Mathematical Notes, vol. 80, no. 4, 2006, pp. 593–598.
Translated from Matematicheskie Zametki, vol. 80, no. 4, 2006, pp. 630–635.

BRIEF COMMUNICATIONS

Sums of Independent Functions in Symmetric Spaces with the Kruglov Property

S. V. Astashkin and F. A. Sukochev

Received April 10, 2006

Key words: *symmetric space, identically distributed functions, Poisson distribution, Kruglov operator, Orlicz space, Hilbert space, Lorentz space, Boyd indices.*

1. INTRODUCTION

It follows from the classical Khintchin inequality that, for any $p \in [1,\infty)$, the Rademacher system

$$\{r_n\}_{n=1}^{\infty}, \qquad r_n(t) = \operatorname{sgn}\sin(2^n \pi t), \quad 0 \le t \le 1,$$

is equivalent in $L_p[0,1]$ to a sequence of disjunctive "shifts" $\{\bar r_n(\,\cdot\,) := r_n(\,\cdot - n+1)\}_{n=1}^{\infty}$ in $L_2(0,\infty)$. A remarkable generalization of this inequality to the class of symmetric spaces X of functions on $[0,1]$ was obtained by Johnson and Schechtman [1]. They determined a symmetric space Z_X^2 on $[0,\infty)$ related to a given X and showed that an arbitrary sequence $\{f_k\}_{k=1}^{\infty}$ of independent mean-zero functions, i.e., of functions such that

$$\int_0^1 f_k(t)\,dt = 0,$$

is equivalent in X to the sequence of their disjunctive "shifts" $\{\bar f_k(\,\cdot\,) := f_k(\,\cdot - k+1)\}_{k=1}^{\infty}$ in Z_X^2 under the condition that X contains $L_p[0,1]$ for some $p<\infty$. We note that, in this case, the main problem is to prove the inequality

$$\Big\| \sum_{k=1}^{n} f_k \Big\|_X \le C \Big\| \sum_{k=1}^{n} \bar f_k \Big\|_{Z_X^2} \tag{1}$$

(see the right-hand side of inequality (3) in [1, Theorem 1]).

Our method for obtaining estimates of type (1) is based on the study of a certain linear operator $\mathscr{K}$ (called the *Kruglov operator*) on $L_1[0,1]$, which we recently introduced in [2] (see also [3]). Our approach is closely related to that developed somewhat earlier by Braverman [4] and based on the use of several ideas of Kruglov and the Kruglov probability constructions [5]. Moreover, our approach is closely related to stochastic integration with respect to symmetrized Poisson processes, which was used in [6] (see also [7]) to investigate whether a given symmetric space X on $[0,1]$ is isomorphic to some symmetric space on the half-axis. We first introduce the necessary definitions in Secs 2 and 3. Then, in Sec. 4, we present the main result stating that, for a wide class of symmetric spaces X, the fact that inequality (1) holds for an arbitrary sequence $\{f_k\}_{k=1}^{\infty} \subseteq X$ of independent mean-zero functions is equivalent to the boundedness of the Kruglov operator $\mathscr{K}$ (see Theorems 1 and 2). As compared to [1], the advantage of our approach is that we can obtain inequality (1) also for several symmetric spaces X that do not contain $L_p[0,1]$ for any $p<\infty$. The corresponding examples of exponential Orlicz spaces are given in Remark 1 (we note that Theorem 1 from [1] cannot be used for this class of spaces).

In Secs. 5 and 6, we present several applications of the results obtained. In Sec. 5, we describe symmetric spaces X on $[0,1]$ in which an arbitrary sequence $\{f_k\}_{k=1}^{\infty} \subset X$ of independent identically distributed functions spans a Hilbert subspace. In particular, we use one of its corollaries to give a negative answer to the question posed by Carothers and Dilworth [8] of whether any such sequence can span a Hilbert subspace in the Lorentz space $L_{2,q}$, $0 < q < 2$. We note that a similar answer to this question was earlier announced in [9], but the proof given there is not complete.

The results presented in Sec.6 concern the well-known problem of whether a given symmetric space X on $[0,1]$ can isomorphically be presented as a symmetric space on the half-axis (this problem was first posed in [10] and studied in detail in [6] and [7]). First, it turns out that, under the condition that the operator $\mathscr{K}$ is bounded, the space X contains a subspace isomorphic to some symmetric space on the half-axis. Earlier, a similar statement was proved only for a significantly narrower class of symmetric spaces with a nontrivial lower Boyd index. Second, Theorem 1 allows one to obtain a much simpler proof of the well-known theorem (see [6, Sec. 8] or [7, p. 203] stating that X is isomorphic to a symmetric space on the half-axis if the Boyd indices of the space X are nontrivial.

2. DEFINITIONS AND NOTATION

The Banach space $(E, \|\cdot\|_E)$ of real-valued functions Lebesgue measurable on the interval $[0,\alpha)$, $0 < \alpha \le \infty$, is said to be *symmetric* (or *rearrangement invariant*) if the relations $y \in E$ and $x^*(t) \le y^*(t)$, $t \in [0,\alpha)$, imply that $x \in E$ and $\|x\|_E \le \|y\|_E$. From now on, λ will denote the Lebesgue measure and $x^*(t)$ will be a nonincreasing right-continuous *permutation* $|x(s)|$, i.e.,

$$x^*(t) = \inf\bigl\{\tau \ge 0 : \lambda(\{s \in [0,\alpha) : |x(s)| > \tau\}) \le t\bigr\}, \qquad t > 0.$$

If E is a symmetric space (s.s.) on the interval $[0,\alpha)$, then the *dual* (or *associated*) space $E^\times$ consists of all measurable functions y such that

$$\|y\|_{E^\times} := \sup\biggl\{\int_0^\alpha |x(t)y(t)|\,dt : x \in E,\ \|x\|_E \le 1\biggr\} < \infty.$$

If E^* is the space conjugate to E, then $E^\times \subset E^*$ and $E^\times = E^*$ if and only if E is a separable space. The natural embedding of E in its second dual $E^{\times\times}$ is an isometric surjection if and only if E is the *maximal* space (or has the *Fatou property*), i.e., the relations

$$\{f_n\}_{n=1}^{\infty} \subseteq E, \qquad f \in S[0,\alpha), \qquad f_n \to f \quad \text{a.e. on } [0,\alpha), \qquad \sup_n \|f_n\|_E < \infty,$$

imply that

$$f \in E \qquad \text{and} \qquad \|f\|_E \le \liminf_{n\to\infty} \|f_n\|_E.$$

The norm $\|\cdot\|_E$ of the symmetric space E is said to be *order-semicontinuous* if the unit ball of E is closed in E with respect to the convergence almost everywhere. The last statement is equivalent to the fact that the natural embedding of E in its second dual is an isometry.

For any symmetric space E on $[0,\alpha)$, the following continuous embeddings hold:

$$L_1[0,\alpha) \cap L_\infty[0,\alpha) \subseteq E \subseteq L_1[0,\alpha) + L_\infty[0,\alpha).$$

If E is a symmetric space, then the function $\varphi_E(t) := \|\chi_A\|_E$, where $\lambda(A) = t$ and χ_A is the characteristic function of this set, is called the *fundamental function* of E. Another characteristic of a symmetric space is given by its Boyd indices. We recall their definition for spaces on $[0,1]$.

If $0 < \tau < \infty$, then the extension operator $\sigma_\tau x(t) := x(t/\tau)\chi_{(0,\min(1,\tau))}(t)$ is bounded in any symmetric space X. The numbers

$$\alpha_X := \lim_{\tau\to 0} \frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln\tau} \qquad \text{and} \qquad \beta_X := \lim_{\tau\to\infty} \frac{\ln\|\sigma_\tau\|_{X\to X}}{\ln\tau},$$

are called the *Boyd indices* of the space X; they always satisfy the condition $0 \le \alpha_X \le \beta_X \le 1$.

The L_p-spaces, $1 \le p \le \infty$, and their generalization, the Orlicz spaces, are important examples of symmetric spaces. Let Φ be the Orlicz function on $[0,\infty)$, i.e., let Φ be a continuous convex increasing function on $[0,\infty)$ such that $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. The Orlicz space $L_\Phi = L_\Phi[0,\alpha)$ consists of all functions f measurable on $[0,\alpha)$ and satisfying the condition that the norm

$$\|f\|_{L_\Phi} = \inf\biggl\{\rho > 0 : \int_0^\alpha \Phi\biggl(\frac{|f(t)|}{\rho}\biggr)\,dt \le 1\biggr\}$$

is finite. The Lorentz space $L_{p,q}$, $1 < p < \infty$, $1 \le q \le \infty$, is formed by all measurable $x(t)$ satisfying the condition that the expressions

$$\|x\|_{p,q} = \biggl(\int_0^\alpha \bigl(x^*(t)t^{1/p}\bigr)^q\,\frac{dt}{t}\biggr)^{1/q} \quad \text{if } q < \infty \qquad \text{and} \qquad \|x\|_{p,\infty} = \sup_{0<t<\alpha}\,(x^*(t)t^{1/p})$$

are finite. The functional $\|\cdot\|_{p,q}$ is the norm if $1 \le q \le p$, and this functional is equivalent to the norm if $q > p$. If X is a symmetric space on $[0,1]$, then

$$Z_X^2 := \{f \in (L_1 + L_\infty)(0,\infty) : \|f\|_{Z_X^2} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_{L_2[1,\infty)} < \infty\}.$$

It is easy to show that the quasinorm $\|\cdot\|_{Z_X^2}$ is equivalent to the norm and hence Z_X^2 is a symmetric space on $[0,\infty)$.

As is well known, the theory of symmetric spaces has much in common with the theory of interpolation operators. Recall that the Banach space X is called an *interpolation* space with respect to the Banach pair (X_0, X_1) if any linear operator T bounded in the spaces X_0 and X_1 is also bounded in X (for details concerning symmetric spaces and interpolation operators, see the monographs [7], [11], [12]).

3. KRUGLOV PROPERTY AND THE OPERATOR $\mathscr{K}$

Let f be a measurable function (a random variable) on $[0,1]$. By $\pi(f)$ we denote the random variable $\sum_{i=1}^N f_i$, where f_i are independent copies of f and N is a random variable Poisson distributed with parameter 1 and independent of the sequence $\{f_i\}$. Another (equivalent) definition of $\pi(f)$ is based on the use of its characteristic function

$$\theta_{\pi(f)}(t) = \exp\biggl(\int_{-\infty}^{\infty} (e^{itx} - 1)\,d\mathscr{F}_f(x)\biggr),$$

where $\mathscr{F}_f$ is the distribution function of f. The following property was first considered by Kruglov in [5] and was intensively studied and used by Braverman [4].

Definition 1. We shall say that a symmetric space X on $[0,1]$ has the *Kruglov property* (which is denoted as $X \in \mathbb{K}$) if the fact that $f \in X$ implies that $\pi(f) \in X$.

In [2] (see also [3]), the authors of the present paper defined a positive linear operator $\mathscr{K}$ closely related to the Kruglov property (hence it was called the *Kruglov operator* there). More precisely, if X is a symmetric space on $[0,1]$, then the operator $\mathscr{K}$ is a bounded operator in X if and only if $X \in \mathbb{K}$ [2, Lemma 3.3]. Simultaneously, the boundedness of $\mathscr{K}$ is a sufficient condition (and a necessary condition under rather wide assumptions) for the equivalence of sums of positive independent and identically distributed (as the sums) disjunctive functions [2, Theorem 3.5]. If $\{f_k\}_{k=1}^\infty$ is a sequence of functions defined on $[0,1]$, then, in what follows, we let $\{\bar f_k\}_{k=1}^\infty$ denote a sequence of disjunctive identically distributed (as f_k) functions on $[0,\infty)$.

4. COMPARISON OF SUMS OF MEAN-ZERO INDEPENDENT AND DISJUNCTIVE FUNCTIONS IN SYMMETRIC SPACES

The main result in this section can be treated as a generalization of the Khintchin inequality to the class of symmetric spaces with the Kruglov property and to arbitrary systems of independent mean-zero functions.

Theorem 1. *Suppose that X and Y are symmetric spaces on $[0,1]$, $X \subseteq Y$, and at least one of the following conditions is satisfied*:

(i) *the operator $\mathscr{K}: X \mapsto Y$ is bounded*;

(ii) *the norm of Y is order-semicontinuous and $\mathscr{K}: X \mapsto Y^{\times\times}$ is bounded.*

Then there exists an $C > 0$ such that the condition

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_Y \le C\Big\|\sum_{k=1}^{n} \bar{f}_k\Big\|_{Z_X^2} \tag{2}$$

holds for each sequence of independent mean-zero functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$.

Remark 1. As we mentioned in the introduction, Theorem 1 gives a sharper version of the corresponding part of Theorem 1 in [1] for normed spaces. In particular, let us consider the Orlicz spaces

$$\exp L_p := L_{N_p}, \qquad \text{where} \quad N_p(t) := e^{t^p} - \sum_{k=0}^{[1/p]} \frac{t^{kp}}{k!}.$$

We note that $\exp L_p \in \mathbb{K}$ if $0 < p < 1$ (see [5] or [4]) and hence Theorem 1 can be used in these spaces. On the contrary, since $L_q \not\subseteq \exp L_p$ for any $q \in (0,\infty)$ and $p \in (0,1)$, Theorem 1 in [1] "does not work" in this case.

If the norm of Y is order-semicontinuous, the converse statement also holds. More precisely, Theorem 1 and an analysis of the proof [2, Theorem 3.5] lead to the following result.

Theorem 2. *Suppose that X and Y are symmetric spaces on $[0,1]$, $X \subseteq Y$, and the norm of Y is order-semicontinuous. Then the following conditions are equivalent*:

(i) *the operator $\mathscr{K}$ is a bounded operator from X into Y*;

(ii) *there exists a constant $C > 0$ such that the inequality*

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_Y \le C\Big\|\sum_{k=1}^{n} \bar{f}_k\Big\|_X \tag{3}$$

holds for each sequence $\{f_k\}_{k=1}^n \subseteq X$ of independent symmetrically and identically distributed functions satisfying the condition

$$\sum_{k=1}^{n} \lambda(\{f_k \neq 0\}) \le 1. \tag{4}$$

As was shown in [13] (see also [7, pp. 134–136]), the Khintchin inequality holds in a symmetric space X if and only if X contains the separable part of the Orlicz space

$$\exp L_2 = L_{N_2}, \qquad N_2(t) := \exp t^2 - 1.$$

In the general situation, when the class of sequences of independent mean-zero functions is considered, such a "minimal" symmetric spaces does not exist any longer. More precisely, we have the following statement, which is, in a sense, converse to the statement of the first part of Theorem 1 in [1].

Corollary 1. *Suppose that a symmetric space E has the following property: for any maximal symmetric space $X \supseteq E$, there exists a $C > 0$ such that* (3) *holds for each sequence $\{f_k\}_{k=1}^n \subseteq X$ of independent symmetrically and identically distributed functions satisfying condition* (4). *Then E contains L_p for some $p \in [1, \infty)$.*

5. ℓ_2-ESTIMATES FOR SEQUENCES OF INDEPENDENT IDENTICALLY DISTRIBUTED FUNCTIONS

Here we consider the following problem: in what symmetric spaces X on $[0,1]$ does each sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent identically distributed functions (briefly: an i.i.d. sequence) span a Hilbert subspace? In other words, does a constant $C > 0$ such that

$$C^{-1}\biggl(\sum_{k=1}^n a_k^2\biggr)^{1/2} \le \biggl\|\sum_{k=1}^n a_k f_k\biggr\|_X \le C\biggl(\sum_{k=1}^n a_k^2\biggr)^{1/2}, \qquad n \in \mathbb{N}, \quad \{a_k\}_{k=1}^n \subset \mathbb{R}. \tag{5}$$

exist for each such a sequence? In fact, we are interested only in the right-hand inequality in (5), because the left-hand inequality holds in any symmetric space [4, Lemma 1, p. 52]. Moreover, if (5) holds, then it is easy to see [4, p. 71] that the space X satisfies the conditions

$$\text{(a)}\quad X^{\times\times} \supset \exp L_2; \qquad \text{(b)}\quad f_1 \in L_2; \qquad \text{(c)}\quad \int_0^1 f_1(t)\,dt = 0.$$

Thus, we can assume that $X \subseteq L_2$ and that the sequence $\{f_k\}_{k=1}^\infty$ consists of mean-zero functions. The following statement is the main result in this section.

Theorem 3. *Suppose that X is a symmetric space on $[0,1]$ and also an interpolation space with respect to the pair (L_2, L_∞). If either*

(i) $\mathscr{K} : X \mapsto X$,

or

(ii) *the norm of X is order-semicontinuous and $\mathscr{K} : X \mapsto X^{\times\times}$,*

then there exists an $C > 0$ such that

$$\biggl\|\sum_{k=1}^\infty a_k f_k\biggr\|_X \le C\|f_1\|_X\|a\|_{\ell_2}$$

for any i.i.d. sequence $\{f_k\}_{k=1}^\infty \subset X$ of mean-zero functions and any $a = (a_k)_{k=1}^\infty \in \ell_2$.

One can show that the condition stating that inequality (5) holds for an arbitrary i.i.d. sequence $\{f_k\}_{k=1}^\infty \subset X$ of mean-zero functions, which is formally weaker than the statement of Theorem 3, is, in fact, equivalent to this statement.

Corollary 2. *If the condition $X \subsetneqq L_2$ for a symmetric space and the condition $\varphi_X(u) = u^{1/2}$ are satisfied, then there exists an i.i.d. sequence $\{f_k\}_{k=1}^\infty \subset X$ of mean-zero functions such that, in X, this sequence spans a subspace that is not isomorphic to the space ℓ_2.*

Remark 2. In particular, if X is the Lorentz space $L_{2,q}$, $1 \le q < 2$, then it follows from the last corollary that there exists an i.i.d. sequence $\{f_k\}_{k=1}^\infty \subset L_{2,q}$ such that, in $L_{2,q}$, it spans a subspace that is not isomorphic to ℓ_2. Thus, we give a negative answer to the question posed in [8, p. 157]. A similar answer was earlier given in [9]; but the proof of this result in that paper was not complete.

Moreover, the same example shows that conditions (a)–(c) on the symmetric space X, stated at the beginning of this section, do not guarantee that (5) holds for any arbitrary i.i.d. sequence $\{f_k\}_{k=1}^\infty \subset X$ of mean-zero functions. Thus, a negative answer is given to the question posed in [4, p. 71].

6. ISOMORPHISMS BETWEEN SYMMETRIC SPACES ON $[0,1]$ AND $[0,\infty)$

Theorem 1 can be used to investigate whether a given symmetric space on a finite interval contains a subspace isomorphic to some symmetric space on the half-axis. The general isomorphism problem between symmetric spaces on $[0,1]$ and symmetric spaces on $[0,\infty)$ (other than the L_p-spaces) was first stated by Mityagin [10]. This and other closely related questions were intensively studied in [6] (see also the monograph [7]) following the approach based on the use of stochastic integration with respect to symmetrized Poisson processes. Our approach is technically simpler, and a similar argument was used earlier in [14] in the special case of the Lorentz spaces $L_{p,q}$. The first part of the following theorem gives a sharper version of the earlier results concerning the isomorphic embedding of some symmetric space on the half-axis in a given symmetric space X on $[0,1]$, in which the condition that the lower Boyd index of X is nontrivial is replaced by a weaker condition that the Kruglov operator $\mathscr{K}$ is bounded. The second part of the theorem is well known (see [6, Sec. 8] or [7, p. 203]), but our proof of this part (based on the use of Theorem 1) is much simpler.

Theorem 4. *Let X be a symmetric space on $[0,1]$. Then*

(i) *if $\mathcal{K}$ is a bounded operator in X, then X contains a subspace isomorphic to the symmetric space Z_X^2;*

(ii) *if the Boyd indices of X are nontrivial, i.e., if $0<\alpha_X\le\beta_X<1$, then the spaces X and Z_X^2 are isomorphic.*

ACKNOWLEDGMENTS

This research was supported by the Australian Scientific Council.

BIBLIOGRAPHY

1. W. B. Johnson and G. Schechtman, *Ann. Probab.*, **17** (1989), 789–808.
2. S. V. Astashkin and F. A. Sukochev, *Israel J. Math.*, **145** (2005), 125–156.
3. S. V. Astashkin and F. A. Sukochev, *Mat. Zametki* [*Math. Notes*], **76** (2004), no. 4, 483–489.
4. M. Sh. Braverman, *Independent Random Variables and Rearrangement Invariant Spaces*, Cambridge University Press, Cambridge, 1994.
5. V. M. Kruglov, *Teor. Veroyatnost. i Primenen.* [*Theory Probab. Appl.*], **15** (1970), no. 2, 331–336.
6. W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric Structures in Banach Spaces*, Mem. Amer. Math. Soc., vol. 19, 1979.
7. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. II. Function Spaces*, Ergeb. Math. Grenzgeb., vol. 97, Springer-Verlag, Berlin–New York, 1979.
8. N. L. Carothers and S. J. Dilworth, *J. Funct. Anal.*, **84** (1989), 146–159.
9. S. Ya. Novikov, *Izv. Vyssh. Uchebn. Zaved. Fiz.* [*Russian Phys. J.*] (1992), no. 4, 36–38.
10. B. S. Mityagin, *Uspekhi Mat. Nauk* [*Russian Math. Surveys*], **25** (1970), 59–103.
11. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978.
12. C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.
13. V. A. Rodin and E. M. Semenov, *Anal. Math.*, **1** (1975), 207–222.
14. N. L. Carothers and S. J. Dilworth, *Proc. Amer. Math. Soc.*, **194** (1988), 221–226.

S. V. Astashkin
Samara State University

F. A. Sukochev
Flinders University, Bedford Park, SA, Australia