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EXTRAPOLATORY DESCRIPTION FOR THE LORENTZ AND MARCINKIEWICZ SPACES "CLOSE" TO L_∞

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Abstract: We introduce the notion of a rearrangement invariant extrapolation space on $[0,1]$. We obtain some sufficient conditions (also necessary in some cases) for the Marcinkiewicz and Lorentz spaces to be extrapolation spaces. We generalize and improve the previous results, which enables us to determine the possible limits of such description of spaces and thereby to establish assertions of the Yano-type theorem. In particular, we present some examples of the spaces "close" to L_∞ in a sense but lacking this description.

Keywords: rearrangement invariant space, Lorentz space, Marcinkiewicz space, extrapolation, interpolation

Introduction

It is well known that many important operators of analysis, for example the maximal Hardy–Littlewood operator, the singular Hilbert operator, and the conjugate function operator in harmonic analysis, act boundedly in the L_p-spaces for $1 < p < \infty$ but unboundedly at the "endpoints" of this scale, i.e., in L_∞ and L_1 (or at least in one of those spaces). This circumstance was one of the reasons giving rise to the extrapolation assertions the first of which is apparently the classical Yano theorem (see [1] or [2, Chapter 12, Theorem 4.41]). Its part on the operators bounded in the spaces "close" to L_∞ can be stated as follows: If T is a bounded linear operator in $L_p = L_p[0,1]$ for all sufficiently large $p < \infty$ and the estimate $\|Tx\|_{L_p} \le Cp^\alpha \|x\|_{L_p}$ holds for some $\alpha > 0$, where $C > 0$ is independent of $x = x(t)$ and α, then T acts boundedly from L_∞ to the Zygmund space $\operatorname{Exp} L^{1/\alpha}$ with the norm

$$\|x\|_{\operatorname{Exp} L^{1/\alpha}} = \sup_{0<t\le 1} \log_2^{-\alpha}\left(\frac{2}{t}\right) x^*(t).$$

Here and in the sequel $x^*(t)$ is the nonincreasing rearrangement of $|x(t)|$.

The next important step in the development of extrapolation theory is connected with the names of Jawerth and Milman; developing the general approaches of the theory, they revealed the "source" of the Yano-type theorems; namely, existence of the extrapolation constructions making it possible to describe the limit spaces of such theorems [3–5]. In particular, they demonstrated (for example, see [5, pp. 22–23]) that

$$\|x\|_{\operatorname{Exp} L^{1/\alpha}} \asymp \sup_{1<p<\infty} (p^{-\alpha} \|x\|_{L_p}). \tag{1}$$

It is easy to see that the Yano theorem is an immediate consequence of (1) and another elementary extrapolation relation $\|x\|_{L_\infty} = \sup_{1<p<\infty} \|x\|_{L_p}$.

More general spaces and constructions were considered in [6, 7]. In particular, extrapolatory description was obtained for the exponential Orlicz spaces $\operatorname{Exp} L^\Phi$ (Φ is an Orlicz function on $(0,\infty)$) consisting of all functions $x = x(t)$, $t \in [0,1]$, satisfying

$$\|x\|_{\operatorname{Exp} L^\Phi} = \inf\left\{u > 0 : \int_0^1 \left(\exp \Phi\left(\frac{|x(t)|}{u}\right) - 1\right) dt \le 1\right\} < \infty.$$

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Namely [7, Corollary 1], for an arbitrary Orlicz function Φ we have

$$\|x\|_{\operatorname{Exp} L^{\Phi}} \asymp \sup_{1<p<\infty} \frac{\|x\|_{\Phi(p)}}{p}. \tag{2}$$

The exponential Orlicz spaces generalize Zygmund spaces, while constituting a subclass of the rearrangement invariant Marcinkiewicz spaces which play an important role in many applications of operator theory. This brings about the question of the possibility of extrapolatory description for rearrangement invariant spaces, in particular, Marcinkiewicz and Lorentz spaces. Lately, such description turned out to be interesting not only in its own right. It proved to be an important tool for studying convergence of series over orthonormal systems in the spaces "close" to L_∞. In this connection, we mention the articles [8–10] which are far from filling a complete list of references.

The article is organized as follows: In § 1 we give the basic definitions and some preliminaries from the theory of rearrangement invariant spaces. Moreover, here we introduce the definition of extrapolation spaces ("close" to L_∞) with respect to the L_p scale $(1 < p < \infty)$.

The main results, presented in § 2, deal with the extrapolatory description for Marcinkiewicz spaces. Here we study two versions of this description. One of them is universal in the sense that every extrapolation Marcinkiewicz space admits such a description. We obtain some necessary and sufficient conditions for a Marcinkiewicz space to be an extrapolation space in terms of the properties of the generating function of the space under study. On the one hand, these results are the far-reaching generalizations of (1) and (2) and, on the other hand, they enable us to find the possible limits of this description for spaces and so of the assertions similar to the Yano theorem. In particular, we present some examples of the spaces "close" to L_∞ in a sense but lacking this description.

In § 3 we consider similar questions for the Lorentz spaces predual to Marcinkiewicz spaces. The necessary and sufficient conditions found here for some quite definite extrapolatory description of these spaces improve the recent results of S. F. Lukomskii [10] who applied them to studying convergence of the Fourier series.

§ 1. Definitions and Preliminaries

Henceforth by an embedding of one Banach space into the other we mean a continuous embedding; i.e., $X_1 \subset X_0$ means that $x \in X_1$ implies $x \in X_0$ and $\|x\|_{X_0} \le C\|x\|_{X_1}$ for some $C > 0$. Writing $X_1 = X_0$ for some Banach spaces, we mean that they coincide and their norms are equivalent. The relation $F_1 \asymp F_2$ means that $cF_1 \le F_2 \le CF_1$ for some $c > 0$ and $C > 0$; moreover, these constants are independent of all or a part of the arguments of F_1 and F_2.

Denote by $\mathscr{F}$ the class of increasing continuous concave functions ψ on $[0,1]$ such that $\psi(0) = 0$ and $\psi(t) > 0$ $(0 < t \le 1)$. Recall that, for every continuous quasiconcave function $\psi(t)$ on $[0,1]$ (i.e., $\psi(t)$ is positive and increasing and $\psi(t)/t$ is decreasing for $t > 0$) such that $\psi(0) = 0$, there is $\varphi \in \mathscr{F}$ for which $\varphi(t) \asymp \psi(t)$ [11, p. 70].

Henceforth we speak of rearrangement invariant (symmetric) function spaces; a detailed exposition of this theory can be found in the monographs [11–13]. Recall that a Banach space X of measurable functions on $[0,1]$ is called a *rearrangement invariant space* if the following are satisfied:

(a) $y = y(t) \in X$ and $|x(t)| \le |y(t)|$ imply that $x = x(t) \in X$ and $\|x\| \le \|y\|$;

(b) if $y = y(t) \in X$ and $x^*(t) = y^*(t)$ then $x \in X$ and $\|x\| = \|y\|$ ($x^*(t)$ denotes the rearrangement of $|x(t)|$, i.e., the nonincreasing function equimeasurable with $|x(t)|$).

The simplest and important example of rearrangement invariant spaces is given by the L_p-spaces $(1 \le p \le \infty)$ with the usual norm

$$\|x\|_p = \left(\int_0^1 |x(t)|^p\,dt\right)^{1/p} \ (1 \le p < \infty) \quad \text{and} \quad \|x\|_\infty = \operatorname*{ess\,sup}_{0\le t\le 1} |x(t)|.$$

The embedding $L_p \subset L_q$ holds (with constant 1) for $p > q$. Moreover, L_∞ is the smallest space among all rearrangement invariant spaces on $[0, 1]$, as mentioned in the introduction, and the norm of an arbitrary function $x(t)$ in L_∞ is expressed in terms of its norms in L_p for $p < \infty$: $\|x\|_\infty = \sup_{1<p<\infty} \|x\|_p$.

Other examples of the rearrangement invariant spaces are the Lorentz and Marcinkiewicz spaces. Let $\varphi(t) \in \mathscr{F}$. The Marcinkiewicz space $M(\varphi)$ consists of all measurable functions $x(t)$ on $[0, 1]$ with the finite norm

$$\|x\|_{M(\varphi)} = \sup_{0<s\le 1} \frac{\varphi(s)}{s} \int_0^s x^*(t)\, dt.$$

The norm in the Lorentz space $\Lambda_p(\varphi)$ $(1 \le p < \infty)$ is defined as follows:

$$\|x\|_{\Lambda_p(\varphi)} = \left(\int_0^1 (x^*(t))^p \, d\varphi(t) \right)^{\frac{1}{p}}.$$

The important characteristic of a rearrangement invariant space X is its fundamental function $\phi_X(t) = \|\chi_{(0,t)}\|_X$, where $\chi_{(0,t)}$ denotes the characteristic function (indicator) of the interval $(0, t)$. In particular, $\phi_{M(\varphi)}(t) = \varphi(t)$ and $\phi_{\Lambda_p(\varphi)}(t) = (\varphi(t))^{\frac{1}{p}}$. Moreover, $\Lambda(\varphi) := \Lambda_1(\varphi)$ is the smallest space and $M(\varphi)$ is the largest space among all rearrangement invariant spaces with the fundamental function $\varphi(t)$ [11, pp. 160, 162].

The dilation function of a positive function $\varphi(t)$, $t \in (0, 1]$, is defined by the relation

$$\mathscr{M}_\varphi(s) = \sup_{0<t\le \min(1,1/s)} \frac{\varphi(st)}{\varphi(t)}, \quad 0 < s < \infty.$$

Furthermore, the numbers

$$\gamma_\varphi = \lim_{s\to 0+} \frac{\log \mathscr{M}_\varphi(s)}{\log s} \quad \text{and} \quad \delta_\varphi = \lim_{s\to\infty} \frac{\log \mathscr{M}_\varphi(s)}{\log s}$$

are called the *lower* and *upper dilation indices* of $\varphi(t)$. If $\varphi \in \mathscr{F}$ then $0 \le \gamma_\varphi \le \delta_\varphi \le 1$ [11, p. 76].

Henceforth we are interested in the rearrangement invariant spaces X such that $X \subset L_p$ for all $p < \infty$. In particular, we will wonder when the norm in these spaces is expressed in terms of the L_p-norm. Moreover, we can construct rearrangement invariant spaces of the above-indicated form by ourselves. Namely, if F is an ideal Banach space of functions on $[1, \infty)$ then we can consider the set $\mathscr{L}_F$ of all functions $x(t)$ for which $\xi = \xi(p) := \|x\|_p \in F$. It is easy to show that this set (with the usual identification of the functions coinciding almost everywhere) becomes a rearrangement invariant space under the norm:

$$\|x\|_{\mathscr{L}_F} := \|\xi\|_F.$$

Instead of a function space as a parameter of an extrapolation space, we can also take an ideal Banach space of sequences, requiring that the sequences $\{\|x\|_n\}_{n=1}^\infty$ belong to this space.

DEFINITION 1. Let X be a rearrangement invariant space on $[0, 1]$ such that $X \subset L_p$ for all $p < \infty$. We say that X is an *extrapolation space* (as $p \to \infty$) if there is an ideal Banach space F for which $X = \mathscr{L}_F$. We denote the set of all extrapolation spaces by $\mathscr{E}$.

If F is an ideal Banach space of functions on $[1, \infty)$ and $w = w(p)$ is a positive function (weight) then we denote by $F(w)$ the ideal space of functions $x = x(t)$ with the norm $\|x\|_{F(w)} = \|xw\|_F$. There is a similar definition for sequence spaces.

§ 2. Extrapolatory Description for Marcinkiewicz Spaces

In this section we wonder when a Marcinkiewicz space $M(\varphi)$ coincides with a space of the form $\mathscr{L}_F$, i.e., when $M(\varphi)$ is an extrapolation space.

Let $\varphi(t) \in \mathscr{F}$ and $\widetilde{\varphi}(t) = t/\varphi(t)$. In the case of $\lim_{t\to 0+} \widetilde{\varphi}(t) > 0$ we can assume that $\varphi(t) = t$ and then $M(\varphi) = L_1$. Otherwise the function $\widetilde{\varphi}$ is absolutely continuous [11, p. 71] and therefore has the derivative $\widetilde{\varphi}'$. Suppose that

$$\lim_{t\to 0+} \widetilde{\varphi}(t) = 0 \text{ and } \widetilde{\varphi}' \in L_p \quad \text{for all } p < \infty. \tag{3}$$

It follows from the definition of the norm of a Marcinkiewicz space that

$$\int_0^s x^*(t)\,dt \le \|x\|_{M(\varphi)} \int_0^s \widetilde{\varphi}'(t)\,dt, \quad 0 < s \le 1.$$

However, this is an inequality between $\mathscr{K}$-functionals on the functions $x(t)$ and $\widetilde{\varphi}'(t)$ in the pair of spaces (L_1, L_∞) (for example, see [11, p. 108]). Since L_p is an interpolation space between (L_1, L_∞) with constant 1, applying [11, p. 130], we conclude that $x \in L_p$ for all $1 \le p < \infty$ and

$$\|x\|_p \le \|x\|_{M(\varphi)} \|\widetilde{\varphi}'\|_p, \quad 1 \le p < \infty,$$

or

$$\sup_{p\ge 1} \frac{\|x\|_p}{\|\widetilde{\varphi}'\|_p} \le \|x\|_{M(\varphi)}.$$

The last inequality means that

$$M(\varphi) \subset \mathscr{L}_{F^\varphi}, \tag{4}$$

where $F^\varphi = L_\infty(1/\|\widetilde{\varphi}'\|_p)$; moreover, the norm of (4) is equal to 1.

Theorem 1. *Let $\varphi \in \mathscr{F}$. The following are equivalent:*
(1) $M(\varphi) \in \mathscr{E}$;
(2) $M(\varphi) = \mathscr{L}_{F^\varphi}$;
(3) there is $C > 0$ such that

$$\varphi(t) \le C \sup_{p\ge 1} \frac{t^{\frac{1}{p}}}{\|\widetilde{\varphi}'\|_p}, \quad 0 < t \le 1. \tag{5}$$

Proof. First of all note that the relations of (3) are consequences of each of the conditions (1)–(3) (in particular, in the case of (1) and (2) they follow from the definition of extrapolation space). Therefore, in all three cases we can assume (4) valid. Now, it is easy to see that (5) is an inequality between the fundamental functions of $M(\varphi)$ and $\mathscr{L}_{F^\varphi}$. Since $M(\varphi)$ is the largest space among all rearrangement invariant spaces with a given fundamental function [11, p. 162], inequality (5) is equivalent to the embedding $\mathscr{L}_{F^\varphi} \subset M(\varphi)$. Consequently, by the previous remark, (5) is valid if and only if $M(\varphi) = \mathscr{L}_{F^\varphi}$. Therefore, (2) and (3) are equivalent.

Since the implication (2) $\Rightarrow$ (1) is obvious, we are left with proving the converse. Assume there is an ideal Banach space F_1 such that $M(\varphi) = \mathscr{L}_{F_1}$. If $a = a(p) \in F^\varphi$ then $|a(p)| \le \|a\|_{F^\varphi} \|\widetilde{\varphi}'\|_p$, $p < \infty$. Since $\widetilde{\varphi}' \in M(\varphi)$, the function $b = b(p) = \|\widetilde{\varphi}'\|_p$ belongs to F_1 and consequently (since F_1 is an ideal space) from the inequality $|a(p)| \le \|a\|_{F^\varphi} b(p)$ we obtain $a \in F_1$ and $\|a\|_{F_1} \le \|a\|_{F^\varphi} \|b\|_{F_1}$, whence $F^\varphi \subset F_1$. Thereby $\mathscr{L}_{F^\varphi} \subset \mathscr{L}_{F_1} = M(\varphi)$. Since the reverse embedding to (4) is always valid, the theorem is proved. $\square$

Now, suppose that the upper dilation index of φ is nontrivial, i.e., $\delta_\varphi < 1$. Then, by [11, p. 156],

$$\|x\|_{M(\varphi)} \asymp \sup_{0<u\le 1} \varphi(u) x^*(u),$$

whence, from monotonicity of $x^*(t)$ and quasiconcavity of $\varphi(t)$, we obtain

$$\|x\|_{M(\varphi)} \asymp \sup_{p\ge 1} \varphi(2^{-p}) x^*(2^{-p}). \tag{6}$$

In this case $x_0(t) = \frac{1}{\varphi(t)} \in M(\varphi)$. Therefore, it follows from (4) and the inequality $\widetilde{\varphi}' \le \frac{1}{\varphi}$ that $\|1/\varphi\|_p \asymp \|\widetilde{\varphi}'\|_p$ with constants independent of $p \in [1,\infty)$. Consequently, we can replace (5) of Theorem 1 for $\delta_\varphi < 1$ with the condition

$$\varphi(t) \le C \sup_{p\ge 1} \frac{t^{1/p}}{\|1/\varphi\|_p}, \quad 0 < t \le 1. \tag{7}$$

Corollary 1. *Let $\varphi \in \mathscr{F}$ be such that $\delta_\varphi < 1$. Then $M(\varphi) \in \mathscr{E}$ if and only if*

$$\sup_{t\in(0;1]} \left\{ \inf_{p\ge 1} \left(\int_0^{1/t} \left(\frac{\varphi(t)}{\varphi(st)} \right)^p ds \right)^{\frac{1}{p}} \right\} < \infty.$$

Proof. Since, under the conditions of the corollary, (5) and (7) are equivalent, by Theorem 1, $M(\varphi) \in \mathscr{E}$ if and only if

$$\sup_{t\in(0;1]} \left\{ \inf_{p\ge 1} \left(\varphi(t) \left(\frac{1}{t} \int_0^1 \frac{1}{\varphi(u)^p} \, du \right)^{\frac{1}{p}} \right) \right\} < \infty.$$

Using the change of variable $u = st$, we obtain the assertion. □

Embedding (4), which is always valid, can be considered as some "upper approximation" of the Marcinkiewicz space by an extrapolation space. Find a similar "lower approximation" in the case when $\delta_\varphi < 1$. Since

$$\|x\|_p \ge \left(\int_0^{2^{-p}} (x^*(t))^p \, dt \right)^{1/p} \ge \frac{1}{2} x^*(2^{-p}),$$

from (6) we derive

$$\|x\|_{M(\varphi)} \le C \sup_{p\ge 1} \varphi(2^{-p}) \|x\|_p \tag{8}$$

with some constant $C > 0$ independent of x. It follows from (8) and (4) that

$$\mathscr{L}_{F_\varphi} \subset M(\varphi) \subset \mathscr{L}_{F^\varphi}, \tag{9}$$

where $F_\varphi = L_\infty(\varphi(2^{-p}))$.

In particular, $M(\varphi)$ is an extrapolation space, if $\mathscr{L}_{F_\varphi} = \mathscr{L}_{F^\varphi}$. At the same time, as follows from Theorem 1, for the last equality to hold, it is sufficient that $M(\varphi)$ and $\mathscr{L}_{F_\varphi}$ coincide. We will find out when this happens.

Lemma 1. *If $\delta_\varphi < 1$ then $M(\varphi) = \mathscr{L}_{F_\varphi}$ if and only if the condition*

$$\int_0^1 \frac{dt}{(\varphi(t))^p} \le C^p \frac{1}{(\varphi(2^{-p}))^p}, \quad p \ge 1, \tag{10}$$

holds with some constant $C > 0$ independent of p.

Proof. Since $\widetilde{\varphi}' \le \frac{1}{\varphi}$, it follows from (10) that $\varphi(2^{-p}) \le C_1 \frac{1}{\|\widetilde{\varphi}'\|_p}$. Consequently, $F^\varphi \subset F_\varphi$, whence $\mathscr{L}_{F^\varphi} \subset \mathscr{L}_{F_\varphi}$. Combining this with (9), we conclude that $\mathscr{L}_{F_\varphi} = M(\varphi) = \mathscr{L}_{F^\varphi}$.

Now, let $M(\varphi) = \mathscr{L}_{F_\varphi}$. Since $x_0(t) = \frac{1}{\varphi(t)} \in M(\varphi)$ by (6), from the embedding $M(\varphi) \subset \mathscr{L}_{F_\varphi}$ we obtain

$$\varphi(2^{-p})\|1/\varphi\|_p \leq \sup_{p\geq 1} \varphi(2^{-p})\|x_0\|_p \leq C_1\|x_0\|_{M(\varphi)} = C.$$

Since the inequalities are equivalent to (10), the lemma is proved. □

Show that (10) is equivalent to a more explicit property of φ.

Definition 2. We say that a function $\varphi(t) \in \mathscr{F}$ *satisfies the* Δ^2*-condition* ($\varphi \in \Delta^2$) if there is $\alpha > 0$ such that

$$\varphi(t) \leq \alpha\varphi(t^2) \tag{11}$$

for all $t \in [0, 1]$.

Lemma 2. *Let $\varphi(t) \in \mathscr{F}$. For $\varphi(t)$ to satisfy the Δ^2-condition, it is necessary and sufficient that* (10) *be valid.*

Proof. If (10) is valid then, recalling that $\varphi(t)$ is increasing, we obtain

$$\frac{2^{-2p}}{\varphi(2^{-2p})^p} \leq \int_0^{2^{-2p}} \frac{dt}{(\varphi(t))^p} \leq \int_0^1 \frac{dt}{(\varphi(t))^p} \leq C^p \frac{1}{(\varphi(2^{-p}))^p},$$

whence $\varphi(2^{-p}) \leq 4C\varphi(2^{-2p})$ for all $p \geq 1$.

After the change $2^{-p} = t$ the last inequality becomes (11) for $t \leq \frac{1}{2}$. Hence, by quasiconcavity of $\varphi(t)$, (11) holds for all $t \in [0, 1]$ with the constant $8C$.

Now, take $\varphi \in \Delta^2$. Then the inequality $\varphi(t) \leq \alpha^n\varphi(t^{2^n})$ holds for all $n \in \mathbb{N}$; in particular, for $t = 2^{-p}$ we have

$$\varphi(2^{-p}) \leq \alpha^n\varphi(2^{-2^n p}), \quad p \geq 0.$$

Now, choose $n_0 \in \mathbb{N}$ so that the inequality $2^n - (n+1)\log_2\alpha \geq n$ holds for all $n \geq n_0$. If $p \geq 1$ then

$$\begin{aligned}
\int_0^1 \frac{dt}{(\varphi(t))^p} &\leq 2^{2^{n_0}p} \int_0^{2^{-2^{n_0}p}} \frac{dt}{(\varphi(t))^p} = 2^{2^{n_0}p} \sum_{n=n_0}^{+\infty} \int_{2^{-2^{n+1}p}}^{2^{-2^n p}} \frac{dt}{(\varphi(t))^p} \\
&\leq 2^{2^{n_0}p} \sum_{n=n_0}^{+\infty} \frac{2^{-2^n p}}{(\varphi(2^{-2^{n+1}p}))^p} \leq 2^{2^{n_0}p} \sum_{n=n_0}^{+\infty} \alpha^{(n+1)p} 2^{-2^n p} \frac{1}{(\varphi(2^{-p}))^p} \\
&\leq 2^{2^{n_0}p} \frac{1}{(\varphi(2^{-p}))^p} \sum_{n=n_0}^{+\infty} 2^{-np} = 2^{2^{n_0}p} \frac{2^{-n_0 p}}{1-2^{-p}} \frac{1}{(\varphi(2^{-p}))^p} \leq C^p \frac{1}{(\varphi(2^{-p}))^p},
\end{aligned}$$

where $C = 2^{2^{n_0}}$; thus, (10) is satisfied. □

Using Lemmas 1 and 2, we can easily obtain

Theorem 2. *Let $\varphi(t) \in \mathscr{F}$ and $\delta_\varphi < 1$. The following are equivalent:*
(1) $\varphi \in \Delta^2$;
(2) $\varphi(2^{-p}) \asymp 1/\|\widetilde{\varphi}'\|_p$ $(p \geq 1)$;
(3) $M(\varphi) = \mathscr{L}_{F_\varphi}$;
(4) $M(\varphi) = \mathscr{L}_{f_\varphi}$, *where* $f_\varphi = l_\infty(\varphi(2^{-n}))$.

Proof. Equivalence of (1)–(3) follows from Lemmas 1 and 2, since the reverse inequality to (10) is always valid. From monotonicity of $f_x(p) = \|x\|_p$ $(p \geq 1)$ and quasiconcavity of $\varphi(t)$ we derive $\mathscr{L}_{F_\varphi} = \mathscr{L}_{f_\varphi}$; hence, each of these conditions is equivalent to (4). □

It follows from Theorem 2 that, for $\varphi \in \Delta^2$, both embeddings in (9) become equalities. If $\varphi(t)$ does not satisfy the Δ^2-condition then, at least for $\delta_\varphi < 1$, the embedding $\mathscr{L}_{F_\varphi} \subset M(\varphi)$ becomes strict. Is it possible that the equality $M(\varphi) = \mathscr{L}_{F\varphi}$ remains valid in this case? We give an example showing that the answer is affirmative.

Example 1. Consider a function $\varphi(t) \in \mathscr{F}$ such that $\varphi(t) \asymp \psi(t) = e^{1-\log^\theta e/t}$ for some $\theta \in (0,1)$. Such a function in the class $\mathscr{F}$ exists indeed, since $\psi(t) > 0$, $\psi'(t) > 0$, and $\psi''(t) < 0$ for $t \in (0, t_0)$ (where t_0 is sufficiently small) and $\lim_{t\to 0+} \psi(t) = 0$.

Show that $M(\varphi) \in \mathscr{E}$. It is easy to see that $\delta_\varphi = 0 < 1$. Therefore, it suffices to prove (7) with $\varphi(t) = \psi(t)$.

Estimate the L_p-norm of $\frac{1}{\psi}$ as follows:

$$\left\|\frac{1}{\psi}\right\|_p^p = \int_0^1 e^{p\log^\theta e/t - p}\, dt = e^{-p+1}\int_1^{+\infty} e^{ps^\theta - s}\, ds.$$

Represent the integrand in the form $e^{ps^\theta - s} = e^{ps^\theta - \beta s} e^{-(1-\beta)s}$, where $\beta \in (0,1)$ is a parameter, and find the maximum of $f_\beta(s) = ps^\theta - \beta s$. Since $f'_\beta(s) = \theta p s^{\theta-1} - \beta$, $s_0 = (\theta p/\beta)^{\frac{1}{1-\theta}}$ is a maximum point of $f_\beta(s)$. Consequently, for $s \in [1,\infty)$, we have $f_\beta(s) \le f_\beta(s_0) = (1-\theta)p^{\frac{1}{1-\theta}}(\theta/\beta)^{\frac{\theta}{1-\theta}}$ and

$$\begin{aligned}\left\|\frac{1}{\psi}\right\|_p^p &\le e^{-p+1}\exp\left((1-\theta)p^{\frac{1}{1-\theta}}\left(\frac{\theta}{\beta}\right)^{\frac{\theta}{1-\theta}}\right)\int_0^\infty e^{-(1-\beta)s}\, ds \\ &= e^{-p+1}\exp\left((1-\theta)p^{\frac{1}{1-\theta}}\left(\frac{\theta}{\beta}\right)^{\frac{\theta}{1-\theta}}\right)\frac{1}{1-\beta}.\end{aligned}$$

Thus,

$$\left\|\frac{1}{\psi}\right\|_p \le \exp\left((1-\theta)\left(\frac{\theta p}{\beta}\right)^{\frac{\theta}{1-\theta}}\right)\left(\frac{1}{1-\beta}\right)^{\frac{1}{p}}.$$

Put $\beta = \frac{p^\gamma}{1+p^\gamma}$, where $\gamma = \frac{\theta}{1-\theta}$. Then

$$(1/(1-\beta))^{\frac{1}{p}} = (1+p^\gamma)^{\frac{1}{p}} \le 2p^{\frac{\gamma}{p}} < C < \infty$$

for all $p \ge 1$. Moreover, since $(1+x)^\gamma \le 1 + C_\gamma x$ for all $x \in [0,1]$, we have

$$(1-\theta)\left(\frac{\theta p}{\beta}\right)^{\frac{\theta}{1-\theta}} = \left(\frac{p}{\beta}\right)^\gamma (1-\theta)\theta^{\frac{\theta}{1-\theta}} = p^\gamma\left(1 + \frac{1}{p^\gamma}\right)^\gamma (1-\theta)\theta^{\frac{\theta}{1-\theta}} \le (1-\theta)\theta^{\frac{\theta}{1-\theta}}(p^\gamma + C_\gamma)$$

and consequently

$$\|1/\psi\|_p \le C_1\exp((1-\theta)\theta^{\frac{\theta}{1-\theta}}p^{\frac{\theta}{1-\theta}}). \tag{12}$$

To prove (7), verify that the inequality

$$\psi(t)\|1/\psi\|_p \le C_2 t^{1/p}$$

holds for $p = \frac{1}{\theta}\log^{1-\theta} e/t$ with the constant $C_2 = C_1 e$. Indeed, on the one hand, by (12),

$$\psi(t)\|1/\psi\|_p \le C_1\exp((1-\theta)\log^\theta e/t)\exp(1-\log^\theta e/t) = C_2\exp(-\theta\log^\theta e/t).$$

On the other hand,

$$t^{1/p} = \exp(\log t/p) = \exp(\theta\log^{\theta-1} e/t(1-\log e/t)) \ge \exp(-\theta\log^\theta e/t).$$

Thus, $M(\varphi) \in \mathscr{E}$. At the same time, we see easily that $\varphi \notin \Delta^2$.

For the function $\varphi(t)$ of Example 1 we have $\delta_\varphi = 0$. The same is true for all functions satisfying the Δ^2-condition. In this connection, the question arises: Is there a Marcinkiewicz space $M(\varphi)$ such that simultaneously $M(\varphi) \in \mathscr{E}$ and $\delta_\varphi > 0$? A partial answer to this question is contained in

Proposition 1. *Let $M(\varphi) \in \mathscr{E}$. If $\delta_\varphi < 1$ then $\delta_\varphi = 0$.*

Proof. Take $\delta_\varphi \in (0,1)$. Then $\mathscr{M}_\varphi(s) > s^\theta$ for some $\theta > 0$ if $s > 1$. Consequently, for every $s > 1$, there is $t_0 = t_0(s) \in (0, 1/s)$ such that

$$\frac{\varphi(st_0)}{\varphi(t_0)} > s^\theta. \tag{13}$$

Since $\delta_\varphi < 1$, the condition $M(\varphi) \in \mathscr{E}$ is equivalent to (7) or the inequality

$$\varphi(t) \le C_1 \sup_{p \ge 2/\theta} \frac{t^{1/p}}{\|1/\varphi\|_p}, \quad 0 < t \le 1. \tag{14}$$

Choose $s > 1$ so that $s^{\theta/2} > C_1$. Since $\theta p - 1 \ge \theta p/2$ for $p \ge 2/\theta$, by (13), we have

$$\left\|\frac{1}{\varphi}\right\|_p^p = \int_0^1 \frac{du}{\varphi(u)^p} \ge \int_0^{t_0} \frac{du}{\varphi(u)^p} \ge \frac{t_0}{\varphi(t_0)^p} > \frac{s^{\theta p} t_0}{\varphi(st_0)^p} = s^{\theta p - 1} \frac{st_0}{\varphi(st_0)^p} \ge (s^{\theta/2})^p \frac{st_0}{\varphi(st_0)^p}$$

for such p's. Hence, we obtain

$$\varphi(t_1) > C_1 \frac{t_1^{1/p}}{\|1/\varphi\|_p}$$

for $t_1 = st_0$ and all $p \ge 2/\theta$, which contradicts (14). $\square$

Corollary 2. *For every $\theta \in (0,1)$, there is a function $\varphi \in \mathscr{F}$ such that $\delta_\varphi = \theta$, $M(\varphi) \subset L_p$ for all $p < \infty$, and $M(\varphi) \notin \mathscr{E}$.*

Proof. We start with constructing a function $c(t) \asymp \frac{1}{\varphi(t)}$ and searching the latter in the form

$$c(t) = \sum_{i=0}^{+\infty} c_i \chi_{(2^{-i-1};2^{-i}]}(t), \quad \text{where } 1 = c_0 \le c_1 \le \dots \le c_n \le \dots.$$

For the function $\psi(t) = \frac{1}{c(t)}$ to be equivalent to some function $\varphi(t) \in \mathscr{F}$, it is sufficient that the conditions

$$c_i \le c_{i+1} \le 2c_i,\ i = 0, 1, \dots, \quad \text{and} \quad c_i \to \infty \text{ as } i \to \infty \tag{15}$$

are satisfied.

Indeed, if (15) is valid then $\psi(t)$ increases and for $k \ge i$ we obtain $c_k \le 2^{k-i} c_i$, whence

$$2^k \psi(2^{-k}) \ge 2^i \psi(2^{-i}).$$

Now, if $t_1 < t_2$ then $t_1 \in [2^{-k-1}, 2^{-k})$ and $t_2 \in [2^{-i-1}, 2^{-i})$, $k \ge i$. Moreover,

$$\frac{\psi(t_2)}{t_2} \le 2^{i+1}\psi(2^{-i}) \le 2^{k+1}\psi(2^{-k}) \le 2^{k+2}\psi(2^{-k-1}) \le 4\frac{\psi(t_1)}{t_1}.$$

Thereby the function $\psi(t)$ is equivalent to some concave function [11, p. 69]. By (15), $\lim_{t \to 0+} \psi(t) = 0$. Eventually, $\psi(t) \asymp \varphi(t) \in \mathscr{F}$.

For $M(\varphi)$ to be embedded into L_p for all $p < \infty$, it is necessary that

$$\|1/\varphi\|_p < \infty, \quad 1 \le p < \infty,$$

or, equivalently,

$$\|c\|_p = \left(\sum_{i=0}^{+\infty} c_i^p 2^{-i-1}\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty. \tag{16}$$

Now, choose $\{c_i\}_{i=0}^{+\infty}$ so that (15) and (16) hold and $\delta_\varphi = \theta \in (0,1)$. First of all, assume that
(0) $n_0 = 1$;
(1) $n_{2j+1} = 2n_{2j}$, $j = 0, 1, \dots$;
(2) $n_{2j+2} = n_{2j+1} + j$, $j = 0, 1, \dots$.
Now, define a sequence $\{c_i\}_{i=0}^{+\infty}$ as follows:
(a) $c_0 = c_1 = 1$;
(b) $c_i = c_{n_{2j}}$ if $n_{2j} \le i \le n_{2j+1}$, $j = 0, 1, \dots$;
(c) $c_i = c_{n_{2j+1}} 2^{\theta(i - n_{2j+1})}$ if $n_{2j+1} \le i \le n_{2j+2}$, $j = 0, 1, \dots$.
Relation (15) follows from the definition of $\{c_i\}_{i=0}^{+\infty}$. To check (16), we carry out some transformations:

$$\sum_{i=2}^{+\infty} c_i^p 2^{-i} = \sum_{k=1}^{+\infty} \sum_{i=n_{2k}}^{n_{2k+1}-1} 2^{\theta p \sum\limits_{j=1}^{k}(n_{2j}-n_{2j-1})} 2^{-i} + \sum_{k=1}^{+\infty} \sum_{i=n_{2k+1}}^{n_{2k+2}-1} 2^{\theta p \sum\limits_{j=1}^{k}(n_{2j}-n_{2j-1})} 2^{\theta p (i - n_{2k+1})} 2^{-i}$$
$$= \sum_{k=1}^{+\infty} \sum_{i=n_{2k}}^{n_{2k+1}-1} 2^{\theta p \frac{k(k-1)}{2} - i} + \sum_{k=1}^{+\infty} \sum_{i=n_{2k+1}}^{n_{2k+2}-1} 2^{\theta p (\frac{k(k-1)}{2} + i - n_{2k+1}) - i}.$$

By (0)–(2), $n_{2k+1} \ge n_{2k} \ge n_{2k-1} \ge 2^k$ for $k > 0$. Therefore,

$$I_1 := \sum_{k=1}^{+\infty} \sum_{i=n_{2k}}^{n_{2k+1}-1} 2^{\theta p \frac{k(k-1)}{2} - i} \le 2 \sum_{k=1}^{+\infty} 2^{\theta p \frac{k(k-1)}{2} - n_{2k}} \le 2 \sum_{k=1}^{+\infty} 2^{\theta p \frac{k(k-1)}{2} - 2^k} < \infty$$

and similarly

$$I_2 := \sum_{k=1}^{+\infty} \sum_{i=n_{2k+1}}^{n_{2k+2}-1} 2^{\theta p (\frac{k(k-1)}{2} + i - n_{2k+1}) - i} \le 2 \sum_{k=1}^{+\infty} 2^{\theta p (\frac{k(k-1)}{2} + k) - 2^k} < \infty.$$

Eventually,

$$\sum_{i=2}^{+\infty} c_i^p 2^{-i} = I_1 + I_2 < \infty;$$

therefore, (16) is valid. Straightforward calculation demonstrates that $\delta_\varphi = \theta$. Consequently, by Proposition 1, $M(\varphi) \notin \mathscr{E}$. $\square$

The following implication played a key role in the proof of the embedding $M(\varphi) \subset \mathscr{L}_{F^\varphi}$ (see (4)) (where $x = x(t)$ is an arbitrary measurable function on $[0,1]$):

$$\int_0^t x^*(s)\,ds \le \int_0^t \widetilde{\varphi}'(s)\,ds \text{ for all } t \in [0,1] \Rightarrow \|x\|_p \le \|\widetilde{\varphi}'\|_p \text{ for all } p \ge 1.$$

As is easily seen, the reverse embedding $\mathscr{L}_{F^\varphi} \subset M(\varphi)$ holds if and only if the converse implication holds:

$$\|x\|_p \le \|\widetilde{\varphi}'\|_p \text{ for all } p \ge 1 \Rightarrow \int_0^t x^*(s)\,ds \le C \int_0^t \widetilde{\varphi}'(s)\,ds \text{ for all } t \in [0,1].$$

By Corollary 2, there is $\varphi \in \mathscr{F}$ such that $M(\varphi) \subset L_p$ for all $p < \infty$, but $M(\varphi) \notin \mathscr{E}$; i.e., the last is false for such function φ. We will give an assertion improving this remark and simultaneously demonstrating the difficulties we face in interconnecting the interpolation and extrapolation constructions.

Proposition 2. *There is no universal constant $C > 0$ such that the inequality*

$$\|x\|_p \le \|y\|_p, \quad p \ge 1,$$

valid for two functions $x, y \in L_\infty$, would imply that

$$\int_0^t x^*(s)\,ds \le C \int_0^t y^*(s)\,ds, \quad 0 < t \le 1. \tag{17}$$

Proof. Take $n \in \mathbb{N}$, $n \ge 3$, $\varepsilon = \frac{1}{2(n-1)^n}$, and $\tau = (2\varepsilon)^{\frac{1}{n}} = \frac{1}{n-1}$. Consider the functions

$$x(t) = \left(1 - \frac{1}{n\varepsilon} t\right) \chi_{[0,n\varepsilon)}(t) \quad \text{and} \quad y(t) = \chi_{[0,\varepsilon)}(t) + \tau \chi_{[\varepsilon,1)}(t).$$

Then $\|y\|_p^p = \varepsilon + \tau^p(1-\varepsilon)$, $\|x\|_p^p = \frac{n\varepsilon}{p+1}$, and

$$\|x\|_p^p \le \|y\|_p^p \tag{18}$$

for all $p \ge 1$. Indeed, we can rewrite the last inequality as follows:

$$\frac{n\varepsilon}{p+1} \le \varepsilon + \tau^p(1-\varepsilon), \quad p \ge 1.$$

If $n \le p+1$ then the inequality is obvious. Assume that $1 \le p \le n-1$. It suffices to show that

$$\frac{n\varepsilon}{p+1} \le \frac{1}{2}\tau^p, \quad 1 \le p \le n-1. \tag{19}$$

Consider the function $\varphi(p) = (p+1)\tau^p$. Since $\varphi'(p) = \tau^p(1 + (p+1)\log\tau)$, $\varphi(p)$ is decreasing if $\tau \le 1/\sqrt{e}$. Therefore, it suffices to verify (19) for $p = n-1$ which is done by simple substitution: $\tau = (2\varepsilon)^{\frac{1}{n}}$. Thereby (18) is valid. Simultaneously, if

$$\mathscr{K}(t, x; L_1, L_\infty) = \int_0^t x^*(s)\,ds \quad (0 \le t \le 1)$$

then

$$\frac{\mathscr{K}(n\varepsilon, x; L_1, L_\infty)}{\mathscr{K}(n\varepsilon, y; L_1, L_\infty)} = \frac{n\varepsilon/2}{\varepsilon + (n-1)\tau\varepsilon} = \frac{n}{2 + 2(n-1)\tau} = \frac{n}{4} \to +\infty \quad \text{as } n \to +\infty,$$

and so (17) is false. $\square$

§ 3. Extrapolatory Description for Lorentz Spaces

S. F. Lukomskii [10] proved the following: If there is $\gamma > 0$ such that the inequality

$$\varphi'(t) \le \gamma t \varphi'(t^2) \tag{20}$$

holds for almost all $t \in [0,1]$ then the Lorentz space $\Lambda_p(\varphi)$ is an extrapolation space; more exactly,

$$\|x\|_{\Lambda_p(\varphi)} \asymp \left(\sum_{n=1}^{+\infty} \|x\|_n^p 2^{-n} \varphi'(2^{-n}) \right)^{\frac{1}{p}}. \tag{21}$$

In this section we find some necessary and sufficient conditions for (21) to be valid and also give an example of $\varphi \in \mathscr{F}$ for which (20) is false whereas (21) holds.

Denote by g_p the space $l_p(2^{-n/p}\varphi'(2^{-n})^{1/p})$ in which the norm coincides with the right-hand side of (21). Its function-space analog is the space $G_p = L_p(2^{-s/p}\varphi'(2^{-s})^{1/p})$ of functions on $[1, \infty)$.

Theorem 3. *The following are equivalent for each $p \in [1, \infty)$:*
(1) $\varphi \in \Delta^2$;
(2) $\Lambda_p(\varphi) = \mathscr{L}_{G_p}$;
(3) $\Lambda_p(\varphi) = \mathscr{L}_{g_p}$.

To prove the theorem, we need Lemma 3 below in which the role of the extrapolation parameter is played by the space $\widetilde{G}_p$ of functions on $[p, \infty)$ with the norm

$$\|y\|_{\widetilde{G}_p} = \Bigg(\int\limits_p^{+\infty} |y(s)|^p 2^{-s} \varphi'(2^{-s})\, ds \Bigg)^{\frac{1}{p}}.$$

Lemma 3. *For every $p \in [1, \infty)$, the rearrangement invariant space $\mathscr{L}_{\widetilde{G}_p}$ is p-convex.*

Proof. We need to prove that

$$\Bigg\| \Bigg(\sum_{k=1}^n |x_k|^p \Bigg)^{\frac{1}{p}} \Bigg\|_{\mathscr{L}_{\widetilde{G}_p}} \le C \Bigg(\sum_{k=1}^n \|x_k\|^p_{\mathscr{L}_{\widetilde{G}_p}} \Bigg)^{\frac{1}{p}} \tag{22}$$

for arbitrary $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n$ from $\mathscr{L}_{\widetilde{G}_p}$. Since, for $s \ge p$, the space $L_s[0;1]$ is p-convex with constant 1 [12, Proposition 1.d.5], we have

$$\Bigg\| \Bigg(\sum_{k=1}^n |x_k|^p \Bigg)^{\frac{1}{p}} \Bigg\|_s \le \Bigg(\sum_{k=1}^n \|x_k\|_s^p \Bigg)^{\frac{1}{p}}.$$

Thereby

$$\Bigg\| \Bigg(\sum_{k=1}^n |x_k|^p \Bigg)^{\frac{1}{p}} \Bigg\|_{\mathscr{L}G_p} \le \Bigg(\int\limits_p^{+\infty} \Bigg(\sum_{k=1}^n \|x_k\|_s^p \Bigg) 2^{-s} \varphi'(2^{-s})\, ds \Bigg)^{\frac{1}{p}} = \Bigg(\sum_{k=1}^n \|x_k\|^p_{\mathscr{L}_{\widetilde{G}_p}} \Bigg)^{\frac{1}{p}},$$

and (22) is proved. $\square$

Proof of Theorem 3. First of all, note that $\mathscr{L}_{\widetilde{G}_p}$ in Lemma 3 coincides with $\mathscr{L}_{G_p}$. Indeed, the embedding $\mathscr{L}_{G_p} \subset \mathscr{L}_{\widetilde{G}_p}$ is obvious. On the other hand,

$$\begin{aligned}
\Bigg(\int\limits_1^{+\infty} \|x\|_s^p\, 2^{-s} \varphi'(2^{-s})\, ds \Bigg)^{\frac{1}{p}} &\le \Bigg(\int\limits_1^{p} \|x\|_s^p\, 2^{-s} \varphi'(2^{-s})\, ds \Bigg)^{\frac{1}{p}} \\
+ \Bigg(\int\limits_p^{+\infty} \|x\|_s^p\, 2^{-s} \varphi'(2^{-s})\, ds \Bigg)^{\frac{1}{p}} &\le 2 \Bigg(\int\limits_{p+1}^{2p} \|x\|_s^p\, 2^{-s} \varphi'(2^{-s})\, ds \Bigg)^{\frac{1}{p}} \\
&+ \Bigg(\int\limits_p^{+\infty} \|x\|_s^p\, 2^{-s} \varphi'(2^{-s})\, ds \Bigg)^{\frac{1}{p}} \le 3\, \|x\|_{\mathscr{L}_{\widetilde{G}}},
\end{aligned}$$

whence we find that $\mathscr{L}_{G_p} \supset \mathscr{L}_{\widetilde{G}_p}$. Therefore, in the proof below, instead of $\mathscr{L}_{G_p}$ we can consider $\mathscr{L}_{\widetilde{G}_p}$.

It is easy to show that

$$\mathscr{L}_{\widetilde{G}_p} \subset \Lambda_p(\varphi) \tag{23}$$

for every $\varphi \in \mathscr{F}$. Indeed, since $x^*(2^{-s}) \le 2\|x\|_s$ $(s \ge 1)$, we have

$$\|x\|^p_{\Lambda_p(\varphi)} \le 2^p \int\limits_0^{1/2^p} x^*(t)^p \, d\varphi(t) = 2^p \log 2 \int\limits_p^{+\infty} x^*(2^{-s})^p 2^{-s} \varphi'(2^{-s}) \, ds \le 2^{p+1} \log 2 \|x\|^p_{\mathscr{L}_{\widetilde{G}_p}}.$$

As is well known [12, p. 54], if X is a p-convex Banach function space then we can consider the Banach space $X_{(p)}$ of all functions $x = x(t)$ such that $|x(t)|^{\frac{1}{p}} \in X$ with the norm $\|x\|_{X_{(p)}} = \||x|^{\frac{1}{p}}\|^p_X$. By Lemma 3, this procedure applies to $\mathscr{L}_{\widetilde{G}_p}$; moreover, $(\Lambda_p(\varphi))_{(p)} = \Lambda(\varphi)$. Thereby, by (23),

$$(\mathscr{L}_{\widetilde{G}_p})_{(p)} \subset \Lambda(\varphi). \tag{24}$$

The Lorentz space $\Lambda(\varphi)$ is the smallest space among all rearrangement invariant spaces with the same fundamental function [11, p. 160]. Therefore, the reverse embedding to (24), i.e., $\Lambda(\varphi) \subset (\mathscr{L}_{\widetilde{G}_p})_{(p)}$ or, equivalently, $\Lambda_p(\varphi) \subset \mathscr{L}_{\widetilde{G}_p}$, is valid if and only if

$$\psi_p(t) \le C\varphi(t), \quad 0 < t \le 1, \tag{25}$$

where $\psi_p(t)$ is the fundamental function of $(\mathscr{L}_{\widetilde{G}_p})_{(p)}$. Using the definition of $(\mathscr{L}_{\widetilde{G}_p})_{(p)}$, we infer the inequality

$$\int\limits_p^{+\infty} t^{\frac{p}{s}} 2^{-s} \varphi'(2^{-s}) \, ds \le C\varphi(t), \quad 0 < t \le 1.$$

Since both sides represent quasiconcave functions, it suffices to validate this relation for $t = 2^{-j}$, $j \ge 2p$. After substitution we obtain

$$\int\limits_{-\infty}^{-p} 2^{\frac{jp}{s}} d\varphi(2^s) \le C\varphi(2^{-j}).$$

Since

$$\int\limits_{-\infty}^{-j} 2^{\frac{jp}{s}} d\varphi(2^s) \le \varphi(2^{-j}),$$

the previous inequality is equivalent to the inequality

$$\int\limits_{-j}^{-p} 2^{\frac{jp}{s}} d\varphi(2^s) \le C\varphi(2^{-j}). \tag{26}$$

Prove that (26) holds if and only if $\varphi \in \Delta^2$.

Assume (26) valid. Then, on the one hand,

$$\int\limits_{-j}^{-j/2} 2^{\frac{jp}{s}} d\varphi(2^s) \le C\varphi(2^{-j})$$

for $j \ge 2p$ and, on the other hand,

$$\int\limits_{-j}^{-j/2} 2^{\frac{jp}{s}} d\varphi(2^s) \ge 2^{-2p}(\varphi(2^{-j/2}) - \varphi(2^{-j})).$$

Hence,

$$\varphi(2^{-j/2}) \le (2^{2p}C + 1)\varphi(2^{-j}), \quad j \ge 2p;$$

now, by the standard arguments involving the quasiconcavity of φ , we arrive at $\varphi \in \Delta^2$.

Suppose, vice versa, that $\varphi \in \Delta^2$. First of all, note that in this case it suffices to prove (25) only for the points $t = 2^{-j}$, where $j = 2^m$, $m \in \mathbb{N}$. Indeed, assume that (25) is valid for such t and take $j \in [2^m, 2^{m+1}]$. Then

$$\psi_p(2^{-j}) \le \psi_p(2^{-2^m}) \le C\varphi(2^{-2^m}) \le C\alpha\varphi(2^{-2^{m+1}}) \le C\alpha\varphi(2^{-j}),$$

where α is the constant from the Δ^2-condition. Thus, in the proof of (26) we may assume that $j = 2^m$, $m \in \mathbb{N}$, and $j \ge p$. In this case we obtain

$$\begin{aligned}
\int_{-j}^{-p} 2^{\frac{jp}{s}}\, d\varphi(2^s) &\le \int_{-j}^{-1} 2^{\frac{jp}{s}}\, d\varphi(2^s) \le \int_{-j}^{-1} 2^{\frac{j}{s}}\, d\varphi(2^s) = \sum_{k=1}^{m} \int_{-2^k}^{-2^{k-1}} 2^{\frac{j}{s}}\, d\varphi(2^s) \\
&\le \sum_{k=1}^{m} 2^{-2^{m-k}} \varphi(2^{-2^{k-1}}) \le \sum_{k=1}^{m} 2^{-2^{m-k}} \alpha^{m-k+1} \varphi(2^{-2^m}) \\
&= \alpha\varphi(2^{-j}) \sum_{l=0}^{m-1} 2^{-2^l + l\log_2 \alpha} < \alpha\varphi(2^{-j}) \sum_{l=0}^{+\infty} 2^{-2^l + l\log_2 \alpha} = C\varphi(2^{-j}),
\end{aligned}$$

where $C = \alpha \sum_{l=0}^{+\infty} 2^{-2^l + l\log_2 \alpha} < \infty$, and (26) is valid. Consequently, $\Lambda_p(\varphi) \subset \mathscr{L}_{\widetilde{G}_p}$.

Thus, (1) and (2) are equivalent. To complete proving, show that $\mathscr{L}_{G_p} = \mathscr{L}_{g_p}$. Indeed, for $s \in [n, n+1]$ we find

$$\frac{1}{2}\|x\|_n^p 2^{-n}\varphi'(2^{-n}) \le \|x\|_s^p 2^{-s}\varphi'(2^{-s}) \le 2\|x\|_{n+1}^p 2^{-n-1}\varphi'(2^{-n-1})$$

and so

$$\begin{aligned}
\|x\|_{\mathscr{L}_{g_p}}^p &= \sum_{n=1}^{+\infty} \|x\|_n^p 2^{-n}\varphi'(2^{-n}) \le 2\sum_{n=1}^{+\infty} \int_n^{n+1} \|x\|_s^p 2^{-s}\varphi'(2^{-s})\, ds \\
&= 2\int_1^{+\infty} \|x\|_s^p\, 2^{-s}\varphi'(2^{-s})\, ds = 2\|x\|_{\mathscr{L}_{G_p}}^p \le 4\sum_{n=1}^{+\infty} \|x\|_{n+1}^p 2^{-n-1}\varphi'(2^{-n-1}) \le 4\|x\|_{\mathscr{L}_{g_p}}^p.
\end{aligned}$$

Therefore, the norms in $\mathscr{L}_{G_p}$ and $\mathscr{L}_{g_p}$ are equivalent. $\square$

Note that if $\varphi \in \mathscr{F}$ satisfies (20) then $\varphi \in \Delta^2$ (it suffices to integrate (20)). We will give an example demonstrating that the converse is false; i.e., there is a function $\varphi \in \mathscr{F}$ belonging to Δ^2 but violating (20).

Example 2. We denote by $\mathscr{F}_1$ the class of piecewise linear functions in $\mathscr{F}$ of the form

$$\varphi(2^{-k}) = \sum_{m \ge k} \Delta_m \quad \text{and} \quad \varphi(t) \text{ is linear on } [2^{-k-1}, 2^{-k}],\ k \in \mathbb{Z}_+.$$

For a such function to belong to $\mathscr{F}$, it is necessary and sufficient that

(1) $\Delta_k > 0$ ($\varphi(t)$ is increasing);

(2) $\sum_{k \ge 0} \Delta_k < \infty$ ($\varphi(t)$ is finite);

(3) $\frac{\Delta_k}{\Delta_{k+1}} \le 2$ ($\varphi(t)$ is concave).

By the properties of the functions in $\mathscr{F}$, it suffices to verify the Δ^2-condition only at the points of the form $t_k = 2^{-k}$. For $\varphi \in \mathscr{F}_1$, this condition is equivalent to the requirement

$$\sum_{k\ge n} \Delta_k \le C \sum_{k\ge 2n} \Delta_k, \quad n \in \mathbb{Z}_+. \tag{27}$$

Condition (20) for $\varphi \in \mathscr{F}_1$ takes the form:

$$\Delta_k \le \gamma_1 \min(\Delta_{2k+1}, \Delta_{2k}), \quad k \in \mathbb{Z}_+.$$

Therefore, φ does not satisfy (20) if

$$\varlimsup_{k\to+\infty} \frac{\Delta_k}{\Delta_{2k}} = +\infty. \tag{28}$$

Thus, it suffices to construct an example of a function $\varphi \in \mathscr{F}_1$ satisfying (27) and (28).

Assume that $n_0 \ge 4$ is a full square natural number satisfying

$$(n + \sqrt{n})^2 \le 2^{\sqrt{n}} n^{\frac{3}{2}} \tag{29}$$

for $n \ge n_0$. Introduce the two sequences of natural numbers:

$$n_i = n_{i-1}^2 \quad \text{and} \quad m_i = \min\bigl\{k \ge n_i : k^2 \le 2^{k-n_i} n_i^{\frac{3}{2}}\bigr\}, \quad i \in \mathbb{N}. \tag{30}$$

By (29) and (30), $m_i \le n_i + \sqrt{n_i}$ and the intervals $[n_i, m_i]$ $(i = 0, 1, \dots)$ are pairwise disjoint. Moreover, none of the intervals of the form $[n, 2n]$ $(n \in \mathbb{N})$ can intersect more than one interval of the form $[n_i, m_i]$. Indeed, since $n_i \ge 4$, we have $n_i^2 > 2(n_i + \sqrt{n_i})$. Consequently, if i is the least number such that $[n_i, m_i] \cap [n, 2n] \ne \varnothing$ then $m_i \ge n$ and

$$n_{i+1} - m_i \ge n_i^2 - n_i - \sqrt{n_i} > n_i + \sqrt{n_i} \ge m_i \ge n,$$

whence $n_{i+1} > m_i + n \ge 2n$. Therefore, $[n_{i+1}, m_{i+1}] \cap [n, 2n] = \varnothing$.

Consider $\varphi(t)$ such that

$$\Delta_k = \begin{cases} \frac{1}{10} & \text{if } k = 0, 1, 2; \\ \frac{1}{k^2} & \text{if } k \ge 3 \text{ and } k \notin [n_i, m_i); \\ 2^{n_i - k} \frac{1}{n_i^{3/2}} & \text{if } k \in [n_i, m_i). \end{cases}$$

We first verify the conditions (1)–(3) for $\varphi(t)$ to belong to $\mathscr{F}_1$. Condition (1) is obvious. Since

$$\sum_{k\ge 0} \Delta_k \le \frac{3}{10} + \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{i=0}^{\infty} \sum_{k=n_i}^{m_i - 1} 2^{n_i - k} \frac{1}{n_i^{3/2}} \le \frac{3}{10} + \frac{\pi^2}{6} + 2\sum_{i=0}^{\infty} \frac{1}{n_i^{3/2}} \le 3 + 2\sum_{k=1}^{\infty} \frac{1}{n_0^k} < \infty,$$

we see that (2) holds as well.

We are left with verifying (3). If $k = 0, 1, 2$ then the condition is obvious. If $k, k+1 \notin [n_i, m_i)$ then

$$\frac{\Delta_k}{\Delta_{k+1}} = \frac{(k+1)^2}{k^2} \le 2$$

for $k \ge 3$. If k, $k+1 \in [n_i, m_i)$ then $\frac{\Delta_k}{\Delta_{k+1}} = 2$ by the definition of Δ_k. In the case $k = n_i - 1$ for some $i = 0, 1, \dots$ we have

$$\frac{\Delta_k}{\Delta_{k+1}} = \frac{n_i^{3/2}}{(n_i - 1)^2} \le \frac{n_i^2}{(n_i - 1)^2} \le 2,$$

since $n_i \geq 4$. We are left with considering the last situation in which $k = m_i - 1$. Using the definition of m_i in (30), we obtain

$$\frac{\Delta_k}{\Delta_{k+1}} = \frac{2^{n_i-(m_i-1)} m_i^2}{n_i^{3/2}} \leq \frac{m_i^2}{(m_i-1)^2} \leq 2,$$

since $m_i \geq 4$.

Thus, $\varphi \in \mathscr{F}_1$. To check (28), it suffices to assert that

$$\lim_{i\to+\infty} \frac{\Delta_{n_i}}{\Delta_{2n_i}} = \infty.$$

Since $n_{i+1} - 2n_i = n_i^2 - 2n_i > 0$ and $m_i \leq n_i + \sqrt{n_i} < 2n_i$, we have $2n_i \in (m_i, n_{i+1})$ and therefore $\Delta_{2n_i} = \frac{1}{(2n_i)^2}$. Consequently,

$$\frac{\Delta_{n_i}}{\Delta_{2n_i}} = \frac{(2n_i)^2}{n_i^{3/2}} = 4\sqrt{n_i} \to +\infty,$$

and thereby (28) is valid.

Checking (27), we can certainly assume that $n \geq 4$. By (30),

$$\frac{1}{k^2} < \frac{2^{n_i-k}}{n_i^{3/2}} = \Delta_k$$

for $k \in [n_i, m_i)$. Therefore, we can estimate the sum on the right-hand side of (27) from below as follows:

$$\sum_{k\geq 2n} \Delta_k \geq \sum_{k\geq 2n} \frac{1}{k^2} \geq \frac{1}{2n}, \quad n = 1, 2, \ldots. \tag{31}$$

Represent the sum on the left-hand side of (27) as

$$\sum_{k\geq n} \Delta_k = \sum_{k=n}^{2n-1} \Delta_k + \sum_{k\geq 2n} \Delta_k. \tag{32}$$

Assume that there is i satisfying $[n_i, m_i] \cap [n, 2n-1] \neq \varnothing$ (as observed above, this index i is unique). Then the first sum on the right-hand side in (32) is estimated as follows:

$$\sum_{k=n}^{2n-1} \Delta_k \leq \sum_{k=n}^{2n-1} \frac{1}{k^2} + \sum_{k=n_i}^{m_i-1} \Delta_k \leq \frac{2}{n} + \frac{2}{n_i}.$$

Since $n < m_i \leq n_i + \sqrt{n_i} < 2n_i$, we have $n_i > \frac{n}{2}$; therefore, by (31),

$$\sum_{k=n}^{2n-1} \Delta_k \leq \frac{6}{n} \leq 12 \sum_{k\geq 2n} \Delta_k.$$

Eventually, using (32), we find that

$$\sum_{k\geq n} \Delta_k \leq 13 \sum_{k\geq 2n} \Delta_k;$$

i.e., (27) is valid.

Remark 1. The above example demonstrates that there exists φ belonging to Δ^2 but violating (20). The results of [10] do not apply to such functions but Theorem 3 of the present article does. At the same time, we should observe that the constructed function $\varphi(t)$ is equivalent to the function $\log^{-1}(10/t)$ which, as is easily verified, satisfies (20). The point is that (20) (unlike the condition $\varphi \in \Delta^2$) is not invariant under passage to equivalent functions.

Remark 2. In Theorem 3 we gave necessary and sufficient conditions for $\Lambda_p(\varphi)$ to coincide with the extrapolation space $\mathscr{L}_{G_p}$, where the parameter G_p has a special form. The authors are unaware whether there are nevertheless $\varphi \in \mathscr{F}$, $\varphi \notin \Delta^2$, such that $\Lambda_p(\varphi) = \mathscr{L}_H$ for some ideal Banach space H.

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