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On the Normability of Marcinkiewicz Classes

S. V. Astashkin*

Samara State University

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Abstract—We obtain simple necessary and sufficient conditions for the normability of Marcinkiewicz classes, which play an important role in various problems of analysis. These conditions are stated in terms of dilation exponents of the function generating the given class as well as in terms of interpolation for operators bounded in the couple (L_1, L_∞).

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1. INTRODUCTION

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be an increasing function such that $\varphi(t) > 0$, $t > 0$. The *Marcinkiewicz class* M_φ is defined as the set of all measurable functions $x = x(t)$ on $[0,\infty)$ such that the functional

$$\|x\|_\varphi = \sup_{t>0} x^*(t)\varphi(t) \tag{1}$$

is finite. From now on, by $x^*(t)$ we denote the nondecreasing rearrangement of the function $|x(t)|$. In particular, for $\varphi(t) = t^{1/p}$, $0 < p < \infty$, we obtain the classical spaces $L_{p,\infty}$ introduced in [1] (see also [2, p. 197]). One can readily show (e.g., as was done for similar classes of integral type in [3]) that $\|\cdot\|_\varphi$ is a quasinorm (i.e.,

$$\|x + y\|_\varphi \le C(\|x\|_\varphi + \|y\|_\varphi), \qquad x, y \in M_\varphi,$$

for some $C > 0$) if and only if the function φ satisfies the Δ_2-*condition*. This condition means that there exists a constant $K > 0$ such that

$$\varphi(2t) \le K\varphi(t) \qquad \text{for} \quad t > 0.$$

It is well known [2, Theor. 2.5.3, p. 156] that the functional (1) is equivalent to the norm in the corresponding symmetric Marcinkiewicz space (see the definitions below) if and only if the upper dilation exponent of φ is nontrivial. Here we consider the more general question about the normability of the class M_φ. A similar problem was studied in [4] for the sets

$$\Lambda^{p,\infty}(\omega), \qquad \text{where} \quad \omega\colon [0,\infty) \to [0,\infty) \quad \text{and} \quad \omega(s) > 0 \quad \text{for } s > 0;$$

these sets consist of measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|_{\Lambda^{p,\infty}(\omega)} = \sup_{t>0} f^*(t)\biggl(\int_0^t \omega(s)\,ds\biggr)^{1/p} < \infty. \tag{2}$$

It was proved in [4] that the normability of $\Lambda^{p,\infty}(\omega)$ is equivalent to each of the following conditions:

*E-mail: astashkn@ssu.samara.ru

(i) The maximal Hardy–Littlewood operator

$$Mf(x) = \sup_{r>0} (2r)^{-n} \int_{Q(x,r)} |f(y)|\, dy,$$

where $Q(x,r)$ is the cube centered at x with edges of length $2r$ parallel to the coordinate axes, is bounded in $\Lambda^{p,\infty}(\omega)$.

(ii) There exists a constant $C > 0$ such that the function

$$W_p(t) := \biggl(\int_0^t \omega(s)\, ds\biggr)^{1/p}$$

satisfies the inequality

$$\int_0^t \frac{1}{W_p(s)}\, ds \le C \frac{t}{W_p(t)}\,, \qquad t > 0. \tag{3}$$

Note that condition (ii) clearly shows that there is some "redundancy" in definition (2). (The answer depends only on W_p.) In this connection, we prefer to define Marcinkiewicz classes (referred to as *Lorentz spaces of the weak type* in [4]) by relation (1).

The main goal of the present paper is to find simpler and more efficient conditions for the normability of M_φ than (i) and (ii). We claim that normability is equivalent, on the one hand (just as for the above-mentioned problem on the coincidence of M_φ with the corresponding Marcinkiewicz space), to the nontriviality of the upper dilation exponent of φ and, on the other hand, to the interpolation property of M_φ with respect to the couple of spaces $L_1 = L_1(0,\infty)$ and $L_\infty = L_\infty(0,\infty)$. We restrict our considerations to function spaces on the half-line $(0,\infty)$, but all assertions remain valid word for word in the case of $\mathbb{R}^n$. At the end of the paper, we show that the cited results in [4] are immediate corollaries of those proved here. In conclusion, note that the more recent paper [5] gives a detailed study of similar problems for discrete Marcinkiewicz classes.

2. DEFINITIONS AND NOTATION

A detailed exposition of the theory of symmetric function spaces can be found, e.g., in the monographs [2] and [6].

If $x = x(t)$ is a measurable function on $[0,\infty)$, then, by definition,

$$n_{|x|}(\tau) = \mu\{t \in [0,\infty) : |x(t)| > \tau\}, \qquad \tau > 0,$$

where μ is the Lebesgue measure. Functions $x(t)$ and $y(t)$ are said to be *equimeasurable* if we have $n_{|x|}(\tau) = n_{|y|}(\tau)$, $\tau > 0$. In particular, each function $x(t)$ is equimeasurable with its nondecreasing rearrangement $x^*(t) = \inf\{\tau \ge 0 : n_{|x|}(\tau) < t\}$.

A Banach space X of measurable functions defined on $[0,\infty)$ is said to be *symmetric* if the relations $y = y(t) \in X$ and $x^*(t) \le y^*(t)$ imply that $x = x(t) \in X$ and $\|x\|_X \le \|y\|_X$.

The *fundamental function* of a symmetric space X is defined by the formula $\varphi_X(t) = \|\chi_{(0,t)}\|_X$, where, as usual, $\chi_E(s) = 1$ for $s \in E$ and $\chi_E(s) = 0$ for $s \notin E$. Recall that the function $\varphi_X(t)$ is *quasiconcave* on $(0,\infty)$ [2, p. 137]; i.e., it is nonnegative and increasing, and the function $\varphi_X(t)/t$ is decreasing. Each quasiconcave function $f(t)$ on the half-line is equivalent to the least concave majorant $\bar{f}(t)$ of itself; more precisely, $f(t) \le \bar{f}(t) \le 2f(t)$ for $t > 0$ [2, p. 70].

Recently, along with ordinary L_p-spaces, $1 \le p \le \infty$, the classes of symmetric Orlicz, Lorentz, and Marcinkiewicz spaces, important in applications, have been studied. Let us recall the definition of spaces of the last class.

Let $\psi(t)$ be a quasiconcave function on $[0,\infty)$. The *Marcinkiewicz space* $M(\psi)$ consists of all measurable functions $x = x(s)$ on $[0,\infty)$ such that

$$\|x\|_{M(\psi)} = \sup_{0<t<\infty} \frac{\int_0^t x^*(s)\, ds}{\psi(t)} < \infty.$$

This space is the dual of the Lorentz space $\Lambda(\bar{\psi})$ [2, p. 152–154], where the norm is defined as follows:

$$\|x\|_{\Lambda(\bar{\psi})} = \int_0^\infty x^*(s)\, d\bar{\psi}(s).$$

The fundamental function of the space $M(\psi)$ is equal to $t/\psi(t)$.

If $f(s)$ is a nonnegative function on $[0,\infty)$, then the *dilation function* of $f(s)$ is defined as

$$\mathscr{M}_f(t) = \sup_{s>0} \frac{f(st)}{f(s)}\,, \qquad t > 0.$$

Since $\mathscr{M}_f(t)$ is semimultiplicative, it follows that there exist numbers

$$\gamma_f = \lim_{t\to 0+} \frac{\ln \mathscr{M}_f(t)}{\ln t} \qquad \text{and} \qquad \delta_f = \lim_{t\to +\infty} \frac{\ln \mathscr{M}_f(t)}{\ln t}\,,$$

which are called the *lower* and *upper dilation exponents* of f. If f is quasiconcave, then

$$0 \le \gamma_f \le \delta_f \le 1.$$

Let (X_0, X_1) be a *Banach couple*, i.e., a pair of Banach spaces linearly and continuously embedded in a separable linear topological space. We say that a set X is an *interpolation set* with respect to this couple if it is intermediate with respect to it, i.e., if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and each linear operator T bounded in X_0 and in X_1 satisfies $T(X) \subset X$.

Finally, we say that two nonnegative functions $f(t)$ and $g(t)$ on $[0,\infty)$ are *equivalent* if we have

$$cf(t) \le g(t) \le Cf(t), \qquad t \ge 0,$$

for some $c > 0$ and $C > 0$.

3. STATEMENT OF RESULTS AND PROOFS

Theorem 1. *The following conditions are equivalent.*

(a) $\delta_\varphi < 1$.

(b) If a measurable function $x(t)$ defined on $(0,\infty)$ satisfies the inequality

$$\int_0^t x^*(s)\, ds \le \int_0^t \frac{ds}{\varphi(s)}\,, \qquad t > 0,$$

then

$$x^*(t) \le \frac{C}{\varphi(t)}\,, \qquad t > 0,$$

where the constant $C > 0$ is independent of t.

(c) The space M_φ is normable.

(d) The space M_φ is an interpolation set between L_1 and L_∞.

We start from an auxiliary assertion, which is essentially known for functions defined on a closed interval [7, Lemma 2]. (See also [2, Lemma 2.1.4, p. 80]). To make the exposition self-contained, we also present the proof.

Lemma 1. *Suppose that $\varphi(t)$ is a function increasing for $t > 0$. Then the inequality*

$$\frac{1}{t}\int_0^t \frac{ds}{\varphi(s)} \le \frac{C}{\varphi(t)}\,, \qquad t > 0, \tag{4}$$

holds with some $C > 0$ if and only if $\delta_\varphi < 1$.

Proof. If $\delta_\varphi < 1$, then the dilation function $\mathscr{M}_\varphi(t)$ satisfies the inequality

$$\mathscr{M}_\varphi(t) \le C_1 t^{1-\varepsilon}, \qquad t \ge 1,$$

with some $C_1 > 0$ and $\varepsilon > 0$ independent of t. Then

$$\frac{\varphi(t)}{t}\int_0^t \frac{ds}{\varphi(s)} = \int_0^1 \frac{\varphi(t)}{\varphi(ut)}\,du \le \int_0^1 \mathscr{M}_\varphi(1/u)\,du \le C_1\int_0^1 u^{\varepsilon-1}\,du = \frac{C_1}{\varepsilon},$$

and hence inequality (4) holds with $C := C_1/\varepsilon$.

Conversely, let (4) hold. Since $\varphi(t)$ increases, we see that

$$\frac{1}{t}\int_0^t \frac{ds}{\varphi(s)} \ge \frac{1}{\varphi(t)};$$

hence $\varphi(t)$ is equivalent to the concave function

$$\widetilde{\Phi}(t) := \frac{t}{\Phi(t)}\,, \qquad \text{where} \quad \Phi(t) = \int_0^t \frac{ds}{\varphi(s)}\,.$$

Now, by [2, Theorem 2.1.1],

$$\frac{t}{\varphi(t)} \le K\frac{u}{\varphi(u)} \tag{5}$$

for some $K > 0$ and for all t and u, $0 < t < u < \infty$. Suppose that $\delta_\varphi \ge 1$. Then, for each $n \in \mathbb{N}$, there exists a $t_n > 0$ such that

$$\frac{\varphi(nt_n)}{\varphi(t_n)} \ge \frac{n}{2}\,.$$

Consequently, in view of (5),

$$\frac{\varphi(nt_n)}{nt_n}\int_0^{nt_n} \frac{ds}{\varphi(s)} \ge \frac{\varphi(t_n)}{2t_n}\int_{t_n}^{nt_n} \frac{s}{\varphi(s)}\frac{ds}{s} \ge \frac{\ln n}{2K}\,, \qquad n \in \mathbb{N},$$

whence it follows that

$$\sup_{t>0}\frac{\varphi(t)}{t}\int_0^t \frac{ds}{\varphi(s)} = \infty, \tag{6}$$

which contradicts (4). $\square$

Proof of Theorem 1. (a) $\Rightarrow$ (b). If $\delta_\varphi < 1$, then

$$x^*(t) \le \frac{1}{t}\int_0^t x^*(s)\,ds \le \frac{1}{t}\int_0^t \frac{ds}{\varphi(s)} \le \frac{C}{\varphi(t)}\,, \qquad t > 0,$$

by Lemma 1 and by our assumption.

(b) $\Rightarrow$ (c). Let $M(\Phi)$ be the symmetric Marcinkiewicz space with

$$\Phi(t) = \int_0^t \frac{ds}{\varphi(s)}\,.$$

Condition (b) means that

$$M(\Phi) \subset M_\varphi \qquad \text{and} \qquad \|x\|_\varphi \le C\|x\|_{M(\Phi)}.$$

On the other hand, if $x \in M_\varphi$, then

$$\int_0^t x^*(s)\,ds \le \|x\|_\varphi \int_0^t \frac{ds}{\varphi(s)}\,;$$

i.e.,

$$x \in M(\Phi) \qquad \text{and} \qquad \|x\|_{M(\Phi)} \le \|x\|_\varphi.$$

Consequently, $M_\varphi = M(\Phi)$ ($\|\cdot\|_\varphi$ and $\|\cdot\|_{M(\Phi)}$ being equivalent). This proves (c).

(c) $\Rightarrow$ (d). Let $\|\cdot\|$ be a norm on M_φ equivalent to the functional $\|\cdot\|_\varphi$. Thus, M_φ becomes a symmetric space whose fundamental function $\psi(t)$ is equivalent to the function $\varphi(t) = \|\chi_{(0,t)}\|_\varphi$. Hence the norm in the Marcinkiewicz space $M(\widetilde{\psi})$, $\widetilde{\psi}(t) = t/\psi(t)$, is equivalent to

$$\sup_{t>0} \frac{\varphi(t)\int_0^t x^*(s)\,ds}{t}.$$

Thus, in view of the inequality

$$\int_0^t x^*(s)\,ds \ge t x^*(t),$$

the embedding $M(\widetilde{\psi}) \subset M_\varphi$ holds and $\|x\|_\varphi \le C_1\|x\|_{M(\widetilde{\psi})}$. At the same time $M(\widetilde{\psi})$ is the largest of all symmetric spaces with fundamental function ψ [2, p. 162]. Hence

$$M_\varphi \subset M(\widetilde{\psi}) \qquad \text{and} \qquad \|x\|_{M(\widetilde{\psi})} \le C_2\|x\|_\varphi.$$

As a result, $M_\varphi = M(\widetilde{\psi})$ (the norms being equivalent), and hence M_φ is an interpolation set between L_1 and L_∞ [2, p. 155].

(d) $\Rightarrow$ (a). Let M_φ be an interpolation set between L_1 and L_∞. Then, in particular, $M_\varphi \subset L_1 + L_\infty$, and hence, in view of [2, p. 108],

$$\left\|\frac{1}{\varphi}\right\|_{L_1+L_\infty} = \int_0^1 \left(\frac{1}{\varphi(s)}\right)^* ds = \int_0^1 \frac{ds}{\varphi(s)}\,ds < \infty;$$

i.e., the function $1/\varphi$ is locally integrable on $(0,\infty)$.

Suppose that $\delta_\varphi \ge 1$. Then, by Lemma 1, inequality (4) fails for any $C > 0$, and we arrive at (6). In this case, at least one of the following relations holds:

$$A := \sup_{0<t\le 1} \frac{\varphi(t)}{t}\int_0^t \frac{ds}{\varphi(s)} = \infty \qquad \text{or} \qquad B := \sup_{t\ge 1} \frac{\varphi(t)}{t}\int_0^t \frac{ds}{\varphi(s)} = \infty.$$

Suppose that, say, $A = \infty$. Then there exists a $\tau_1 > 0$ such that

$$\frac{\varphi(\tau_1)}{\tau_1}\int_0^{\tau_1} \frac{ds}{\varphi(s)} > 2.$$

Since

$$\lim_{t\to 0+} \frac{\varphi(\tau_1)}{\tau_1 - t}\int_0^{\tau_1} \frac{ds}{\varphi(s)} = \frac{\varphi(\tau_1)}{\tau_1}\int_0^{\tau_1} \frac{ds}{\varphi(s)},$$

it follows that

$$\frac{\varphi(\tau_1)}{\tau_1 - t}\int_t^{\tau_1} \frac{ds}{\varphi(s)} > 1, \qquad 0 < t < \varepsilon, \tag{7}$$

for some $\varepsilon \in (0,\tau_1)$. Next, again since $A = \infty$, we see that there exists a $\tau_2 \in (0,\varepsilon)$ such that

$$\frac{\varphi(\tau_2)}{\tau_2}\int_0^{\tau_2} \frac{ds}{\varphi(s)} > 3.$$

In particular, in view of (7),

$$\frac{\varphi(\tau_1)}{\tau_1 - \tau_2}\int_{\tau_2}^{\tau_1} \frac{ds}{\varphi(s)} > 1.$$

By continuing this process, we construct a sequence

$$\{\tau_n\}_{n=1}^{\infty}, \qquad \tau_1 > \tau_2 > \cdots, \quad \lim_{n\to\infty} \tau_n = 0,$$

so that

$$\sup_{n\in\mathbb{N}} \frac{\varphi(\tau_n)}{\tau_n - \tau_{n+1}} \int_{\tau_{n+1}}^{\tau_n} \frac{ds}{\varphi(s)} = \infty. \tag{8}$$

Note that the same relation results if $B = \infty$, with the only difference that $\tau_1 < \tau_2 < \cdots$ and $\lim_{n\to\infty} \tau_n = \infty$.

Let Δ_n be the interval $(\tau_{n+1}, \tau_n]$ if $A = \infty$ and the interval $(\tau_n, \tau_{n+1}]$ if $B = \infty$. In addition, let

$$u(t) = \sum_{n=1}^{\infty} \frac{\Phi(\tau_n) - \Phi(\tau_{n+1})}{\tau_n - \tau_{n+1}} \chi_{\Delta_n}(t), \qquad \text{where} \quad \Phi(t) = \int_0^t \frac{ds}{\varphi(s)}, \ t > 0.$$

Since $\Phi(t)$ is concave, it follows that $u(t)$ decreases, and hence $u^*(t) = u(t)$. The function

$$U(t) := \int_0^t u(s)\,ds$$

is linear on the intervals Δ_n, $n = 1, 2, \dots$, and $U(\tau_n) = \Phi(\tau_n)$. Since $\Phi(t)$ is concave, we obtain

$$\int_0^t u^*(s)\,ds \le \int_0^t \frac{ds}{\varphi(s)}.$$

Now let us use the fact that, by assumption, M_φ is an interpolation set between L_1 and L_∞ and $1/\varphi \in M_\varphi$. By the Calderón–Mityagin theorem (see [8], [9], or [2, Theorem 2.4.3, p. 130]), we conclude that $u \in M_\varphi$; i.e., $u(t) \le C/\varphi(t)$ with some $C > 0$ independent of $t > 0$. This obviously contradicts relation (8). Thus, the assumption that $\delta_\varphi \ge 1$ is false, and the proof of the last implication (c) $\Rightarrow$ (d) is complete. $\square$

Using this theorem, one can readily obtain the results in [4] concerning the normability of the space $\Lambda^{p,\infty}(\omega)$. (See conditions (i) and (ii) at the beginning of the paper.) Indeed, it is well known [2, Theorem 2.6.10, p. 193] that the maximal Hardy–Littlewood operator M is bounded in a symmetric space X if and only if the upper Boyd index of X satisfies $\beta_X < 1$. The proof of Theorem 1 shows that the normability of the class M_φ (as well as of $\Lambda^{p,\infty}(\omega)$) is equivalent to its coincidence with the symmetric Marcinkiewicz space $M(\widetilde{\psi})$, $\widetilde{\psi}(t) = t/\psi(t)$, where $\psi(t)$ is equivalent to $\varphi(t)$. But the Boyd indices of a Marcinkiewicz space coincide with the respective dilation exponents of the fundamental function [2, p. 138]. Hence $\beta_{M(\widetilde{\psi})} = \delta_\psi$. Since equivalent functions have equal dilation exponents, we have $\delta_\psi = \delta_\varphi$. As a result, the operator M is bounded in M_φ if and only if $\delta_\varphi < 1$; by Theorem 1, this is equivalent to the normability of the class M_φ. As to condition (ii), it suffices to note, in view of Lemma 1, that (3) coincides with (4) for the function $\varphi(t) = W_p(t)$.

REFERENCES

1. G. G. Lorentz, "Some new functional spaces," Ann. of Math. (2) **51** (1), 37–55 (1950).
2. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators* (Nauka, Moscow, 1978) [in Russian].
3. M. J. Carro and J. Soria, "Weighted Lorentz spaces and the Hardy operator," J. Funct. Anal. **112** (2), 480–494 (1993).
4. J. Soria, "Lorentz spaces of weak type," Quart. J. Math. Oxford (2) **49**, 93–103 (1998).
5. M. J. Carro, J. Raposo, and J. Soria, *Recent developments in the theory of Lorentz spaces and weighted inequalities*, `arXiv: math.CA/0010010`.
6. C. Bennett and R. Sharpley, *Interpolation of Operators* (Academic Press, Boston, 1988).
7. N. K. Bari and S. B. Stechkin, "Best approximations and differential properties of two conjugate functions," Trudy Moskov. Mat. Obshch. **5**, 483–522 (1956) [Trans. Moscow Math. Soc.].
8. A. P. Calderón, "Spaces between L_1 and L_∞ and the theorem of Marcinkiewicz," Studia Math. **26**, 273–299 (1966).
9. B. S. Mityagin, "An interpolation theorem for modular spaces," Mat. Sb. **66 (108)** (4), 473–482 (1965) [Math. USSR-Sb.].