=[0,\alpha)$ consists of all measurable functions y for which

$$\|y\|_{X^\times}:=\sup\Big\{\int_J|x(t)y(t)|dt:\ x\in X,\ \|x\|_X\le 1\Big\}<\infty.$$

Basic properties of Köthe duality can be found in [11] (where the Köthe dual is called the *associate* space). If X^* denotes the Banach dual of X, then it is known that $X^\times \subset X^*$, and $X^\times = X^*$ if and only if the norm $\|\cdot\|_X$ is order-continuous, i.e., from the relations $\{x_n\}_{n\geq 1} \subseteq X$ and $x_n \downarrow_n 0$ it follows that $\|x_n\|_X \to 0$. We note that the norm $\|\cdot\|_X$ of a rearrangement-invariant space X on J is order-continuous if and only if X is separable. An r.i. space X on J is said to have the *Fatou property* if, whenever $\{f_n\}_{n\geq 1} \subseteq X$, $f_n \to f$ a.e. on J, and $\sup_n \|f_n\|_X < \infty$, then $f \in X$ and $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$. It is well known that an r.i. space X has the Fatou property if and only if the natural embedding of X into its Köthe bidual $X^{\times\times}$ is a surjective isometry. Such spaces are also called *maximal.* Somewhat weaker than the notion of Fatou property of an r.i. space X is the notion of a Fatou norm. A norm $\|\cdot\|_X$ on X is said to be a *Fatou* (or *order subcontinuous) norm* if the unit ball of X is closed in E with respect to convergence almost everywhere. The norm on an r.i. space X is a Fatou norm if and only if the natural embedding $X \hookrightarrow X^{\times\times}$ of X into its Köthe bidual is an isometry.

Recall that for $\tau > 0$, the dilation operator σ_τ is defined by

$$\sigma_\tau x(t) = x(t/\tau) \quad \text{if} \quad J = [0,\infty)$$

and by

$$\sigma_\tau x(t) = \begin{cases} x(t/\tau), & 0 \leq t \leq \min(1,\tau), \\ 0, & \min(1,\tau) < t \leq 1, \end{cases}$$

if $J = [0,1]$, The operators σ_τ are bounded in every r.i. space X. The numbers α_X and β_X given by

$$\alpha_X := \lim_{\tau\to 0} \frac{\log\|\sigma_\tau\|_X}{\log \tau} \quad \text{and} \quad \beta_X := \lim_{\tau\to\infty} \frac{\log\|\sigma_\tau\|_X}{\log \tau}$$

belong to the closed interval $[0,1]$ and are called the *Boyd indices* of X. The Boyd indices of a given r.i. space X are said to be nontrivial if $0 < \alpha_X \leq \beta_X < 1$.

Important examples of r.i. spaces are L_p-spaces ($1 \leq p \leq \infty$) and also their generalizations, the Orlicz spaces. Let Φ be a convex, continuous, increasing function on $[0,\infty)$ such that $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. The Orlicz space $L_\Phi = L_\Phi[0,\alpha)$ consists of all functions $f \in S(0,\alpha)$ for which the norm

$$\|f\|_{L_\Phi} = \inf\left\{\rho > 0 : \int_0^\alpha \Phi(|f(t)|/\rho)\, dt \leq 1\right\}$$

is finite.

We denote by $\chi_E(t)$ the indicator function of a measurable set $E \subset [0,\infty)$ and by $\operatorname{supp} f$ the support of a measurable function f.

2.2. The Kruglov property and the operator $\mathcal{K}$ in r.i. spaces. Let f be a measurable function (or, an r.v.) on $[0,1]$ and let $\mathcal{F}_f$ be its d.f. By $\pi(f)$ we denote any r.v. on $[0,1]$ whose ch.f. is given by

$$\phi_{\pi(f)}(t) = \exp\left(\int_{-\infty}^{\infty} \left(e^{itx} - 1\right) d\mathcal{F}_f(x)\right),$$

or, equivalently, an r.v. $\sum_{i=1}^{N} f_i$, where the f_i are independent copies of f and N is a Poisson random variable with parameter 1 independent of the sequence $\{f_i\}$.

Definition 2.1. *An r.i. space X is said to have the Kruglov property (we write: $X \in \mathbb{K}$) if*

$$f \in X \Longleftrightarrow \pi(f) \in X.$$

This property has been studied and extensively used by Braverman [5]. Note that only the implication $f \in X \Longrightarrow \pi(f) \in X$ is nontrivial since the inverse implication is always satisfied thanks to Prokhorov's inequality [12] (see also [5, p. 11]). It is known that an r.i. space $X \in \mathbb{K}$ if $X \supseteq L_p$ for some $p < \infty$ [5, Theorem 2, p. 16]. In particular, the latter holds for an r.i. space X with a nontrivial lower Boyd index, i.e., if $\alpha_X > 0$.

Moreover, some exponential Orlicz spaces that do not contain L_p for any $p < \infty$ also possess this property (see Remark 3.5 below).

There exists an operator $\mathcal{K}$ which is closely related to the Kruglov property (see [3] and also [4]). First, we define an auxiliary operator $\mathcal{K}_1$ with values in $S(\Omega, \mathcal{P})$, where $(\Omega, \mathcal{P}) := \prod_{k=0}^{\infty}([0,1]\lambda_k)$ (here λ_k is Lebesgue measure on $[0,1]$ for every $k \geq 0$). Let $\{E_n\}$ be a sequence of pairwise disjoint subsets of $[0,1]$ such that $\lambda(E_n) = \frac{1}{e \cdot n!}$, $n \in \mathbb{N}$. For a given $f \in S([0,1], \lambda)$, we set

$$\mathcal{K}_1 f(\omega_0, \omega_1, \omega_2, \dots) := \sum_{n=1}^{\infty} \sum_{k=1}^{n} f(\omega_k) \chi_{E_n}(\omega_0).$$

It is well known that there exists a measure preserving isomorphism $\delta : (\Omega, \mathcal{P}) \to ([0,1], \lambda)$. For a function $g \in S(\Omega, \mathcal{P})$, we set $R(g)(x) := g(\delta^{-1}x)$, $x \in [0,1]$. It is easy to see that

$$\lambda\{x \in [0,1] : \ R(g)(x) < t\} = \mathcal{P}\{\omega \in \Omega : \ g(\omega) < t\}, \quad t \in \mathbb{R}.$$

Therefore, $\mathcal{K} := R\mathcal{K}_1$ is a positive linear operator from $S([0,1], \lambda)$ into $S([0,1], \lambda)$ such that the distribution functions of $\mathcal{K}f$ and $\mathcal{K}_1 f$ are the same for any measurable function f on $[0,1]$.

Since we work here with r.i. spaces, our main object of interest is the distribution of the function $\mathcal{K}f$; this allows us to restate the definition of the operator $\mathcal{K}$ from a somewhat different (and frequently, more convenient) viewpoint. Assume that $f \in S([0,1], \lambda)$ and $\{f_{n,k}\}_{k=1}^{n}$, $n \in \mathbb{N}$, is a sequence of measurable functions on $[0,1]$ such that for every $n \in \mathbb{N}$,

(i) $f_{n,1}, \ f_{n,2}, \dots, f_{n,n}, \ \chi_{E_n}$ is a sequence of independent r.v.'s;

(ii) $\mathcal{F}_{f_{n,k}} = \mathcal{F}_f, \quad k = 1, 2, \dots, n$.

In this case, for every $n \in \mathbb{N}$, we write

$$\mathcal{K}' f(x) := \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{n,k}(x) \chi_{E_n}(x), \quad x \in [0,1].$$

A straightforward verification shows that the d.f. of $\mathcal{K}f$ is the same as the d.f. of $\mathcal{K}'f$. Therefore, the operator $\mathcal{K}$ may be identified (in a certain sense) with the operator $\mathcal{K}'$.

Recall that the operator $\mathcal{K}$ maps an r.i. space X boundedly into itself if and only if X has the Kruglov property (see [3, Lemma 3.3]). In [3], the action of the linear operator $\mathcal{K}$ on various classes of r.i. spaces is studied. In this paper, our main task is to compare series of independent, mean zero random variables in r.i. spaces having the Kruglov property with their disjoint counterparts.

3. Estimates of sums of independent, mean zero r.v.'s in rearrangement-invariant spaces having the Kruglov property

Let X and Y be r.i. spaces on $[0,1]$ such that $X \subseteq Y$. Theorem 6.1 of [3] asserts that if either *(i)* the operator $\mathcal{K} : X \mapsto Y$ is bounded or *(ii)* the space Y has the Fatou norm and the operator $\mathcal{K} : X \mapsto Y^{\times\times}$ is bounded, then there exists $C > 0$ such that the following inequality holds for every $n \in \mathbb{N}$ and for any sequence of independent r.v.'s $\{f_k\}_{k=1}^n$:

$$\left\| \sum_{k=1}^{n} f_k \right\|_Y \leq C \left\| \sum_{k=1}^{n} \bar{f}_k \right\|_{Z_X^1},$$

where

$$Z_X^p := \{f \in (L_1 + L_\infty)(0, \infty) \ : \ \|f\|_{Z_X^p} := \|f^* \chi_{[0,1]}\|_X + \|f^* \chi_{[1,\infty)}\|_{L_p[1,\infty)} < \infty\}, \quad 1 \leq p < \infty,$$

and $\{\bar{f}_k\}_{k=1}^\infty$ is a sequence of pairwise disjoint functions on $[0, \infty)$ such that the d.f's for $\bar{f}_k$ and f_k, $k \in \mathbb{N}$, coincide. In this section, we prove a similar inequality (with Z_X^2 in place of Z_X^1) for sequences $\{f_k\}_{k=1}^n$ of independent, mean zero (i.e., such that $\int_0^1 f_k(t)dt = 0$, $k = 1, 2, \dots, n$) r.v.'s. It was remarked in the Introduction that this result strengthens [1, Theorem 1] in the case of normed spaces (for more details, see Remark 3.5 below).

Theorem 3.1. *Let X and Y be r.i. spaces on $[0,1]$ such that $X \subseteq Y$. If one of the following conditions is satisfied:*

(i) *the operator $\mathcal{K} : X \mapsto Y$ is bounded;*

(ii) *the space Y has a Fatou norm and $\mathcal{K} : X \mapsto Y^{\times\times}$ is bounded,*

then there exists a constant $C > 0$ such that the following inequality holds for every $n \in \mathbb{N}$ and for any sequence of independent mean zero variables $\{f_k\}_{k=1}^n \subset X$:

$$\left\| \sum_{k=1}^{n} f_k \right\|_Y \le C \left\| \sum_{k=1}^{n} \bar{f}_k \right\|_{Z_X^2} . \tag{3.1}$$

An important ingredient of the proof of Theorem 3.1 is contained in the following proposition.

Proposition 3.2. *If X and Y are r.i. spaces on $[0,1]$ such that $X \subseteq Y$ and $\mathcal{K} : X \mapsto Y$, then the following implication holds for any sequence $\{f_k\}_{k=1}^\infty$ of independent, symmetrically distributed r.v.'s:*

$$\text{if } \sum_{k=1}^{\infty} \bar{f}_k \in Z_X^2, \quad \text{then } \sum_{k=1}^{\infty} f_k \ \text{ converges a. e. on } [0,1] \ \text{ and } \sum_{k=1}^{\infty} f_k \in Y. \tag{3.2}$$

Proof. Denote by $F_k(\cdot)$ the d.f. of the r.v. f_k and let $\{g_k\}_{k=1}^\infty$ be a sequence of independent r.v.'s such that the d.f.'s of the r.v.'s g_k and $\pi(f_k)$ coincide for every $k = 1, 2, 3, \ldots$. Assume for a moment that the series $\sum_{k=1}^{\infty} g_k$ converges a.e. on $[0,1]$ to a r.v. $g \in Y$. Then the assertion of Proposition 3.2 can be obtained as follows. By Prokhorov's inequality ([12] or [5, p. 8]),

$$\lambda \left\{ t \in [0,1] : \left| \sum_{k=1}^{n} f_k(t) \right| \ge x \right\} \le 8\lambda \left\{ t \in [0,1] : \left| \sum_{k=1}^{n} g_k(t) \right| \ge \frac{x}{2} \right\} \tag{3.3}$$

for all $n \in \mathbb{N}$ and $x > 0$. Recalling (see, e.g., [13, Theorem 2.1.1, p. 29]) that a series of independent r.v.'s converges in measure if and only if this series converges a.e., we deduce from (3.3) that $\sum_{k=1}^{\infty} f_k$ converges a.e. to some r.v. f on $[0,1]$. Furthermore, the convergence in measure guarantees that there exists an increasing sequence of natural numbers $\{n_m\}_{m\ge1}$ such that

$$\lambda\{t : |f(t)| \ge \tau\} \ \le \ \lambda \left\{ t : \left| \sum_{k=1}^{n_m} f_k(t) \right| \ge \tau - \frac{1}{m} \right\} \ + \ \frac{1}{m}$$

for every $\tau > 0$ and $m \ge 1$, and

$$\lambda \left\{ t : \left| \sum_{k=1}^{n_m} g_k(t) \right| \ge \frac{\tau}{2} - \frac{1}{2m} \right\} \ \le \ \lambda \left\{ t : |g(t)| \ge \frac{\tau}{2} - \frac{1}{2m} \right\} \ + \ \frac{1}{m}.$$

These inequalities and (3.3) imply that

$$\begin{aligned} \lambda\{t : |f(t)| \ge \tau\} \ &\le \ 8\lambda \left\{ t : \left| \sum_{k=1}^{n_m} g_k(t) \right| \ge \frac{\tau}{2} - \frac{1}{2m} \right\} \ + \ \frac{1}{m} \\ &\le \ 8\lambda \left\{ t : |g(t)| \ge \frac{\tau}{2} - \frac{1}{2m} \right\} \ + \ \frac{2}{m}, \end{aligned}$$

or, passing to the limit as $m \to \infty$, that

$$\lambda\{t : |f(t)| \ge \tau\} \ \le \ 8\lambda \left\{ t : |g(t)| \ge \frac{\tau}{2} \right\}.$$

Since $g \in Y$, we immediately infer from the estimate above that $f \in Y$; this completes the proof of implication (3.2).

To prove that the series $\sum\limits_{k=1}^{\infty} g_k$ converges a.e. on $[0,1]$ and that its sum g belongs to the space Y, we need some auxiliary information concerning the ch.f. of the r.v. $S_n := \sum_{k=1}^{n} g_k$, $n \geq 1$. Since the r.v. g_k (as well as the r.v. f_k) is symmetrically distributed (see, e.g., [5, p. 11] or [3]), its ch.f. φ_k is a real-valued function given by

$$\varphi_k(t) = \exp\left(\int_{-\infty}^{\infty} (\cos(tx) - 1)\, dF_k(x)\right), \ t \in \mathbb{R}, \quad k = 1, 2, \dots . \tag{3.4}$$

Furthermore, the distribution of g_k, $k \geq 1$, is a mixture of the discrete Poisson distribution with parameter 1 and a family of convolutions of the F_k's (see, e.g., [5] or [3]), which is frequently referred to as a generalized (or compound) Poisson distribution (see, e.g., [14, Chap. 12] and [15, Chap. 17, p. 555]). The generalized Poisson distribution is infinitely divisible [15, Chap. 17, §1]; thus, by the Lévy-Khinchine formula (see, e.g., [5, p. 72]),

$$\varphi_k(t) = \exp\left(\int_{-\infty}^{\infty} (\cos(tx) - 1)\frac{1+x^2}{x^2}\, dH_k(x)\right), \ k = 1, 2, \dots , \tag{3.5}$$

where $H_k(x)$ is a bounded, nondecreasing, left continuous function on $\mathbb{R}$, which is called the (Lévy-Khinchine) spectral function. We may assume that $H_k(-\infty) = 0$. Comparing (3.4) and (3.5), we see that

$$H_k(x) = \int\limits_{-\infty}^{x} \frac{y^2}{1+y^2}\, dF_k(y), \ k = 1, 2, \dots . \tag{3.6}$$

We need the following auxiliary lemma.

Lemma 3.3. *Under the assumptions of Proposition* 3.2, *the series*

$$H(x) := \sum_{k=1}^{\infty} H_k(x) \tag{3.7}$$

converges uniformly on $\mathbb{R}$, *and* $H(\infty) = \sum\limits_{k=1}^{\infty} H_k(\infty) < \infty$.

Proof of Lemma 3.3. First of all, we note that since the function $\bar{f}(t) := \sum\limits_{k=1}^{\infty} \bar{f}_k(t) \in Z_X^2$, the function $F(x) := \sum\limits_{k=1}^{\infty} F_k(x) = \lambda\{t > 0 : \bar{f}(t) < x\}$ is finite for all $x < 0$. Since f_k is symmetrically distributed, at any continuity point $x \in \mathbb{R}$ of the function $F_k(x)$,

$$F_k(-x) = 1 - F_k(x), \ k = 1, 2, \dots . \tag{3.8}$$

Therefore, there exists a dense set $E \subseteq (0, \infty)$ such that

$$n_{\bar{f}}(x) := \lambda\{t > 0 : |\bar{f}(t)| > x\} = 2F(-x) \quad \text{for any} \quad x \in E.$$

Combining this fact with (3.6) and (3.8), we deduce that

$$\sum_{k=1}^{n} H_k(\infty) = -2\int_0^{\infty} \frac{y^2}{1+y^2}\, d\left(\sum_{k=1}^{n} F_k(-y)\right) \leq -2\int_0^{\infty} \frac{y^2}{1+y^2}\, dF(-y) = -\int\limits_0^{\infty} \frac{y^2}{1+y^2}\, dn_{\bar{f}}(y) \tag{3.9}$$

for all $n \in \mathbb{N}$. Without loss of generality, we assume that $\lambda\{t > 0 : |\bar{f}(t)| = \tau\} = 0$ for any $\tau \geq 0$. Let us consider the following two mutually exclusive cases:

(a) there exists $y_0 > 0$ such that $n_{\bar{f}}(y_0) = 1$;

(b) $n_{\bar{f}}(y) < 1$ for all $y > 0$, i.e., $\lambda\{\operatorname{supp} \bar{f}\} < 1$.

In case (a), it follows from (3.9) that

$$\begin{aligned}\sum_{k=1}^{n} H_k(\infty) &= -\int_0^{y_0} \frac{y^2}{1+y^2} dn_{\bar f}(y) - \int_{y_0}^{\infty} \frac{y^2}{1+y^2} dn_{\bar f}(y) \\ &\le -\int_0^{y_0} y^2 dn_{\bar f}(y) - \int_{y_0}^{\infty} dn_{\bar f}(y) \\ &= \|\bar f^* \mathbf{1}_{[1,\infty)}\|_{L_2}^2 + n_{\bar f}(y_0) \le \|\bar f\|_{Z_X^2}^2 + 1 < \infty.\end{aligned}$$

In case (b), similar (and simpler) estimates show that $\sum_{k=1}^{n} H_k(\infty) \le 1$. Thus, $\sum_{k=1}^{\infty} H_k(\infty) < \infty$. Since the function $H_k(x)$, $k \ge 1$, is increasing, it follows that series (3.7) converges uniformly on $\mathbb{R}$. Further, $H(x) \ge \sum_{k=1}^{n} H_k(x)$ for any $n \in \mathbb{N}$ and any $x > 0$. Now, first letting $x \to \infty$ and then letting $n \to \infty$, we conclude that

$$H(\infty) \ge \sum_{k=1}^{\infty} H_k(\infty).$$

Since the inverse inequality is obvious, this completes the proof of Lemma 3.3. $\square$

We continue the proof of Proposition 3.2. Let $S_n := \sum_{k=1}^{n} g_k$ and let ψ_n be the ch.f. of the r.v. S_n, $n = 1, 2, \ldots$. Since the g_k are independent,

$$\psi_n(t) = \prod_{k=1}^{n} \varphi_k(t) = \exp\left(\int_{-\infty}^{\infty} (\cos(tx) - 1) \frac{1+x^2}{x^2} d\left(\sum_{k=1}^{n} H_k(x)\right)\right).$$

Since the functions $H_k(x)$, $k = 1, 2, \ldots$, are increasing and $\cos(tx) - 1 \le 0$, the sequence $\{\psi_n(t)\}_{n=1}^{\infty}$ is monotonically decreasing for every fixed real t. By Lemma 3.3,

$$\lim_{n\to\infty} \psi_n(t) = \psi(t) := \exp\left(\int_{-\infty}^{\infty} (\cos(tx) - 1) \frac{1+x^2}{x^2} dH(x)\right). \tag{3.10}$$

Now we show that $\psi(t)$ is a ch.f. of some r.v.To this end, it is sufficient to verify that $\psi(t)$ is continuous at 0 (see [16, Chap. 6, § 5, Theorem 4]).

Let $\varepsilon > 0$ be given and fix $x_0 \ge 1$ so that $H(-x_0) + H(\infty) - H(x_0) < \frac{\varepsilon}{8}$. It is easy to see that there exists a constant $c = c(x_0)$ such that $(1 - \cos(tx))\frac{1+x^2}{x^2} \le c \cdot t^2$ for all $|x| \le x_0$. Thus, if $|t|$ is small enough, then

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} (\cos(tx) - 1) \frac{1+x^2}{x^2} dH(x)\right| \\ =&\int_{-x_0}^{x_0} (1 - \cos(tx)) \frac{1+x^2}{x^2} dH(x) + \int_{|x|\ge x_0} (1 - \cos(tx)) \frac{1+x^2}{x^2} dH(x) \\ \le& ct^2 (H(x_0) - H(-x_0)) + 4 \cdot (H(-x_0) + H(\infty) - H(x_0)) \\ \le& c \cdot H(\infty) t^2 + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Thus, ψ is a ch.f. of some r.v. g whose d.f. is the weak limit of distribution functions of the sums S_n (see [15, Chap. 15, §3, Theorem , p. 508]). We note that g has infinitely divisible distribution (see [15, Chap. 17, Corollary to Theorem 2, p. 557]). Applying the Itô-Nisio theorem [13, Theorem 2.1.1, pp. 29–32], we infer that the series $\sum_{k=1}^{\infty} g_k$ converges a.e. on $[0, 1]$ to the r.v. g. This completes the first part of the proof of Proposition 3.2. To complete the proof, it remains to show that $g \in Y$.

It follows from (3.6) and (3.8) that

$$H_k(\infty) - H_k(x) = \int\limits_x^\infty \frac{y^2}{1+y^2} dF_k(y) = \int\limits_x^\infty \frac{y^2}{1+y^2} d(1-F_k(-y))$$
$$= \int\limits_{-\infty}^{-x} \frac{y^2}{1+y^2} dF_k(y) = H_k(-x),\ k=1,2,\dots,$$

for all x from a dense subset of $(0,\infty)$. Hence (again taking into account (3.6)), we conclude that

$$H(-x) + H(\infty) - H(x) = 2\sum_{k=1}^\infty H_k(-x) \le 2\sum_{k=1}^\infty F_k(-x) = n_{\bar f}(x). \tag{3.11}$$

Since the function $H(x)$ is left continuous, there exists a measurable function u on $(0,\infty)$ such that

$$\lambda\{s>0\ :\ u(s)<x\} = H(x), \quad x\in\mathbb{R}.$$

By the assumption, $\bar f \in Z_X^2$; therefore, it follows from (3.11) that $u\in Z_X^2$. This immediately implies that $v\in Z_X^2$, where v is given by

$$v(s) = u(H(\infty)\cdot s), \quad s>0.$$

It immediately follows from the definition of the space Z_X^2 that $v^*(t)\chi_{[0,1]}(t)\in X$. Moreover, since

$$\lambda\{s>0\ :\ v(s)<x\} = \frac{H(x)}{H(\infty)}, \quad x\in\mathbb{R},$$

$v^*(t) = v^*(t)\chi_{[0,1]}(t)\in X$. Thus, there exists a r.v. $h\in X$ whose d.f. $\mathcal{F}_h(x)$ is given by

$$\mathcal{F}_h(x) := \frac{H(x)}{H(\infty)}, \quad x\in\mathbb{R}.$$

Now we show that the last fact and the assumption that $\mathcal{K}: X\mapsto Y$ imply that $g\in Y$ (see a somewhat similar reasoning in the proof of [5, Lemma 10, p. 72]).

Let us introduce the following notation:

$$U(t,x) := (\cos(tx)-1)\,\frac{1+x^2}{x^2}, \quad t,x\in\mathbb{R}, \tag{3.12}$$

$$\begin{aligned}\nu_1(t) &:= \exp\left(\int_{|x|<1} U(t,x)dH(x)\right), \quad t\in\mathbb{R},\\ \nu_2(t) &:= \exp\left(\int_{|x|\ge1} U(t,x)dH(x)\right), \quad t\in\mathbb{R}.\end{aligned} \tag{3.13}$$

It follows from (3.10) that $\psi(t) = \nu_1(t)\cdot\nu_2(t)$, $t\in\mathbb{R}$; thus, $g \overset{d}{=} g_1+g_2$, where g_1 and g_2 are independent r.v.'s with the ch.f.'s ν_1 and ν_2, respectively. To show that $g\in Y$, it is sufficient to prove that $g_1\in Y$ and $g_2\in Y$.

It follows from the Lévy-Khinchine formula that the r.v. g_1 has infinitely divisible distribution and that its corresponding spectral function is constant on each of the intervals $(-\infty,-1)$ and $(1,\infty)$. Thus, it follows from [17, Theorem 4] that

$$\liminf_{x\to\infty} \frac{-\log\lambda\{t\in[0,1]:|g_1(t)|\ge x\}}{x\log(1+x)} = 1.$$

The estimate above implies that

$$\lambda\{t\in[0,1]:|g_1(t)|\ge x\} \ \le\ \exp(-\frac{1}{2}x\log(1+x))$$

for sufficiently large $x > 0$. Consequently, g_1 belongs to the Orlicz space L_{M_1}, where $M_1(t) = e^{|t|\log(e+|t|)} - 1$, $t \in \mathbb{R}$ (see also [3, Lemma 4.3]). Since $X \supseteq L_\infty$ (see [11, Chap. II, §4]) and $\mathcal{K} : X \mapsto Y$, we deduce from [3, Theorem 4.4] that $Y \supseteq L_{M_1}$; in particular, $g_1 \in Y$.

To show that $g_2 \in Y$, we write

$$\nu_2(t) := \exp\left(\int_{-\infty}^{\infty} (\cos(tx) - 1) dW(x)\right), \tag{3.14}$$

where

$$W(x) := \begin{cases} \int\limits_{-\infty}^{x} \frac{1+y^2}{y^2} dH(y) & \text{if } \infty < x \leq -1, \\ W(-1) & \text{if } -1 < x \leq 1, \\ W(-1) + \int_1^x \frac{1+y^2}{y^2} dH(y) & \text{if } x > 1. \end{cases}$$

Clearly, $W(x)$ is a nonnegative, bounded, nondecreasing, left continuous function on $\mathbb{R}$ with $W(-\infty) = 0$.

It is sufficient to consider only the case where $H(\infty) > 0$. The latter assumption implies that $W(\infty) > 0$. Let us consider the d.f. $\mathcal{F}_W(\cdot)$ given by the equality $\mathcal{F}_W(x) = \frac{W(x)}{W(\infty)}$. By [5, Proposition 12, p. 74], the inclusion $g_2 \in Y$ holds if and only if $w \in Y$, where w is an r.v. with the ch.f.

$$\theta(t) = \exp\left(\int_{-\infty}^{\infty} (\cos(tx) - 1) d\mathcal{F}_W(x)\right).$$

Let w' be an r.v. with d.f. $\mathcal{F}_W(x)$. Since the r.v. h (with d.f. $\mathcal{F}_h(x) = \frac{H(x)}{H(\infty)}$) belongs to X, $w' \in X$. Indeed, since $1 < (1+y^2)y^{-2} \leq 2$ for $y \geq 1$,

$$\lambda\{t : |w'(t)| \geq x\} = \frac{1}{W(\infty)} \int_{|y| \geq x} \frac{1+y^2}{y^2} dH(y) \leq \frac{2H(\infty)}{W(\infty)} \lambda\{t : |h(t)| \geq x\}$$

for all $x \geq 1$. By (3.15), $w = \pi(w')$. Thus, combining the assumption that $\mathcal{K} : X \mapsto Y$, the condition $w' \in X$, and [3, Theorem 3.5], we deduce that $w \in Y$ and $g_2 \in Y$. This completes the proof of Proposition 3.2.

The following lemma, combined with Proposition 3.2, establishes an important particular case of Theorem 3.1.

Lemma 3.4. *If X and Y are r.i. spaces on $[0,1]$ such that $X \subseteq Y$ and implication (3.2) holds for any sequence of independent, symmetrically distributed r.v.'s $\{f_k\}_{k=1}^{\infty} \subset X$, then there exists $C > 0$ such that*

$$\left\| \sum_{k=1}^{\infty} f_k \right\|_Y \leq C \left\| \sum_{k=1}^{\infty} \bar{f}_k \right\|_{Z_X^2} \tag{3.16}$$

for any such sequence.

Proof. Define the following subspace of the space Z_X^2 :

$$Z_{X,s}^2 := \{f \in Z_X^2 \ : f^{(k)}(x) := f(x+k-1), \ \ 0 \leq x \leq 1, \\ \text{is symmetrically distributed for every} \quad k \geq 1\}.$$

We first show that the subspace $Z_{X,s}^2$ is closed in Z_X^2. Let $\{f_n\} \subset Z_{X,s}^2$ and let $f_n \to f \in Z_X^2$ in $\|\cdot\|_{Z_X^2}$. Clearly, $f_n^{(k)} \to f^{(k)}$ in measure for every $k = 1, 2, \ldots$. Now we fix an index $k \in \mathbb{N}$ and denote by E the set of all points $x \in (0,\infty)$ at which all the d.f. $F_n(x)$ of the r.v. $f_n^{(k)}$, $n = 1, 2, \ldots$, and the d.f. $F(x)$ of the r.v. $f^{(k)}$ are continuous. Clearly, E is a dense subset of $(0,\infty)$. Since convergence in measure yields weak convergence of distributions [15, Chap. 8, § 3, Lemma 2], $F_n(x) \to F(x)$ for every $x \in E$. Since the d.f.'s $F_n(\cdot)$, $n = 1, 2, \ldots$, are symmetric, $F(x) = 1 - F(-x)$, $x \in E$. Since E is dense in $(0,\infty)$, the last equality holds for all points $x > 0$ of continuity of $F(x)$. In other words, $f^{(k)}$ has symmetric distribution for any $k \in \mathbb{N}$; thus, $Z_{X,s}^2$ is closed.

Let us consider a sequence $\{L_n\}_{n\geq 1}$ of operators from $Z^2_{X,s}$ into $Y(\Omega,\mu)$ defined by

$$(L_n f)(w_1, w_2, \dots) := \sum_{k=1}^{n} f^{(k)}(w_k),\ \ f \in Z^2_{X,s},\ \ n \in \mathbb{N},$$

where $f^{(k)}(x) = f(x+k-1)$, $x \in [0,1]$, and $(\Omega, \mathcal{P}) = \prod_{k=1}^{\infty}([0,1], \lambda_k)$ (λ_k is Lebesgue measure on $[0,1]$). Note that the embeddings $L_n(Z^2_{X,s}) \subseteq Y(\Omega,\mathcal{P})$, $n \geq 1$, are guaranteed by the assumption that $X \subseteq Y$. Moreover, there exists a constant $C_1 < \infty$ such that

$$\|L_n f\|_{Y(\Omega,\mathcal{P})} \leq n \max_{k=1,\dots,n} \|f^{(k)}\|_Y \leq C_1 n \|f^* \mathbf{1}_{[0,1]}\|_X \leq C_1 n \|f\|_{Z^2_X}$$

for any $n = 1, 2, \dots$ and $f \in Z^2_{X,s}$. Thus, the linear operators L_n, $n \geq 1$, are bounded. By the assumption, the series $\sum_{k=1}^{\infty} f^{(k)}(w_k)$ converges $\mathcal{P}$-a.e. on Ω, and its sum belongs to the space Y. Consequently, it follows from [5, Proposition 11, p. 6] that

$$\sup_n \|L_n f\|_{Y(\Omega,\mathcal{P})} \leq C(Y) \left\| \sum_{k=1}^{\infty} f^{(k)}(w_k) \right\|_Y < \infty$$

for some constant $C(Y) > 0$ and for any $f \in Z^2_{X,s}$. Therefore, the sequence $\{L_n\}_{n\geq 1}$ is pointwise bounded on the Banach space $Z^2_{X,s}$. By the Banach-Steinhaus principle, $\sup_{n\geq 1} \|L_n\|_{Z^2_X \to Y} < \infty$. This means that there exists $C_2 > 0$ such that

$$\left\| \sum_{k=1}^{n} f^{(k)}(w_k) \right\|_{Y(\Omega,\mathcal{P})} \leq C_2 \|f\|_{Z^2_X}$$

for every $n \in \mathbb{N}$ and $f \in Z^2_{X,s}$. Using [5, Proposition 11, p. 6] once more, we conclude that

$$\left\| \sum_{k=1}^{\infty} f^{(k)}(w_k) \right\|_{Y(\Omega,\mathcal{P})} \leq C_2 C(Y) \|f\|_{Z^2_X}.$$

The latter inequality is equivalent to (3.16).

Proof of Theorem 3.1. Assume that assumption *(i)* holds. Let $\{f_k\}_{k=1}^n \subset X$ be a sequence of independent, mean zero r.v.'s. Using a standard symmetrization trick, we consider another sequence $\{f'_k\}_{k=1}^n$ of independent, mean zero r.v.'s (which is also independent with respect to the sequence $\{f_k\}_{k=1}^n$) such that $f'_k \overset{d}{=} f_k$, and define r.v.'s $h_k := f_k - f'_k$, $k = 1, \dots, n$. Clearly, $\{h_k\}_{k=1}^n$ is a sequence of independent, symmetrically distributed r.v.'s. By [5, Proposition 11, p. 6],

$$\left\| \sum_{k=1}^{n} f_k \right\|_Y \leq C(Y) \left\| \sum_{k=1}^{n} h_k \right\|_Y.$$

Noting that

$$\lambda\{t > 0 : \sum_{k=1}^{n} |\bar{h}_k(t)| > x\} \leq 2\lambda\{t > 0 : \sum_{k=1}^{n} |\bar{f}_k(t)| > x\}$$

for all $x \geq 0$, we immediately deduce (3.1) from Proposition 3.2 and Lemma 3.4.

Assume that assumption *(ii)* holds. The same reasoning as above shows that there exists a constant $C > 0$ such that

$$\left\| \sum_{k=1}^{n} f_k \right\|_{Y^{\times\times}} \leq C \left\| \sum_{k=1}^{n} \bar{f}_k \right\|_{Z^2_X}$$

for every sequence $\{f_k\}_{k=1}^n \subset X$ of independent, mean zero r.v.'s. It remains to note that since the norm in Y has the Fatou property, the inequality above is equivalent to (3.1).

Remark 3.5. As was mentioned in the Introduction, Theorem 3.1 strengthens the corresponding part of [1, Theorem 1] for normed spaces. In particular, setting $\operatorname{Exp} L_p := L_{N_p}$, where

$$N_p(t) := e^{|t|^p} - \sum_{k=0}^{[1/p]} \frac{|t|^{kp}}{k!}, \quad t > 0,$$

we note that $\operatorname{Exp} L_p \in \mathbb{K}$ if $0 < p \le 1$ (see [6], or [5], or [3]); therefore, Theorem 3.1 is applicable to these spaces. However, since $L_q \not\subseteq \operatorname{Exp} L_p$ for any $q \in (0,\infty)$ and $p \in (0,1)$, it follows that [1, Theorem 1] is not applicable in this setting.

Furthermore, it is also important to point out that if Y has a Fatou norm, then the converse to the assertion of Theorem 3.1 holds. Indeed, the application of Theorem 3.1 and a careful analysis of the proof of [3, Theorem 3.5] show that the following result holds.

Theorem 3.6. *Let $X \subseteq Y$ be r.i. spaces on $[0,1]$ such that Y has a Fatou norm. Then the following conditions are equivalent:*
(i) *the operator $\mathcal{K}$ acts boundedly from X into $Y^{\times\times}$;*
(ii) *there exists a constant $C > 0$ such that*

$$\left\| \sum_{k=1}^{n} f_k \right\|_Y \le C \left\| \sum_{k=1}^{n} \bar{f}_k \right\|_X \tag{3.17}$$

for every sequence $\{f_k\}_{k=1}^n \subseteq X$ of independent, identically and symmetrically distributed r.v.'s satisfying the assumption

$$\sum_{k=1}^{n} \lambda(\{f_k \ne 0\}) \le 1.$$

It is shown in [9] (see also [8, pp. 134–136]) that the Khintchine inequality holds for an r.i. space X if and only if X contains the separable part of the space $\operatorname{Exp} L_2 = L_{N_2}$, where $N_2(t) := \exp t^2 - 1$. However, in the general setting of sequences of independent, mean zero random variables, in contrast to the situation with the Khintchine inequality, there is no "minimal" r.i. space E such that inequality (3.17) holds for every r.i. space $X \supseteq E$. In fact, if E is such an r.i. space, then it contains an L_p-space for some $p < \infty$. This result is the converse (in a certain sense) to the first part of [1, Theorem 1].

Corollary 3.7. *If an r.i. space E is such that for every maximal r.i. space $X \supseteq E$ there exists a constant $C > 0$ such that* (3.17) *holds for every sequence $\{f_k\}_{k=1}^n \subseteq X$ of independent, identically and symmetrically distributed r.v.'s satisfying assumption* (3.18), *then E contains an L_p-space for some $p \in [1,\infty)$.*

Proof. Combine Theorem 3.6 above with [3, Corollaries 5.4 and 5.6].

4. Isomorphism between r.i. spaces on $[0,1]$ and $[0,\infty)$

In this section, we apply Theorem 3.1 to the problem whether a given r.i. space on a finite interval contains a subspace isomorphic to an r.i. space on the semi-axis. The general problem concerning isomorphism between r.i. spaces on $[0,1]$ and $[0,\infty)$ (other than L_p-spaces) was first posed in [10]. This and other related problems were extensively studied in [7] (see also [8]) via the approach using a stochastic integral with respect to a symmetrized Poisson process. Our approach in this paper is technically simpler; a somewhat similar approach was applied earlier in [18] to a special case of r.i. (Lorentz) spaces $L_{p,q}$. The first part of the following theorem strengthens earlier results concerning isomorphic embedding of some r.i. space on the semi-axis into a given r.i. space X on $[0,1]$ by replacing the assumption that $\alpha_X > 0$ by a weaker condition that the operator $\mathcal{K}$ is bounded in X. The second part of the theorem is well known (see [7, Sec. 8] or [8, p. 203]); however, our proof (based on Theorem 3.1) is much simpler.

Theorem 4.1. *Let X be an r.i. space on $[0,1]$. Then the following statements hold.*
(i) *If the operator $\mathcal{K}$ maps X into itself, then X contains a subspace isomorphic to the r.i. space Z_X^2.*
(ii) *If $0 < \alpha_X \le \beta_X < 1$, then the spaces X and Z_X^2 are isomorphic.*

Proof. Let us define on Z_X^2 the operator

$$Qf(t,\omega_1,\omega_2,\dots) = \sum_{k=1}^{\infty} f_k(\omega_k) r_k(t),$$

where

$$f_k(\omega_k) = f(k-1+\omega_k), \quad k = 1, 2, \dots .$$

By Theorem 3.1, the operator Q acts boundedly from Z_X^2 into $X(\Omega \times [0,1])$, where $\Omega = [0,1]^\infty$ with measure $\prod_{k=1}^{\infty} \lambda_k$ (λ_k is Lebesgue measure). The fact that there exists a constant $C > 0$ such that $\|Qf\|_{X(\Omega\times[0,1])} \geq C\|f\|_{Z_X^2}$ follows from the easy part of [1, Theorem 1] (the proof of the left-hand side of inequality (3) in [1, Theorem 1] holds for an arbitrary r.i. space X). Thus, the image $\mathfrak{I}(Q)$ of the operator Q is isomorphic to the space Z_X^2. Since the spaces X and $X(\Omega \times [0,1])$ are isomorphic, this completes the proof of part (i).

To prove part (ii), we first note that the condition $\mathcal{K} : X \mapsto X$ holds because $\alpha_X > 0$ [5, Theorem 3, p. 16]. We claim that the range $\mathfrak{I}(Q)$, which consists of functions $g \in X(\Omega \times [0,1])$ such that the series

$$g(t, \omega_1, \omega_2, \dots) = \sum_{k=1}^{\infty} g_k(\omega_k) r_k(t)$$

converges a.e. in $\Omega \times [0,1]$, is complemented in $X(\Omega \times [0,1])$.

We define a sequence of conditional expectations acting on $L_1(\Omega \times [0,1])$ by setting

$$\mathbb{E}(g|\omega_k)(\omega_k) := \int_0^1 \dots \int_0^1 g(t, \omega_1, \omega_2, \dots) dt d\omega_1, \dots d\omega_{k-1} d\omega_{k+1} \dots , \quad k \geq 1,$$

for every $g \in L_1(\Omega \times [0,1])$; we also define the projection

$$Pg(t, \omega_1, \omega_2, \dots) := \sum_{k=1}^{\infty} \mathbb{E}(g r_k|\omega_k) r_k(t).$$

We show that P is bounded in $L_p(\Omega \times [0,1]) \approx L_p$ for all $1 < p < \infty$. Consider the σ-algebra $\mathcal{F} := \{F \times [0,1] :$ F is a measurable subset in $\Omega\}$ of subsets of the space $\Omega \times [0,1]$. Note that every expectation $\mathbb{E}(\cdot|\omega_k)$, $k \geq 1$, can be obtained as a product of expectations $\mathbb{E}(\cdot|\mathfrak{A}_k)\mathbb{E}(\cdot|\mathfrak{B}_k)$, where the σ-algebras $\mathfrak{A}_k$ and $\mathfrak{B}_k$ are generated by the sets

$$A_k := \Big\{ \prod_{j=1}^{\infty} C_j : \ C_j \text{ is a measurable subset of } [0,1] \text{ if } j \leq k \text{ and } C_j = [0,1] \text{ if } j > k \Big\}$$

and

$$B_k := \Big\{ \prod_{j=1}^{\infty} C_j : \ C_j \text{ is a measurable subset of } [0,1] \text{ if } j \geq k \text{ and } C_j = [0,1] \text{ if } j < k \Big\},$$

respectively.

Now, applying first the Khintchine inequality, then twice the Stein Theorem for conditional expectations [19], and finally using the Khintchine inequality once more, we see that

$$\|Pf\|_{L_p(\Omega\times[0,1])} \asymp \left\| \left(\sum_{k=1}^{\infty} (\mathbb{E}(f r_k|\omega_k))^2 \right)^{1/2} \right\|_{L_p(\Omega)}$$

$$\leq C_p \left\| \left(\sum_{k=1}^{\infty} (\mathbb{E}(f r_k|\mathcal{F}))^2 \right)^{1/2} \right\|_{L_p(\Omega)} \asymp \left\| \sum_{k=1}^{\infty} \mathbb{E}(f r_k|\mathcal{F}) r_k \right\|_{L_p(\Omega\times[0,1])} .$$

Since

$$\sum_{k=1}^{\infty} \mathbb{E}(f r_k|\mathcal{F})(\omega_1, \omega_2, ..) r_k(t) = \sum_{k=1}^{\infty} \int_0^1 f(u, \omega_1, \omega_2, ..) r_k(u)\, du \cdot r_k(t),$$

we conclude (see, for example, [8, p. 137]) that for any $p \in (1,\infty)$ there exists a constant $C'_p < \infty$ such that

$$\int_0^1 |Pf(t,\omega_1,\omega_2,\dots)|^p dt \le C'_p \int_0^1 |f(t,\omega_1,\omega_2,\dots)|^p dt \quad \text{for any} \quad f \in L_p(\Omega \times [0,1]).$$

If we integrate this inequality over Ω, we see that P is bounded in $L_p(\Omega \times [0,1])$ for any $p \in (1,\infty)$. The interpolation Boyd theorem (see [8, 2.b.11]) implies now that the projection P is bounded in $X(\Omega \times [0,1])$. Moreover, it is obvious that $\mathfrak{I}(P) = \mathfrak{I}(Q)$. Hence, $\mathfrak{I}(Q)$ (which is isomorphic to Z_X^2 by part (i)) is complemented in $X(\Omega \times [0,1]) \approx X$. On the other hand, X is obviously isomorphic to a complemented subspace of Z_X^2. Since $X \oplus X$ (respectively, $Z_X^2 \oplus Z_X^2$) is isomorphic to X (respectively, Z_X^2), we may apply Pelczynski's decomposition method [8, p. 172], which completes the proof of the isomorphism between X and Z_X^2.

At conclusion, we inticate a close relationship between a stochastic Poisson integral, which is a key instrument in constructions of [7, Sec. 8] and [8, Sec. 2.f], and the operator $\mathcal{K}$ introduced in [3].

For a complete definition of the operator T of stochastic integration with respect to a symmetrized Poisson process, we refer to [8, p. 205–206]. In the special case where an r.v. f is defined on $[0,1]$, we set $Tf = T'f - T''f$, where

$$T'f(u,v) := \sum_{k=1}^{\infty} \chi_{F'_k}(u) \sum_{j=1}^{k} f(\varphi'_j(v)), \quad u, v \in [0,1].$$

Here $\{F'_k\}_{k\ge 0}$ is a sequence of pairwise disjoint subsets of $[0,1]$, $m(F'_k) = \frac{1}{e\cdot k!}$, $k = 0,1,2,\dots$, and $\{\varphi'_j\}_{j=1}^{\infty}$ is a sequence of independent random variables uniformly distributed on $[0,1]$. The function $T''f$ is defined similarly, and the corresponding sequences $\{F''_k\}_{k\ge 0}$ and $\{\varphi''_j\}_{j=1}^{\infty}$ are independent of the sequences $\{F'_k\}_{k\ge 0}$ and $\{\varphi'_j\}_{j=1}^{\infty}$, respectively. Thus, the functions $T'f$ and $T''f$ are equidistributed and independent.

It is shown in [7, Sec. 8] (see also [8, Sec. 2.f]) that the operator T acts boundedly from Z_X^2 to X provided that $\alpha_X > 0$.

The following theorem shows that this result holds under a weaker assumption on X.

Theorem 4.2. *If X is an r.i. space on $[0,1]$ that is either separable or maximal, then the operator T acts boundedly from Z_X^2 into X if and only if the operator $\mathcal{K}$ is bounded in X.*

Proof. If an r.v. f is defined on $[0,1]$, then the sequence $\{f(\varphi'_j)\}_{j=1}^{\infty}$ from the definition of the operator T is a sequence of independent r.v.'s equidistributed with the r.v. f. Consequently, it follows from the definition of the operator $\mathcal{K}$ that $T'f \overset{d}{=} \mathcal{K}f$; thus, $\|T'f\|_X = \|\mathcal{K}f\|_X$. Moreover, the independence of $T'f$ and $T''f$ yields the equality

$$\begin{aligned}\lambda\{(u,v) : |Tf(u,v)| > \tau\} \;&\ge\; \lambda\{(u,v) : |T'f(u,v)| > \tau \text{ and } u \in F''_0\} \\ &=\; e^{-1}\lambda\{(u,v) : |T'f(u,v)| > \tau\};\end{aligned}$$

hence,

$$\|\mathcal{K}f\|_X \;=\; \|T'f\|_X \;\le\; \|\sigma_e\|_{X\to X}\|Tf\|_X \;\le\; e\|Tf\|_X.$$

Thus, if the operator $T : Z_X^2 \to X$ is bounded, so is the operator $\mathcal{K} : X \to X$.

Conversely, let us assume that $\mathcal{K} : X \to X$ is bounded. If an r.v. f is defined on $[0,1]$, then

$$\|Tf\|_X = \|T'f - T''f\|_X \le 2\|\mathcal{K}f\|_X. \tag{4.1}$$

Let us consider the case of an arbitrary function $f \in Z_X^2$. Since the operators T' and T'' are positive, we may assume that f is nonnegative. Set $f_n := f\chi_{(n-1,n)}$, $n = 1,2,\dots$. It is easy to see that there exist sequences $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ such that $f_n = g_n + h_n$, $g_n h_n = 0$, and $\int_{n-1}^{n} g_n(t)\,dt = \int_{n-1}^{n} h_n(t)\,dt$, $n = 1,2,\dots$. Hence, if $f'_n := g_n - h_n$, then $|f'_n| = f_n$ and $\int_{n-1}^{n} f'_n(t)\,dt = 0$, $n = 1,2,\dots$.

Let now $\{b_n\}_{n=1}^{\infty}$ be a sequence of independent copies of f'_n on $[0,1]$ (i.e., $b_n \overset{d}{=} f'_n$, $n \ge 1$). By Theorem 3.1,

$$\Big\|\sum_{k=1}^{n} b_k\Big\|_X \;\le\; C\Big\|\sum_{k=1}^{n} f'_k\Big\|_{Z_X^2} \;=\; C\Big\|\sum_{k=1}^{n} f_k\Big\|_{Z_X^2}. \tag{4.2}$$

Moreover, the definition of the operator T [8, p. 207] implies that $\{Tf_n\}_{n=1}^{\infty}$ and $\{Tb_n\}_{n=1}^{\infty}$ are two sequences of symmetric independent functions, and Tf_n and Tb_n are equally distributed for any $n = 1, 2, \ldots$. Therefore, by [20, Theorem 5.4.4],

$$\Big\|\sum_{k=1}^{n} Tb_k\Big\|_X \asymp \Big\|\sum_{k=1}^{n} Tf_k\Big\|_X, \quad n = 1, 2, \ldots.$$

Relations (4.1) and (4.2) and the fact that the operator $\mathcal{K}$ is bounded in the space X imply that

$$\Big\|T\Big(\sum_{k=1}^{n} f_k\Big)\Big\|_X = \Big\|\sum_{k=1}^{n} Tf_k\Big\|_X \asymp \Big\|\sum_{k=1}^{n} Tb_k\Big\|_X$$

$$\le \Big\|T\Big(\sum_{k=1}^{n} b_k\Big)\Big\|_X \le 2\Big\|\mathcal{K}\Big(\sum_{k=1}^{n} b_k\Big)\Big\|_X \le 2\|\mathcal{K}\|_{X\to X}\Big\|\sum_{k=1}^{n} b_k\Big\|_X$$

$$\le 2\|\mathcal{K}\|_{X\to X}C\Big\|\sum_{k=1}^{n} f_k\Big\|_{Z_X^2} \le 2\|\mathcal{K}\|_{X\to X}C\|f\|_{Z_X^2}.$$

Since X is either separable or maximal, a standard reasoning shows that the operator $T : Z_X^2 \to X$ is bounded.

Theorem 4.2 and [8, Theorem 2.f.1(i)] yield the following corollary.

Corollary 4.3. *If X is an r.i. space on $[0,1]$ that is either separable or maximal and if $X \in \mathbb{K}$, then the range of the stochastic integration operator with respect to the symmetrized Poisson process is isomorphic to the space Z_X^2.*

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