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BRIEF COMMUNICATIONS

The Real Interpolation Method on Couples of Intersections

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ABSTRACT. Suppose that (X_0, X_1) is a Banach couple, $X_0 \cap X_1$ is dense in X_0 and X_1, $(X_0, X_1)_{\theta,q}$ $(0 < \theta < 1,\ 1 \leqslant q < \infty)$ are the spaces of the real interpolation method, $\psi \in (X_0 \cap X_1)^*$, $\psi \neq 0$, is a linear functional, $N = \operatorname{Ker}\psi$, and N_i stands for N with the norm inherited from X_i $(i = 0, 1)$. The following theorem is proved: the norms of the spaces $(N_0, N_1)_{\theta,q}$ and $(X_0, X_1)_{\theta,q}$ are equivalent on N if and only if $\theta \in (0, \alpha) \cup (\beta_\infty, \alpha_0) \cup (\beta_0, \alpha_\infty) \cup (\beta, 1)$, where α, β, α_0, β_0, α_∞, and β_∞ are the dilation indices of the function $k(t) = \mathscr{K}(t, \psi; X_0^*, X_1^*)$.

KEY WORDS: interpolation space, interpolation of subspaces, interpolation of intersections, real interpolation method, $\mathscr{K}$-functional, dilation index of a function, weighted L_p-space.

1. Introduction. Interpolation of subspaces is still one of the most significant problems in interpolation theory [1]. It has been stated as early as in the monograph [2] by Lions and Magenes. Various aspects of this problem have been treated in [3]–[12]. (The authors are very far from claiming the completeness of this citation list.) An important special case of this general problem is the problem of interpolation of intersections, which can be stated for the real method of interpolation as follows. (The definitions will be given below.) Let (X_0, X_1) be a Banach couple, i.e., a pair of Banach spaces linearly and continuously embedded in a Hausdorff topological vector space $\mathscr{T}$. Every linear subspace N of $\mathscr{T}$ generates a normed (in general, non-Banach) couple $(X_0 \cap N, X_1 \cap N)$, where the norm on $X_i \cap N$ is the restriction of the norm of the space X_i $(i = 0, 1)$. The problem is to find conditions on the triple (X_0, X_1, N) and the parameters $\theta \in (0, 1)$ and $q \in [1, \infty]$ of the real interpolation method under which the natural formula

$$(X_0 \cap N, X_1 \cap N)_{\theta,q} = (X_0, X_1)_{\theta,q} \cap N \tag{1}$$

is valid (in the sense that the spaces coincide as sets and the norms are equivalent).

We consider the case in which N is the kernel of a linear functional $\psi \in (X_0 \cap X_1)^*$. Then formula (1) just means the equivalence of norms of the spaces $N_{\theta,q} = (X_0 \cap N, X_1 \cap N)_{\theta,q}$ and $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ on the subspace N. In [13], [14] and independently in [15], four dilation indices related to the functional ψ viewed as an element of the sum $X_0^* + X_1^*$ of dual spaces were introduced, and the problem was solved under some additional condition on these indices. In this note, we announce results in which this additional condition is dropped owing to the introduction of yet another two similar indices. Thus we obtain a definitive, complete solution of this problem.

An important special case of this problem was treated by Krugljak, Maligranda, and Persson [16], who, under some additional assumptions, found conditions on $\theta \in (0, 1)$, $p \in [1, \infty)$, and the weight functions $w_0(x)$ and $w_1(x)$ ensuring that

$$(L_p(w_0) \cap N, L_p(w_1) \cap N)_{\theta,p} \ = \ (L_p(w_0), L_p(w_1))_{\theta,p} \cap N, \tag{2}$$

where $L_p(w)$ is the weighted L_p-space on $(0, \infty)$ with the usual norm and N is the space of all functions $f\colon (0, \infty) \to \mathbb{R}$ satisfying $\int_0^\infty f(x)\,dx = 0$. In the same paper, a similar problem was posed for the more general couple $(L_{p_0}(w_0), L_{p_1}(w_1))$. Using Theorems 1 and 2 in the present paper, one can readily solve this problem; at the end of the paper, we state one result in this direction. (See [14] for details.)

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2. Definitions and notation. For a normed couple (X_0, X_1) and $t>0$, the Peetre $\mathscr{K}$-functional is defined by the formula

$$\mathscr{K}(t,x;X_0,X_1)=\inf_{x=x_0+x_1,x_i\in X_i}(\|x_0\|_{X_0}+t\|x_1\|_{X_1}),\qquad x\in X_0+X_1.$$

If $0<\theta<1$ and $1\leqslant q<\infty$, then the space $X_{\theta,q}=(X_0,X_1)_{\theta,q}$ of the real interpolation method consists of all $x\in X_0+X_1$ such that

$$\|x\|_{X_{\theta,q}}=\left\{\int_0^\infty (t^{-\theta}\mathscr{K}(t,x:X_0,X_1))^q\,\frac{dt}{t}\right\}^{1/q}<\infty.$$

One can give another definition of this space (with an equivalent norm) in terms of the $\mathscr{J}$-functional

$$\mathscr{J}(t,x;X_0,X_1)=\max(\|x\|_{X_0},t\|x\|_{X_1}),\qquad x\in X_0\cap X_1.$$

Specifically, $X_{\theta,q}$ consists of all $x\in X_0+X_1$ representable in the form $x=\sum_{k\in\mathbb{Z}}2^{\theta k}x_k$ (convergence in X_0+X_1) with the norm

$$\inf\left\{\sum_{k\in\mathbb{Z}}(\mathscr{J}(2^k,x_k;X_0,X_1))^q\right\}^{1/q},\tag{3}$$

where the infimum is taken over all such representations.

If $X_0\cap X_1$ is dense in X_0 and X_1 and $\psi\in(X_0\cap X_1)^*$, then we can consider the Banach couple (X_0^*,X_1^*) of dual spaces, and $\psi\in X_0^*+X_1^*$ [1, §3.7]. In what follows, an important role will be played by the $\mathscr{K}$-functional $k(t)=\mathscr{K}(t,\psi;X_0^*,X_1^*)$ and the functions

$$M(t)=\sup_{s>0}\frac{k(ts)}{k(s)},\quad M_0(t)=\sup_{0<s\leqslant\min(1,1/t)}\frac{k(ts)}{k(s)},\quad M_\infty(t)=\sup_{s\geqslant\max(1,1/t)}\frac{k(ts)}{k(s)}.$$

They are submultiplicative for $t>0$, and therefore, the following numbers, called the *dilation indices* of $k(t)$, are well defined:

$$\alpha=\lim_{t\to0}\frac{\log_2 M(t)}{\log_2 t},\quad \alpha_0=\lim_{t\to0}\frac{\log_2 M_0(t)}{\log_2 t},\quad \alpha_\infty=\lim_{t\to0}\frac{\log_2 M_\infty(t)}{\log_2 t},$$
$$\beta=\lim_{t\to\infty}\frac{\log_2 M(t)}{\log_2 t},\quad \beta_0=\lim_{t\to\infty}\frac{\log_2 M_0(t)}{\log_2 t},\quad \beta_\infty=\lim_{t\to\infty}\frac{\log_2 M_\infty(t)}{\log_2 t}.$$

It is readily seen that $0\leqslant\alpha\leqslant\alpha_0\leqslant\beta_0\leqslant\beta\leqslant1$ and $0\leqslant\alpha\leqslant\alpha_\infty\leqslant\beta_\infty\leqslant\beta\leqslant1$.

3. Main results. Suppose that (X_0,X_1) is a Banach couple such that $X_0\cap X_1$ is dense in X_0 and X_1. Let $\psi\in(X_0\cap X_1)^*$ be a nonzero functional, let $N=\operatorname{Ker}\psi$, and let N_i be the normed space N with norm inherited from X_i $(i=0,1)$.

Theorem 1. *The norms of the interpolation spaces $N_{\theta,q}=(N_0,N_1)_{\theta,q}$ and $X_{\theta,q}=(X_0,X_1)_{\theta,q}$ are equivalent on N if and only if*

$$\theta\in(0,\alpha)\cup(\beta_\infty,\alpha_0)\cup(\beta_0,\alpha_\infty)\cup(\beta,1).\tag{4}$$

Moreover, if $\theta\in(0,\alpha)\cup(\beta_0,\alpha_\infty)\cup(\beta,1)$, then $N_{\theta,q}=(N_0,N_1)_{\theta,q}$ is dense in $X_{\theta,q}=(X_0,X_1)_{\theta,q}$; if $\theta\in(\beta_\infty,\alpha_0)$, then $N_{\theta,q}$ is dense in some subspace of $X_{\theta,q}$ of codimension 1.

Remark 1. The inequalities $\alpha_0\leqslant\beta_0$ and $\alpha_\infty\leqslant\beta_\infty$ imply that at most one of the intervals (β_0,α_∞) and (β_∞,α_0) can be nonempty.

Using the following definition in [15], we can state Theorem 1 in a different way. Keeping the preceding notation and conditions, by $X_{\theta,q,\psi}$ we denote the set of all $x\in X_0+X_1$ representable in the form

$$x=\sum_{k\in\mathbb{Z}}2^{\theta k}x_k,\quad x_k\in N\ (\text{convergence in } X_0+X_1)\tag{5}$$

with the norm (3), where the infimum is taken over all representations (5).

Theorem 2. *The space $X_{\theta,q,\psi}$ is closed in the real interpolation space $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ if and only if condition* (4) *holds.*

Moreover, if $\theta \in (0,\alpha) \cup (\beta_0, \alpha_\infty) \cup (\beta, 1)$, then $X_{\theta,q,\psi} = X_{\theta,q}$; if $\theta \in (\beta_\infty, \alpha_0)$, then $X_{\theta,q,\psi} = X_{\theta,q} \cap \operatorname{Ker} \widetilde{\psi}$, where the functional $\widetilde{\psi}$ is a continuous extension of ψ to the space $X_{\theta,q}$.

Remark 2. Theorem 2 strengthens Proposition 5.6 in [15], which contains the additional condition $\beta_\infty \leqslant \alpha_0$.

Remark 3. As in [13]–[15], we use the idea of reducing the general problem of interpolation of intersections to the study of the shift operator in some weighted l_p sequence space. This approach was developed by Ivanov and Kalton [10] for the comparison of the interpolation spaces $(X_0, X_1)_{\theta,q}$ and $(N_0, X_1)_{\theta,q}$, where $\psi \in X_0^*$ and $N_0 = \operatorname{Ker} \psi$. Here we consider a more general situation, and therefore, Theorem 1 implies the interpolation results in [10] (see [14] and [15]).

4. On interpolation of intersections of weighted L_p-spaces generated by an integral functional. Let $w(x)$ be a positive measurable function on $(0,\infty)$, $1 \leqslant p \leqslant \infty$. As usual, we define $L_p(w)$ as the space of all measurable functions $f\colon (0,\infty) \to \mathbb{R}$ with the norm $\|f\|_{w,p} = (\int_0^\infty |f(x)|^p w(x)\,dx)^{1/p}$ (with the usual modification for $p=\infty$). Consider the linear functional $\varphi(f) = \int_0^\infty f(x)\,dx$ with domain consisting of all measurable functions $f\colon (0,\infty) \to \mathbb{R}$ such that $\int_a^b |f(x)|\,dx < \infty$ for arbitrary $0 < a < b < \infty$ and there exists a limit $\lim_{a\to 0, b\to\infty} \int_a^b f(x)\,dx$. Denote by N its kernel.

Assume that the restriction of the functional φ to the intersection $L_{p_0}(w_0) \cap L_{p_1}(w_1)$ is bounded. Using Theorem 1 and arguing as in [14], we can find necessary and sufficient conditions under which a formula similar to (2) holds for the couple $(L_{p_0}(w_0), L_{p_1}(w_1))$ (without any restrictions on the dilation indices of the Peetre $\mathscr{K}$-functional of φ in the couple $(L_{p_0}(w_0)^*, L_{p_1}(w_1)^*)$). Here we do not dwell upon details and state only the simplest result for the case $p_0 = p_1 = 1$. Let $w(x) = \frac{w_0(x)}{w_1(x)}$, and let $w^{-1}(x)$ be the inverse function of $w(x)$.

Theorem 3. *Suppose that $w_0(x)$ increases, $w_1(x)$ decreases, and $\lim_{x\to 0} w(x) = \lim_{x\to\infty} \frac{1}{w(x)} = 0$. Let $\alpha, \beta, \alpha_0, \beta_0, \alpha_\infty$, and β_∞ be the dilation indices of the function $k(t) = 1/w_0(w^{-1}(1/t))$. The formula*

$$(L_1(w_0) \cap N, L_1(w_1) \cap N)_{\theta,q} = (L_1(w_0), L_1(w_1))_{\theta,q} \cap N$$

is valid if and only if condition (4) *holds.*

References

[1] J. Bergh and J. Löfström, Interpolation spaces. An introduction, Springer-Verlag, Berlin–Heidelberg–New York, 1976.

[2] J.-L. Lions and E. Magenes, Problémes aux limites non homogénes et applications, vol. 1, Dunod, Paris, 1968.

[3] H. Triebel, Math. Nachr., **69** (1975), 57–60.

[4] L. Maligranda, Suppl. Rend. Circ. Mat. Palermo, **10** (1985), 113–118.

[5] R. Wallsten, in: Lecture Notes in Math., vol. 1902, 1988, 410–419.

[6] G. Pisier, Pacific J. Math., **155** (1992), 341–368.

[7] S. Janson, Ark. Mat., **31** (1993), 307–338.

[8] S. V. Kislyakov and Kuankhua Shu, Algebra i Analiz, **8**:4 (1996), 75–109; English transl.: St. Petersburg Math. J., **8**:4 (1997), 593–615.

[9] J. Löfström, Interpolation of subspaces, Technical report no. 10, Univ. of Göteborg, 1997.

[10] S. Ivanov and N. Kalton, Algebra i Analiz, **13**:2 (2001), 93–115; English transl.: St. Petersburg Math. J., **13**:2 (2002), 221–239.

[11] S. V. Astashkin, J. Math. Math. Sci., **25**:7 (2001), 451–465.

[12] S. Kaijser and P. Sunehag, in: Function Spaces, Interpolation Theory and Related Topics (Lund 2000), de Gruyter, Berlin, 2002, 345–353.

[13] S. V. Astashkin, Funkts. Anal. Prilozhen., **39**:2 (2005), 61–64; English transl.: Funct. Anal. Appl., **39**:2 (2005), 131–134.
[14] S. V. Astashkin, Algebra i Analiz, **17**:2 (2005), 33–69; English transl.: St. Petersburg Math. J., **17**:2 (2006), 239–266.
[15] P. Sunehag, J. Approx. Theory, **130** (2004), 78–98.
[16] N. Krugljak, L. Maligranda, and L.-E. Persson, Ark. Mat., **37** (1999), 323–344.

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