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Journal of Functional Analysis 226 (2005) 173–192

JOURNAL OF
Functional
Analysis

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Symmetric kernel of Rademacher multiplier spaces

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Received 6 August 2004; received in revised form 19 January 2005; accepted 27 January 2005

Communicated by G. Pisier

Available online 31 March 2005

Abstract

Let X be a rearrangement invariant (r.i.) function space on $[0,1]$. We consider the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$ of measurable functions x such that $xh \in X$ for every a.e. converging series $h = \sum a_n r_n \in X$, where (r_n) are the Rademacher functions. We show that for a broad class of r.i. spaces X , the space $\Lambda(\mathcal{R}, X)$ is not r.i. In this case, we identify the symmetric kernel $\text{Sym}(\mathcal{R}, X)$ of the Rademacher multiplier space and study when $\text{Sym}(\mathcal{R}, X)$ reduces to L^∞ . In the opposite direction, we find new examples of r.i. spaces for which $\Lambda(\mathcal{R}, X)$ is r.i. We consider in detail the case when X is a Marcinkiewicz or an exponential Orlicz space. © 2005 Elsevier Inc. All rights reserved.

MSC: primary 46E35; 46E30; secondary 47G10

Keywords: Rademacher functions; Rearrangement invariant space; Marcinkiewicz spaces; Orlicz spaces; Interpolation of operators; K -functional

0. Introduction

This paper is concerned with the behaviour of the Rademacher functions (r_n) in function spaces. Let $\mathcal{R}$ denote the set of all functions of the form $\sum a_n r_n$, where the

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¹ Partially supported D.G.I. #BFM2003–06335–C03–01, Spain.

series converges a.e. For a rearrangement invariant (r.i.) space X on $[0,1]$, let $\mathcal{R}(X)$ be the closed linear subspace of X given by $\mathcal{R} \cap X$. The *Rademacher multiplier space* of X is the space $\Lambda(\mathcal{R}, X)$ of measurable functions $x: [0,1] \rightarrow \mathbb{R}$ such that $x \sum a_n r_n \in X$, for every $\sum a_n r_n \in \mathcal{R}(X)$. Endowed with the norm

$$\|x\|_{\Lambda(\mathcal{R}, X)} = \sup \left\{ \left\| x \sum a_n r_n \right\|_X : \sum a_n r_n \in X, \left\| \sum a_n r_n \right\|_X \leq 1 \right\},$$

$\Lambda(\mathcal{R}, X)$ is a Banach function space on $[0,1]$. The space $\Lambda(\mathcal{R}, X)$ can be viewed as the space of operators given by multiplication by a measurable function, from $\mathcal{R}(X)$ into the whole space X .

In [9], it was shown that for a broad class of classical r.i. spaces X (including, for example, the $L^{p,q}$ spaces) the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$ is not r.i. In [10] the case when $\Lambda(\mathcal{R}, X) = L^\infty$ was studied; moreover, a class of Orlicz spaces with $\Lambda(\mathcal{R}, X)$ r.i. but different from L^∞ was identified. In [2], it was shown that $\Lambda(\mathcal{R}, X) = L^\infty$ holds for all r.i. spaces X which are interpolation spaces for the couple (L^∞, L_N) , where L_N is the Orlicz space with $N(t) = \exp(t^2) - 1$.

In this paper, we study the symmetric kernel $\text{Sym}(\mathcal{R}, X)$ of the Rademacher multiplier space, namely, the largest r.i. space embedded into $\Lambda(\mathcal{R}, X)$. It can alternatively be described as the set of all $x \in \Lambda(\mathcal{R}, X)$ such that if y is a function equimeasurable with x then, $y \in \Lambda(\mathcal{R}, X)$. This study is motivated by an extension of the result from [9] mentioned above, where we show that for “most” r.i. spaces X the space $\Lambda(\mathcal{R}, X)$ is not r.i. (Theorem 2.1). This is done in Section 2, where we prove that if X is a r.i. space satisfying the Fatou property and $X \supset L_N$, then $\text{Sym}(\mathcal{R}, X) = X_{\log^{1/2}}$, where $X_{\log^{1/2}}$ is the r.i. space with norm $\|x\|_{\log^{1/2}} = \|x^*(t) \log^{1/2}(e/t)\|_X$ (Corollary 2.11). We consider the problem of which r.i. spaces arise as symmetric kernels of Rademacher multiplier spaces, showing that this is the case for all interpolation spaces for the couple $(L \log^{1/2} L, L^\infty)$ with Fatou property (Theorem 2.17). In Section 3, we characterize when the symmetric kernel reduces to L^∞ (Theorem 3.2). In Section 4, we exhibit a family of r.i. spaces X for which the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$ is r.i. but different from L^∞ (Theorem 4.7). Namely, the exponential Orlicz spaces, which include the previous examples of this situation shown in [9,10]. All these results allow to identify the symmetric kernel for relevant classes of r.i. spaces as Lorentz–Zigmond spaces $L^{p,q}(\log L)^\alpha$, Lorentz spaces $\Lambda^p(\varphi)$ and Marcinkiewicz spaces $M(\varphi)$.

1. Preliminaries

Throughout the paper a r.i. space X is a Banach space of classes of measurable functions on $[0,1]$ such that if $y^* \leq x^*$ and $x \in X$ then $y \in X$ and $\|y\|_X \leq \|x\|_X$. Here x^* is the decreasing rearrangement of x , that is, the right continuous inverse of its distribution function: $n_x(\lambda) = m\{t \in [0,1] : |x(t)| > \lambda\}$, where m is the Lebesgue measure on $[0,1]$. Functions x and y are said to be equimeasurable if $n_x(\lambda) = n_y(\lambda)$, for all $\lambda > 0$; this denoted by $x \simeq y$. The associated space (or Köthe dual) of X is the space X' of all functions y such that $\int_0^1 |x(t)y(t)| dt < \infty$, for every $x \in X$.

It is a r.i. space. The space X' is a subspace of the topological dual X^* . If X' is a norming subspace of X^* , then X is isometric to a subspace of the space $X'' = (X')'$. The space X satisfies the Fatou property if $x_n \in X$, with $\|x_n\|_X \leq M$, for all $n \in \mathbb{N}$, and $0 \leq x_n \leq x_{n+1} \uparrow x$ a.e. imply $x \in X$ and $\|x_n\|_X \rightarrow \|x\|_X$. In this case, X' is a norming subspace of X^* and $X = X''$. We denote by X_0 the closure of L^∞ in X . If X is not L^∞ , then X_0 coincides with the absolutely continuous part of X , that is, the set of all functions $x \in X$ such that $\lim_{m(A) \rightarrow 0} \|x\chi_A\|_X = 0$. Here and next, χ_A is the characteristic function of the set $A \subset [0, 1]$. The fundamental function of X is the function $\varphi_X(t) := \|\chi_{[0,t]}\|_X$.

Important examples of r.i. spaces are Marcinkiewicz, Lorentz and Orlicz spaces. Let $\varphi: [0, 1] \rightarrow [0, +\infty)$ be a quasi-concave function, that is, φ increases, $\varphi(t)/t$ decreases and $\varphi(0) = 0$. The Marcinkiewicz space $M(\varphi)$ is the space of all measurable functions x on $[0, 1]$ for which the norm

$$\|x\|_{M(\varphi)} = \sup_{0 < t \leq 1} \frac{1}{\varphi(t)} \int_0^t x^*(s) ds < \infty.$$

If $\varphi: [0, 1] \rightarrow [0, +\infty)$ is an increasing concave function and $p \in [1, +\infty)$, then the Lorentz space $\Lambda^p(\varphi)$ consists of all measurable functions x on $[0, 1]$ such that

$$\|x\|_{\Lambda^p(\varphi)} = \left(\int_0^1 (x^*(s))^p d\varphi(s) \right)^{1/p} < \infty.$$

The last Riemann–Stieltjes integral may be rewritten in the form

$$\|x\|_{\Lambda^p(\varphi)} = \left(\varphi(+0) \|x\|_{L^\infty}^p + \int_0^1 (x^*(s))^p \varphi'(s) ds \right)^{1/p}.$$

In particular, this implies that $\Lambda^p(\varphi_0) = L^\infty$, for $\varphi_0(t) = 1$.

Let M be an Orlicz function, that is, an increasing convex function on $[0, \infty)$ with $M(0) = 0$. The norm of the Orlicz space L_M is defined as follows:

$$\|x\|_{L_M} = \inf \left\{ \lambda > 0 : \int_0^1 M\left(\frac{|x(s)|}{\lambda}\right) ds \leq 1 \right\}.$$

The fundamental functions of these spaces are $\varphi_{M(\varphi)}(t) = t/\varphi(t)$, $\phi_{\Lambda^p(\varphi)}(t) = \varphi(t)^{1/p}$ and $\varphi_{L_M}(t) = 1/M^{-1}(1/t)$, respectively.

The Marcinkiewicz $M(t/\varphi)$ and Lorentz $\Lambda(\varphi) := \Lambda^1(\varphi)$ spaces are, respectively, the largest and the smallest r.i. spaces with fundamental function φ , that is, if the fundamental function of a r.i. space X is equal to φ , then $\Lambda(\varphi) \subset X \subset M(t/\varphi)$ [15, Theorems II.5.5 and II.5.7].

If ψ is a positive function defined on $[0,1]$, then its lower dilation index is

$$\gamma_\psi = \lim_{t \rightarrow 0+} \frac{\log \left(\sup_{0 < s \leq 1} \frac{\psi(st)}{\psi(s)} \right)}{\log t}.$$

For each $t > 0$, the dilation operator $\sigma_t x(s) := x(s/t) \chi_{[0,1]}(s/t)$, $s \in [0, 1]$, is bounded in any r.i. space X . Moreover, $\|\sigma_t\|_{X \rightarrow X} \leq \max(1, t)$ [15]. Important characteristics of a r.i. space X are the lower and upper Boyd indices

$$\alpha_X = \lim_{t \rightarrow +0} \frac{\ln \|\sigma_t\|_{X \rightarrow X}}{\ln t} \quad \text{and} \quad \beta_X = \lim_{t \rightarrow \infty} \frac{\ln \|\sigma_t\|_{X \rightarrow X}}{\ln t}.$$

The Rademacher functions are $r_n(t) = \text{sign} \sin(2^n \pi t)$, $t \in [0, 1]$, $n \geq 0$. We have already introduced the space $\mathcal{R}(X) := \mathcal{R} \cap X$ where $\mathcal{R}$ is the set of all a.e. converging series $\sum a_n r_n$, that is, $(a_n) \in \ell^2$ [21, Theorem V.8.2]. For $X = L^p$, $1 \leq p < \infty$, Khintchin inequality shows that $\mathcal{R}(X)$ is isomorphic to ℓ^2 . If $X = L^\infty$, then $\mathcal{R}(X) \approx \ell^1$. The Orlicz space L_N , for $N(t) = \exp(t^2) - 1$, will be of major importance in our study. A result of Rodin and Semenov shows that $\mathcal{R}(X) \approx \ell^2$ if and only if $(L_N)_0 \subset X$ [20]. Hence, for spaces X satisfying this condition we have, for constants $C_1, C_2 > 0$ depending on X and not on (a_n) ,

$$C_1 \|(a_n)\|_2 \leq \left\| \sum a_n r_n \right\|_X \leq C_2 \|(a_n)\|_2. \quad (1)$$

We will express this situation by writing $\left\| \sum a_n r_n \right\|_X \asymp \|(a_n)\|_2$. The fundamental function of L_N is (equivalent to) $\varphi(t) = \log^{-1/2}(e/t)$. Since $N(t)$ increases very rapidly, L_N coincides with the Marcinkiewicz space with fundamental function φ , that is, $M(\psi)$ for $\psi(t) = t \log^{1/2}(e/t)$ [17]. This together with [15, Theorem II.5.3], gives

$$\|x\|_{L_N} \asymp \sup_{0 < t \leq 1} x^*(t) \log^{-1/2}(e/t). \quad (2)$$

In particular, for every $0 < t \leq 1$ we have

$$x^*(t) \leq C \|x\|_{L_N} \log^{1/2}(e/t). \quad (3)$$

Hence, for a r.i. space X , $L_N \subset X$ is equivalent to $\log^{1/2}(e/t) \in X$. From (1) and (3) we have, for every $0 < t \leq 1$

$$\left(\sum a_n r_n \right)^*(t) \leq K \|(a_n)\|_2 \log^{1/2}(e/t). \quad (4)$$

For facts related to r.i. spaces see [6,15,16].

2. Symmetric kernel of the Rademacher multiplier space

In [9] it was proved that if X is a r.i. space such that $L_N \subset X$, the fundamental function φ_X satisfies the inequality $\varphi_X(st) \leq C \varphi_X(s) \varphi_X(t)$ for a constant $C > 0$ and all $s, t \in [0, 1]$, and the lower Boyd and fundamental indices coincide then, the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$ is not r.i. A detailed analysis of the proof shows that the following more general result holds.

Theorem 2.1. *Let X be a r.i. space with $\gamma_{\varphi_X} > 0$, where γ_{φ_X} is the lower dilation index of φ_X , then $\Lambda(\mathcal{R}, X)$ is not r.i. In particular, the result holds if the lower Boyd index $\alpha_X > 0$.*

Proof. If $\gamma_{\varphi_X} > 0$, then for $0 < \alpha < \gamma_{\varphi_X}$ there exists a constant $C > 0$ such that

$$\frac{\varphi_X(st)}{\varphi_X(s)} \leq C t^\alpha \quad (5)$$

for $0 < s, t \leq 1$. Hence, $\varphi_X(t) \leq C \varphi_X(1) t^\alpha$ and so, $\Lambda(t^\alpha) \subset \Lambda(\varphi_X)$. Note that $L^p \subset L^{1/\alpha, 1} := \Lambda(t^\alpha)$ for $p > 1/\alpha$; see [6, IV.4.2]. Since $L_N \subset L^p$ for all finite p , and $\Lambda(\varphi_X) \subset X$, we have $L_N \subset X$. So, $\mathcal{R}(X) \approx \ell^2$.

Lemma 1 in [9] shows that for every $n \geq 1$ there exist measurable sets B_n and D_n of measure $n2^{-n}$, such that for any $(b_i) \in l_2$

$$c \cdot \frac{\|\chi_{B_n}\|_{\Lambda(\mathcal{R}, X)}}{\|\chi_{D_n}\|_{\Lambda(\mathcal{R}, X)}} \leq \frac{\left\| \chi_{[0, n2^{-n}]} \sum_{n+1}^{\infty} b_i r_i \right\|_X}{n^{1/2} \varphi_X(n2^{-n})} + \left(\frac{\varphi_X(2^{-n})}{\varphi_X(n2^{-n})} \right)^{1/2} \quad (6)$$

with a constant $c > 0$ independent of n .

From (5) we have $\varphi_X(2^{-n})/\varphi_X(n2^{-n}) < C(1/n)^\alpha$. Hence, the second term in the right-hand side of (6) tends to zero as $n \rightarrow \infty$.

We now consider the first term in the right-hand side of (6). Since

$$m \left(\left\{ t : \chi_{[0, n2^{-n}]}(t) \left| \sum_{n+1}^{\infty} b_i r_i(t) \right| > \lambda \right\} \right) = \frac{n}{2^n} m \left(\left\{ t : \left| \sum_{n+1}^{\infty} b_i r_i(t) \right| > \lambda \right\} \right),$$

we have

$$\left(\chi_{[0, n2^{-n}]} \sum_{n+1}^{\infty} b_i r_i \right)^* (t) = \left(\sum_{n+1}^{\infty} b_i r_i \right)^* (2^n t/n), \quad 0 \leq t \leq n/2^n.$$

The embedding $\Lambda(\varphi_X) \subset X$ and (4) give

$$\begin{aligned} \left\| \chi_{[0, n2^{-n}]} \sum_{n+1}^{\infty} b_i r_i \right\|_X &\leq \left\| \chi_{[0, n2^{-n}]} \sum_{n+1}^{\infty} b_i r_i \right\|_{\Lambda(\varphi_X)} \\ &= \int_0^{n/2^n} \left(\sum_{n+1}^{\infty} b_i r_i \right)^* \left(\frac{2^n t}{n} \right) \varphi_X'(t) dt \\ &\leq K \| (b_i) \|_2 \int_0^{n/2^n} \log^{1/2} \left(\frac{ne}{2^n t} \right) \varphi_X'(t) dt. \end{aligned}$$

Integrating by parts this last integral, we get

$$\varphi_X \left(\frac{n}{2^n} \right) - \lim_{t \rightarrow 0} \log^{1/2} \left(\frac{ne}{2^n t} \right) \varphi_X(t) + \frac{1}{2} \int_0^{n/2^n} \frac{\log^{-1/2} \left(\frac{ne}{2^n t} \right) \varphi_X(t)}{t} dt. \quad (7)$$

From (5), the second term in (7) is zero. Since $\log^{-1/2} \left(\frac{ne}{2^n t} \right) \leq 1$, for $0 < t \leq n/2^n$, the last term in (7) is bounded by $C\varphi_X(n/2^n)$; see [15, p. 57].

Consequently, the first term in (6) is bounded by $C/n^{1/2}$, which tends to zero as $n \rightarrow \infty$. Hence, the space $\Lambda(\mathcal{R}, X)$ is not r.i.

The last assertion follows from $\gamma_{\varphi_X} \geq \alpha_X$. $\square$

From Theorem 2.1 it follows that for “most” r.i. spaces X the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$ is not r.i. In this case we are interested in identifying the largest r.i. space embedded into $\Lambda(\mathcal{R}, X)$. The problem of finding the largest r.i. space embedded into a Banach function space E has been considered in [4] in the case when E is reflexive. Under certain conditions, the largest r.i. space embedded into E coincides with the set of all functions in E which remain in E after rearrangement.

Definition 2.2. Let E be a Banach lattice of measurable functions on $[0, 1]$. The symmetric kernel of E is

$$\text{Sym}(E) = \{x \in E : \text{if } y \simeq x, \text{ then } y \in E\}.$$

Proposition 2.3. Let E be a Banach lattice of measurable functions on $[0, 1]$. Suppose that E' is a norming subspace of E^* . The symmetric kernel of E endowed with the norm

$$\|x\|_{\text{Sym}(E)} := \sup_{z: z \simeq x} \|z\|_E$$

is the largest r.i. space embedded into E . In particular, the result holds if E has the Fatou property.

Proof. For $x \in E$, let $f_x(y) := \int_0^1 x(t)y(t) dt$, for $y \in E'$. Let $x \in \text{Sym}(E)$ and $y \in E'$. There exists a function x_0 equimeasurable with x , so $x_0 \in E$, such that

$$\sup_{z: z \simeq x} f_z(y) = f_{x_0}(y) \leq \|x_0\|_X \|y\|_{X'} < \infty;$$

see [6, II.2.6]. Hence, the set $\{f_z : z \simeq x\}$ of continuous linear functionals over E' is pointwise bounded. Since E' is a closed subspace of E^* , by the Uniform Boundedness Principle, this set is uniformly bounded

$$\sup_{z: z \simeq x} \sup_{\|y\|_{E'} \leq 1} \int_0^1 z(t)y(t) dt < \infty.$$

Since E' is a norming subspace of E^* , we have $\sup_{z: z \simeq x} \|z\|_E < \infty$. Thus, $\|\cdot\|_{\text{Sym}(E)}$ is well defined. It is routine to check that it is a complete norm. By construction $\text{Sym}(E)$ is r.i. and if Y is a r.i. space with $Y \subset E$, then $Y \subset \text{Sym}(E)$. $\square$

For the symmetric kernel of the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$ we are able to prove an analogous result without any assumptions on X . To simplify notation we will write $\text{Sym}(\mathcal{R}, X)$ for $\text{Sym}(\Lambda(\mathcal{R}, X))$.

Proposition 2.4. *Let X be a r.i. space. The symmetric kernel of $\Lambda(\mathcal{R}, X)$ endowed with the norm*

$$\|x\|_{\text{Sym}(\mathcal{R}, X)} := \sup_{z: z \simeq x} \|z\|_{\Lambda(\mathcal{R}, X)}$$

is the largest r.i. space embedded into $\Lambda(\mathcal{R}, X)$.

Proof. Let $x \in \text{Sym}(\mathcal{R}, X)$. For every $z \simeq x$ consider the operator T_z given by $T_z y(t) := z(t)y(t)$, where $y \in \mathcal{R}(X)$. Since $z \in \Lambda(\mathcal{R}, X)$, we have $T_z: \mathcal{R}(X) \rightarrow X$ boundedly. Given $y \in \mathcal{R}(X)$ and $\varepsilon > 0$, there exists a measure preserving transformation $\omega: [0, 1] \rightarrow [0, 1]$ such that $\| |y| - y^*(\omega) \|_\infty \leq \varepsilon$; see [15, p. 60]. Then,

$$\begin{aligned} \|T_z^* y^*\|_X &= \|z^*(\omega) y^*(\omega)\|_X \\ &\leq \|z^*(\omega) y\|_X + \|z^*(\omega) (|y| - y^*(\omega))\|_X \\ &\leq \|x^*(\omega) y\|_X + \varepsilon \|x\|_X, \end{aligned}$$

since $x \in \text{Sym}(\mathcal{R}, X)$ and $y \in \mathcal{R}(X)$ imply that $x^*(\omega)y \in X$. This together with

$$(T_z y)^*(2t) \leq T_z^* y^*(t), \quad 0 < t < \frac{1}{2},$$

see [15, p. 67], and the boundedness of the dilation operators gives

$$\sup_{z \simeq x} \|T_z y\|_X \leq 2 \|T_z^* y^*\|_X \leq 2 (\|x^*(\omega)y\|_X + \varepsilon \|x\|_X) < \infty.$$

So, the family $\{T_z : z \simeq x\}$ is pointwise bounded and hence, by the Uniform Boundedness Principle it is uniformly bounded. The result follows. $\square$

Proposition 2.4 and general properties of rearrangements [15, p. 69] imply the following useful characterization of the symmetric kernel $\text{Sym}(\mathcal{R}, X)$ of the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$.

Corollary 2.5. *Let X be a r.i. space. Then, $x \in \text{Sym}(\mathcal{R}, X)$ if and only if $x^*(\sum a_n r_n)^* \in X$ for every $\sum a_n r_n \in \mathcal{R}(X)$, and in this case*

$$\|f\|_{\text{Sym}(\mathcal{R}, X)} \asymp \sup_{\|\sum a_n r_n\|_X \leq 1} \left\| f^* \left(\sum a_n r_n \right)^* \right\|_X.$$

The Rademacher multiplier space and its symmetric kernel inherit certain properties from the space X .

Proposition 2.6. *Let X be a r.i. space with the Fatou property. Then the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$ and its symmetric kernel $\text{Sym}(\mathcal{R}, X)$ have the Fatou property.*

Proof. Let (x_n) be a bounded sequence in $\Lambda(\mathcal{R}, X)$ of nonnegative functions increasing to x . For any $z \in \mathcal{R}(X)$ and $y \in X'$, by the Monotone Convergence Theorem, $\int_0^1 |x_n(t)z(t)y(t)| dt$ increases to $\int_0^1 |x(t)z(t)y(t)| dt$ so,

$$\int_0^1 |x(t)y(t)z(t)| dt \leq \sup_n \|x_n z\|_X \|y\|_{X'} \leq \sup_n \|x_n\|_{\Lambda(\mathcal{R}, X)} \|z\|_X \|y\|_{X'}.$$

Hence, $xz \in X''$. Since X has the Fatou property, then $X = X''$, so $xz \in X$. Thus, $x \in \Lambda(\mathcal{R}, X)$.

The result for $\text{Sym}(\mathcal{R}, X)$ follows similarly using Corollary 2.5. $\square$

The following definition will be useful in identifying the symmetric kernel of the Rademacher multiplier space for many classical r.i. spaces.

Definition 2.7. Let X be a r.i. space with $L_N \subset X$. We denote by $X_{\log^{1/2}}$ the space

$$X_{\log^{1/2}} = \left\{ x : \|x\|_{\log^{1/2}} = \|x^*(t) \log^{1/2}(e/t)\|_X < \infty \right\}.$$

Note that condition $L_N \subset X$ implies $\log^{1/2}(e/t) \in X$.

Recall that given Banach spaces X_0, X_1 continuously embedded in a common Hausdorff topological vector space, a Banach space X is an interpolation space with respect to the couple (X_0, X_1) if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and for every linear operator T with $T: X_i \rightarrow X_i$ continuously, $i = 0, 1$, we have $T: X \rightarrow X$ continuously. We denote by $\mathcal{I}(X_0, X_1)$ the set of all interpolation spaces with respect to (X_0, X_1) . The K -functional of $x \in X_0 + X_1$ is defined, for $t > 0$ as

$$K(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i\}.$$

The couple (X_0, X_1) is K -monotone if every $X \in \mathcal{I}(X_0, X_1)$ satisfies that $x \in X$ and $K(\cdot, y; X_0, X_1) \leq K(\cdot, x; X_0, X_1)$ imply $y \in X$ and $\|y\|_X \leq C\|x\|_X$. In this case, there exists a Banach lattice E of two-sided sequences satisfying $(\min(1, 2^k))_{-\infty}^{\infty} \in E$, such that

$$\|x\|_X \asymp \|(K(2^k, x; X_0, X_1)_k)\|_E;$$

see [8, Theorem 4.4.5].

The functional $\|\cdot\|_{\log^{1/2}}$ is only a quasinorm. However, if $X \in \mathcal{I}(L^1, L^\infty)$ then it is equivalent to a norm and $X_{\log^{1/2}}$ becomes a r.i. space. Indeed, (L^1, L^∞) is a K -monotone couple and $K(t, x; L^1, L^\infty) = \int_0^t x^*(s) ds$; see [8, Theorem 2.6.9]. This, together with $\log^{1/2}(e/t)$ being decreasing and Hardy's lemma (see [6, II.3.6]) gives the subadditivity of $\|x^* \log^{1/2}(e/t)\|_X$. In particular, $X_{\log^{1/2}}$ is r.i. if X has the Fatou property or is separable. Note that if X has the Fatou property then, as in Proposition 2.6, $X_{\log^{1/2}}$ also has the Fatou property.

Spaces similar to $X_{\log^{1/2}}$ have already appeared in the literature: in the study of operators of weak type $(1,1)$, Dmitriev and Semenov given a r.i. space X considered the space of functions $x \in X$ such that $\|x^*(t) \log(e/t)\|_X < \infty$ [12].

Theorem 2.8. *Let X be a r.i. space with $L_N \subset X$. Then*

$$X_{\log^{1/2}} \subset \text{Sym}(\mathcal{R}, X) \subset (X'')_{\log^{1/2}}.$$

Proof. Since $L_N \subset X$ we have $\mathcal{R}(X) \approx \ell^2$. Let x be such that $x^* \log^{1/2}(e/t) \in X$. For $(a_n) \in \ell^2$, from (4) we have, for every $0 < t \leq 1$,

$$x^*(t) \left(\sum a_n r_n \right)^* (t) \leq K \| (a_n) \|_2 x^*(t) \log^{1/2}(e/t).$$

It follows that $x^* \left(\sum a_n r_n \right)^* \in X$. Hence, from Corollary 2.5 we have $x \in \text{Sym}(\mathcal{R}, X)$ and

$$\|x\|_{\text{Sym}(\mathcal{R}, X)} \leq K \|x\|_{X_{\log^{1/2}}}.$$

Suppose now that $x \in \text{Sym}(\mathcal{R}, X)$. Let $x_n = n^{-1/2} \sum_1^n r_i$. From (1), $\|x_n\|_X \leq C_2$. From the Central Limit Theorem we have

$$\log^{1/2} \left(\frac{1}{\sqrt{2\pi t}} \right) \leq \lim_n x_n^*(t), \quad 0 < t < \frac{1}{\sqrt{2\pi}};$$

see [16, p. 136]. Thus, for a certain constant $C > 0$

$$x^*(t) \log^{1/2}(e/t) \leq C \lim_n x^*(t) x_n^*(t). \quad (8)$$

Since $x \in \text{Sym}(\mathcal{R}, X)$, from Corollary 2.5 we have $x^* x_n^* \in X$ and $\|x^* x_n^*\|_X \leq C_2 \|x\|_{\text{Sym}(\mathcal{R}, X)}$. Thus, (8) implies that $x^*(t) \log^{1/2}(e/t) \in X''$ and

$$\|x^*(t) \log^{1/2}(e/t)\|_{X''} \leq C \liminf \|x^* x_n^*\|_X \leq C C_2 \|x\|_{\text{Sym}(\mathcal{R}, X)}. \quad \square$$

Remark 2.9. Observe that the imbedding $\text{Sym}(\mathcal{R}, X) \subset (X'')_{\log^{1/2}}$ still holds if $(L_N)_0 \subset X$, that is, if $L_N \subset X''$.

Remark 2.10. From Theorem 2.8, since $\text{Sym}(\mathcal{R}, L^1) = (L^1)_{\log^{1/2}} = L \log^{1/2} L$, it follows that for every r.i. space X :

$$L^\infty \subset \text{Sym}(\mathcal{R}, X) \subset L \log^{1/2} L.$$

If X has the Fatou property then $X = X''$. Hence, we have the following consequence of Theorem 2.8.

Corollary 2.11. Let X be a r.i. space with $L_N \subset X$ and the Fatou property. Then

$$\text{Sym}(\mathcal{R}, X) = \left\{ x : \|x\|_{\log^{1/2}} = \|x^* \log^{1/2}(e/t)\|_X < \infty \right\}.$$

We apply Corollary 2.11 to identify the symmetric kernel of the Rademacher multiplier space for relevant classes of spaces X .

Example 2.12. For $X = L^p$ with $1 \leq p < \infty$, we have

$$\|x\|_{\text{Sym}(\mathcal{R}, X)} \asymp \left(\int_0^1 (x^*(t) \log^{1/2}(e/t))^p dt \right)^{1/p}.$$

Hence, $\text{Sym}(\mathcal{R}, L^p)$ is the Zygmund space $L^p(\log L)^{1/2}$; see [6, IV.6.11].

Example 2.13. The spaces in Example 2.12 are included in the case when X is a Lorentz–Zygmund space $L^{p,q}(\log L)^\alpha$; see [5]. We consider either $1 < p < \infty$, $1 \leq q < \infty$ and $\alpha \in \mathbb{R}$, or $p = q = 1$ and $\alpha \geq 0$. In this case,

$$\|x\|_{\text{Sym}(\mathcal{R}, X)} \asymp \left(\int_0^1 (t^{1/p} \log^{1/2+\alpha}(e/t) x^*(t))^q \frac{dt}{t} \right)^{1/q}.$$

Hence, $\text{Sym}(\mathcal{R}, L^{p,q}(\log L)^\alpha)$ is the Lorentz–Zygmund space $L^{p,q}(\log L)^{1/2+\alpha}$.

Example 2.14. For $X = L^{p,\infty}(\log L)^\alpha$ with $1 < p < \infty$ and $\alpha \in \mathbb{R}$ we have

$$\|x\|_{\text{Sym}(\mathcal{R}, X)} \asymp \sup_{0 < t \leq 1} t^{1/p} \log^{1/2+\alpha}(e/t) x^*(t).$$

Hence, $\text{Sym}(\mathcal{R}, L^{p,\infty}(\log L)^\alpha)$ is the Lorentz–Zygmund space $L^{p,\infty}(\log L)^{1/2+\alpha}$. In particular, for $\alpha = 0$ we have the weak- L^p spaces for which $\text{Sym}(\mathcal{R}, L^{p,\infty}) = L^{p,\infty}(\log L)^{1/2}$.

Example 2.15. For $X = \Lambda^p(\varphi)$, where $1 \leq p < \infty$ and φ is an increasing concave function, the condition $L_N \subset \Lambda^p(\varphi)$ is precisely $\int_0^1 \log^{p/2}(e/t) d\varphi(t) < \infty$. Then

$$\|x\|_{\text{Sym}(\mathcal{R}, X)} \asymp \left(\int_0^1 (x^*(t))^p \log^{p/2}(e/t) d\varphi(t) \right)^{1/p}.$$

The function $\varphi'(t) \log^{p/2}(e/t)$ is nonnegative, integrable and decreasing. Hence, $\text{Sym}(\mathcal{R}, \Lambda^p(\varphi))$ is the Lorentz space $\Lambda^p(\psi)$, where $\psi(s) = \int_0^s \varphi'(t) \log^{p/2}(e/t) dt$. In particular, for $p = 1$ we have $\text{Sym}(\mathcal{R}, \Lambda(\varphi)) = \Lambda(\psi)$ for $\psi'(t) = \varphi'(t) \log^{1/2}(e/t)$.

Example 2.13 shows that there exists X such that $\text{Sym}(\mathcal{R}, X) = L^p$, indeed, the Zygmund space $X = L^p(\log L)^{-1/2}$. The general question in this regard is: What r.i. spaces Z arise as symmetric kernels of Rademacher multiplier spaces? Remark 2.10 shows that necessarily $Z \subset L \log^{1/2} L$.

The next lemma (in the spirit of Krein et al. [15, (2.40), p. 75]) is needed later.

Lemma 2.16. Let $\alpha > 0$ and $\gamma \in \mathbb{R}$. Then, for constants depending on α and γ but independent of the function x on $[0, 1]$, we have

$$\int_0^1 (x^*(\cdot) \log^\gamma(e/\cdot))^*(t) \log^\alpha(e/t) dt \asymp \int_0^1 x^*(t) \log^{\gamma+\alpha}(e/t) dt.$$

Proof. We need to consider only the case when $\gamma < 0$. Since $\alpha > 0$, we have $\log^\alpha(e/t)$ decreasing, so

$$\int_0^1 x^*(t) \log^{\gamma+\alpha}(e/t) dt \leq \int_0^1 (x^*(\cdot) \log^\gamma(e/\cdot))^*(t) \log^\alpha(e/t) dt.$$

For the other inequality, following the proof of Krein et al. [15, (2.40), p. 75] it is enough to show that the result holds for $x = \chi_{(0,a]}$, $0 < a < 1$. In this case

$$\begin{aligned} \int_0^a (\log^\gamma(e/\cdot))^*(t) \log^\alpha(e/t) dt &= \int_0^a \log^\gamma\left(\frac{e}{a-t}\right) \log^\alpha(e/t) dt \\ &\leq \log^\gamma(e/a) \int_0^a \log^\alpha(e/t) dt \asymp \int_0^a \log^{\gamma+\alpha}(e/t) dt, \end{aligned}$$

taking into account that $\int_0^t \log^\beta(e/s) ds \asymp t \log^\beta(e/t)$, for $\beta \in \mathbb{R}$. $\square$

Theorem 2.17. Let Z be a r.i. space with the Fatou property and $Z \in \mathcal{I}(L \log^{1/2} L, L^\infty)$. Then, there exists a r.i. space X such that $\text{Sym}(\mathcal{R}, X) = Z$.

Proof. Given a r.i. Z , we define the space

$$Z_{\log^{-1/2}} = \left\{ x : \|x\|_{\log^{-1/2}} = \|x^*(t) \log^{-1/2}(e/t)\|_Z < \infty \right\}.$$

The functional $\|\cdot\|_{\log^{-1/2}}$ is equivalent to a r.i. norm. To see this we first compute the K -functional for the couple $(L \log^{1/2} L, L^\infty)$. The spaces L^∞ and $L \log^{1/2} L$ are Lorentz spaces. Indeed, $L^\infty = \Lambda(\varphi_0)$, for $\varphi_0(t) = 1$ and $L \log^{1/2} L = \Lambda(\varphi_1)$, for $\varphi_1(t) = t \log^{1/2}(e/t)$. Since φ_0/φ_1 is decreasing, we have $\Lambda(\varphi_0) + \Lambda(\varphi_1) = \Lambda(\min\{\varphi_0, \varphi_1\})$; see [15, Theorem II.5.9]. Thus, for $t \in (0, 1]$,

$$\begin{aligned} K(t, x; L \log^{1/2} L, L^\infty) &= \|x\|_{L \log^{1/2} L} + t \|x\|_\infty \\ &= \int_0^1 x^*(s) d(\min\{\varphi_1(s), t\varphi_0(s)\}) \\ &= \int_0^{s(t)} x^*(s) d(s \log^{1/2}(e/s)), \end{aligned}$$

where $s(t)$ satisfies the equation: $s \log^{1/2}(e/s) = t$; thus, $s(t) \asymp t \log^{-1/2}(e/t)$. Since $d(s \log^{1/2}(e/s)) \asymp \log^{1/2}(e/s) ds$, we have

$$K(t, x; L \log^{1/2} L, L^\infty) \asymp \int_0^{t \log^{-1/2}(e/t)} x^*(s) \log^{1/2}(e/s) ds \quad (0 < t \leq 1). \quad (9)$$

Since $Z \in \mathcal{I}(L \log^{1/2} L, L^\infty)$ and this couple is K -monotone [11] there exists a Banach lattice E of two-sided sequences satisfying $(\min(1, 2^k))_{k=-\infty}^\infty \in E$ such that

$$\|x\|_Z \asymp \|(K(2^k, x; L \log^{1/2} L, L^\infty))_{k=-\infty}^\infty\|_E.$$

Let $\{e_k\}_{k=-\infty}^\infty$ be the standard basis in the sequence space E . Note that, for $t \geq 1$, we have $K(t, x; L \log^{1/2} L, L^\infty) = \|x\|_{L \log^{1/2} L}$. Since the K -functional is concave, we have $K(1, x; L \log^{1/2} L, L^\infty) \leq 2 K(1/2, x; L \log^{1/2} L, L^\infty)$; see [6, V.1.2]. Then,

$$\begin{aligned} \left\| \sum_{k=0}^\infty K(2^k, x; L \log^{1/2} L, L^\infty) e_k \right\|_E &= K(1, x; L \log^{1/2} L, L^\infty) \left\| \sum_{k=0}^\infty e_k \right\|_E \\ &\leq C K(1/2, x; L \log^{1/2} L, L^\infty) \|e_{-1}\|_E \\ &\leq C \left\| \sum_{k=-\infty}^{-1} K(2^k, x; L \log^{1/2} L, L^\infty) e_k \right\|_E. \end{aligned}$$

Consequently,

$$\|x\|_Z \asymp \left\| \sum_{k=-\infty}^{-1} K(2^k, x; L \log^{1/2} L, L^\infty) e_k \right\|_E. \quad (10)$$

From (9), (10) and Lemma 2.16 it follows that

$$\begin{aligned} \|x\|_{\log^{-1/2}} &\asymp \left\| \sum_{k=-\infty}^{-1} \int_0^{2^k/\sqrt{-k}} (x^*(\cdot) \log^{-1/2}(e/\cdot))^*(t) \log^{1/2}(e/t) dt e_k \right\|_E \\ &\asymp \left\| \sum_{k=-\infty}^{-1} \int_0^{2^k/\sqrt{-k}} x^*(t) dt e_k \right\|_E. \end{aligned}$$

This expression shows that $\|\cdot\|_{\log^{-1/2}}$ is equivalent to a r.i. norm. Moreover, it implies that $Z_{\log^{-1/2}}$ is an interpolation space for the couple (L^1, L^∞) .

It is not hard to see, as in Proposition 2.6, that $X = Z_{\log^{-1/2}}$ has the Fatou property, thus from Corollary 2.11 we have

$$\begin{aligned} \|x\|_{\text{Sym}(\mathcal{R}, X)} &\asymp \|x^* \log^{1/2}(e/t)\|_X \\ &= \|(x^*(\cdot) \log^{1/2}(e/\cdot))^*(t) \log^{-1/2}(e/t)\|_Z = \|x\|_Z, \end{aligned}$$

since the function $x^*(t) \log^{1/2}(e/t)$ is decreasing. $\square$

3. The case $\text{Sym}(\mathcal{R}, X)$ equal to L^∞

For $X = L_N$ Corollary 2.11 gives $\text{Sym}(\mathcal{R}, L_N) = L^\infty$. In this section we study the general situation when $\text{Sym}(\mathcal{R}, X)$ reduces to L^∞ . Recall that X_0 is the closure of L^∞ in X and $(L_N)_0$ is the closure of L^∞ in L_N .

Lemma 3.1. *Let X be a r.i. space such that $(L_N)_0 \not\subset X_0$. Then for every $\varepsilon > 0$ there exist a Rademacher sum $x_n = \sum_1^n a_i r_i$ and a set E with $m(E) < \varepsilon$ for which $\|x_n \chi_E\|_X \geq (1 - \varepsilon) \|x_n\|_X$.*

Proof. If $(L_N)_0 \not\subset X_0$, the norms of X_0 and L^1 are not equivalent on $\mathcal{R}(X_0)$ [20]. Then it is easily checked that, for every $\varepsilon > 0$, we have

$$\mathcal{R}(X_0) \not\subset \left\{ x \in X_0 : m(\{t \in [0, 1] : |x(t)| > \varepsilon \|x\|_X\}) \geq \varepsilon \right\}.$$

These last sets were defined by Kadec and Pelczynski in [13] in the case $X = L^p$, see also [18,19] in the case of general r.i. spaces. This implies that for every $\varepsilon > 0$ there exists a Rademacher series $x = \sum a_i r_i \in X_0$ such that

$$m(\{t \in [0, 1] : |x(t)| > \varepsilon \|x\|_X\}) < \varepsilon.$$

Let $x_n = \sum_1^n a_i r_i$. The Rademacher functions form a 1-unconditional basic sequence in X , so $\|x_n\|_X \leq \|x_{n+1}\|_X$; see [7, Proposition 14]. Since X_0 is separable it has absolutely continuous norm. This, together with $x_n \rightarrow x$ a.e., implies that $\|x\|_X \leq \liminf \|x_n\|_X$. Hence, $\|x_n\|_X \rightarrow \|x\|_X$. The a.e. convergence of the series implies that for a certain $n \in \mathbb{N}$ we have:

$$m(\{t \in [0, 1] : |x_n(t)| > \varepsilon \|x_n\|_X\}) < \varepsilon.$$

Set $E = \{t \in [0, 1] : |x_n(t)| > \varepsilon \|x_n\|_X\}$. Observe that $m([0, 1] \setminus E) > 1 - \varepsilon$ and on $[0, 1] \setminus E$ we have $|x_n| \leq \varepsilon \|x_n\|_X$. Then

$$\|x_n\|_X \leq \|x_n \chi_E\|_X + \|x_n \chi_{[0,1] \setminus E}\|_X \leq \|x_n \chi_E\|_X + \varepsilon \|x_n\|_X,$$

and the proof is completed. $\square$

We now characterize when $\text{Sym}(\mathcal{R}, X)$ coincides with L^∞ .

Theorem 3.2. *Let X be a r.i. space. Then, $\text{Sym}(\mathcal{R}, X) = L^\infty$ if and only if $\log^{1/2}(e/t) \notin X_0$.*

Proof. Suppose $\log^{1/2}(e/t) \in X_0$. From (2) it follows that $L_N \subset X$, hence (1) holds. Then, from Corollary 2.5 it follows that, for every $0 < a \leq 1$,

$$\|\chi_{[0,a]}\|_{\text{Sym}(\mathcal{R},X)} \asymp \sup_{\|(a_n)\|_2 \leq 1} \left\| \chi_{[0,a]} \left(\sum a_n r_n \right)^* \right\|_X. \quad (11)$$

If $\|(a_n)\|_2 \leq 1$, from (2)

$$\left(\sum a_n r_n \right)^*(t) \leq C \log^{1/2}(e/t), \quad 0 < t \leq 1,$$

which implies

$$\chi_{[0,a]}(t) \left(\sum a_n r_n \right)^*(t) \leq C \chi_{[0,a]}(t) \log^{1/2}(e/t), \quad 0 < t, a \leq 1.$$

From (11), it follows that

$$\|\chi_{[0,a]}\|_{\text{Sym}(\mathcal{R},X)} \leq C \|\chi_{[0,a]}\|_{\log^{1/2}(e/t)} \|X\|. \quad (12)$$

Since X_0 has absolutely continuous norm and $\log^{1/2}(e/t) \in X_0$ we have $\|\chi_{[0,a]}\|_{\log^{1/2}(e/t)} \|X\| \rightarrow 0$ as $a \rightarrow 0$. Hence, (12) implies $\|\chi_{[0,a]}\|_{\text{Sym}(\mathcal{R},X)} \rightarrow 0$. Therefore, $\text{Sym}(\mathcal{R}, X) \neq L^\infty$.

Suppose now $\log^{1/2}(e/t) \notin X_0$. If $(L_N)_0 \not\subset X_0$, then from Lemma 3.1 and Corollary 2.5 we get $\|\chi_E\|_{\text{Sym}(\mathcal{R},X)} \asymp 1$ for every measurable set $E \subset [0, 1]$. Hence, $\text{Sym}(\mathcal{R}, X) = L^\infty$.

Consider now the case $(L_N)_0 \subset X_0$. Then, $L_N \subset X''$. From Theorem 2.8 and Remark 2.9 it follows that $\text{Sym}(\mathcal{R}, X) \subset (X'')_{\log^{1/2}}$ and

$$\|x\|_{\text{Sym}(\mathcal{R},X)} \geq C \|x^*\|_{\log^{1/2}(e/t)} \|X''\|. \quad (13)$$

Since $(X'')_0 = X_0$, we have $\log^{1/2}(e/t) \notin (X'')_0$. Thus, there exists $\delta > 0$ such that for every $\varepsilon > 0$ we have

$$\|\log^{1/2}(e/t)\chi_{[0,\varepsilon]}\|_{X''} \geq \delta. \quad (14)$$

From (13) and (14) we have that, for every $\varepsilon > 0$

$$\|x\|_{\text{Sym}(\mathcal{R},X)} \geq C x^*(\varepsilon) \|\log^{1/2}(e/t)\chi_{[0,\varepsilon]}\|_{X''} \geq C \delta x^*(\varepsilon).$$

Thus, $\|x\|_{\text{Sym}(\mathcal{R},X)} \geq C \delta \|x\|_\infty$. So, $\text{Sym}(\mathcal{R}, X) = L^\infty$. $\square$

Corollary 3.3. *If X is a separable r.i. space, then $\text{Sym}(\mathcal{R}, X) = L^\infty$ if and only if $L_N \not\subset X$.*

Corollary 3.4. *If X and Y are r.i. spaces with $X \subset Y$, then $\text{Sym}(\mathcal{R}, X) \subset \text{Sym}(\mathcal{R}, Y)$.*

Proof. If $L_N \not\subset X$, then Theorem 3.2 implies $\text{Sym}(\mathcal{R}, X) = L^\infty$ so, $\text{Sym}(\mathcal{R}, X) \subset \text{Sym}(\mathcal{R}, Y)$. If $L_N \subset X \subset Y$, then $\mathcal{R}(X) = \mathcal{R}(Y) \approx \ell^2$. In this case, the definition of the Rademacher multiplier space and $X \subset Y$ imply $\Lambda(\mathcal{R}, X) \subset \Lambda(\mathcal{R}, Y)$. Thus, again $\text{Sym}(\mathcal{R}, X) \subset \text{Sym}(\mathcal{R}, Y)$. $\square$

Example 3.5. Theorem 3.2 allows to answer in the negative the question if the space L_N is the largest r.i. space with $\text{Sym}(\mathcal{R}, X) = L^\infty$. Let the function ψ be piecewise linear, supported on the points $(t_k, \log^{-1/2}(e/t_k))$, where $t_k \in [0, 1]$, $t_0 = 1 > t_1 > t_2 \downarrow 0$. Then ψ is a quasi-concave function satisfying: (1) $0 \leq \psi(t) \leq \log^{-1/2}(e/t)$; (2) $\psi(t_k) = \log^{-1/2}(e/t_k)$, $k \geq 0$. Moreover, we can choose $\{t_k\}$ so that (3) $\inf_{0 < t \leq 1} \psi(t) / \log^{-1/2}(e/t) = 0$. Consider the Marcinkiewicz space $M(t/\psi)$. From (1) and (3) we have $L_N \subsetneq M(t/\psi)$. Condition (2) implies that $\lim_{t \rightarrow 0} \psi(t) \log^{1/2}(e/t) \neq 0$. Since $tx^*(t) \leq \int_0^t x^*(s) ds$, it follows that $\log^{1/2}(e/t) \notin (M(t/\psi))_0$; see [15, Theorem II.5.3 and Lemma II.5.4]. Thus, $\text{Sym}(\mathcal{R}, M(t/\psi)) = L^\infty$.

4. The case $\Lambda(\mathcal{R}, X)$ a r.i. space different from L^∞

In this section, we exhibit a family of spaces X for which $\Lambda(\mathcal{R}, X)$ is a r.i. space different from L^∞ . For this we first study the space $\text{Sym}(\mathcal{R}, X)$ for X a Marcinkiewicz space.

Theorem 4.1. *Let φ be a quasi-concave function with $L_N \subset (M(\varphi))_0$. Consider the following conditions:*

(i) *There exists a constant $C > 0$ such that, for every $0 \leq t \leq 1$*

$$\int_0^t \frac{\varphi(s)}{s} \log^{-1}(e/s) ds \leq C \varphi(t);$$

(ii) *$\text{Sym}(\mathcal{R}, M(\varphi))$ is a Marcinkiewicz space;*

Then, (i) implies (ii). If $\log^{-1/2}(e/t)\varphi(t)/t$ is decreasing then, (ii) implies (i).

Proof. Suppose (i) holds. Since $L_N \subset M(\varphi)$, from Corollary 2.11 we have $\text{Sym}(\mathcal{R}, M(\varphi)) = (M(\varphi))_{\log^{1/2}}$. Since $M(\varphi) \in \mathcal{I}(L^1, L^\infty)$, it follows that $\text{Sym}(\mathcal{R}, M(\varphi))$ is a r.i. space. Its fundamental function is

$$\varphi_{\text{Sym}(\mathcal{R}, M(\varphi))}(t) \asymp \sup_{0 < s \leq t} \frac{s \log^{1/2}(e/s)}{\varphi(s)}. \quad (15)$$

Denote by $\Phi(t)$ the function in the right-hand side of (15). It is increasing and $\Phi(t)/t$ is decreasing. Since $L_N \subset (M(\varphi))_0$ we have $\lim_{t \rightarrow 0} t \log^{1/2}(e/t)/\varphi(t) = 0$, so $\lim_{t \rightarrow 0} \Phi(t) = 0$. Thus, Φ is quasi-concave and so, $(M(\varphi))_{\log^{1/2}} \subset M(t/\Phi)$.

For the opposite inclusion, let $x \in M(t/\Phi)$ have norm one. Then, $\int_0^t x^*(s) ds \leq t/\Phi(t)$ for $0 < t \leq 1$. Thus, using $t \log^{1/2}(e/t)/\varphi(t) \leq \Phi(t)$, we have

$$\begin{aligned} \int_0^t x^*(s) \log^{1/2}(e/s) ds &= \int_0^t \log^{1/2}(e/s) d\left(\int_0^s x^*(u) du\right) \\ &\leq \int_0^t \log^{1/2}(e/s) d(s/\Phi(s)) \\ &= \log^{1/2}(e/s) \frac{s}{\Phi(s)} \Big|_0^t + \frac{1}{2} \int_0^t \frac{\log^{-1/2}(e/s)}{\Phi(s)} ds \\ &\leq \varphi(t) + \frac{1}{2} \int_0^t \frac{\varphi(s)}{s} \log^{-1}(e/s) ds \\ &\leq \left(1 + \frac{C}{2}\right) \varphi(t), \end{aligned}$$

since $t \log^{1/2}(e/t)/\Phi(t) \leq \varphi(t)$. Hence, $x \in (M(\varphi))_{\log^{1/2}}$ and $\|x\|_{(M(\varphi))_{\log^{1/2}}} \leq 1 + C/2$.

Suppose now that (ii) holds and $\log^{-1/2}(e/t)\varphi(t)/t$ is decreasing. From (15), $\Phi(t) = t \log^{1/2}(e/t)/\varphi(t)$. Then, $\text{Sym}(\mathcal{R}, M(\varphi)) = M(t/\Phi) = M(\varphi(t) \log^{-1/2}(e/t))$. Let $\psi(t)$ be the least concave majorant of the function $\varphi(t) \log^{-1/2}(e/t)$. Using integration by parts, we have

$$\begin{aligned} \int_0^t \psi'(s) \log^{1/2}(e/s) ds &\geq \frac{1}{2} \int_0^t \frac{\psi(s)}{s} \log^{-1/2}(e/s) ds \\ &\geq \frac{1}{2} \int_0^t \frac{\varphi(s)}{s} \log^{-1}(e/s) ds. \end{aligned}$$

Since $\varphi(t) \log^{-1/2}(e/t) \leq \psi(t) \leq 2\varphi(t) \log^{-1/2}(e/t)$, [15, p. 49], then $M(\psi) = M(\varphi(t) \log^{-1/2}(e/t))$. On the other hand, $\text{Sym}(\mathcal{R}, M(\varphi)) = (M(\varphi))_{\log^{1/2}}$. Thus, $\psi' \in M(\psi)$ implies $\psi' \in (M(\varphi))_{\log^{1/2}}$. So, $\int_0^t \psi'(s) \log^{1/2}(e/s) ds \leq C \varphi(t)$. Hence (i) follows. $\square$

Remark 4.2. Condition $\varphi' \in L(\log \log L)$ is necessary for (i) in Theorem 4.1 to hold. This follows from

$$\begin{aligned} \int_0^1 \varphi'(s) \log \log(e/s) ds &= \int_0^1 \frac{1}{s} \int_0^s \varphi'(t) dt \log^{-1}(e/s) ds \\ &= \int_0^1 \frac{\varphi(s)}{s} \log^{-1}(e/s) ds \leq C \varphi(1). \end{aligned}$$

Remark 4.3. The proof of Theorem 4.1 shows $\text{Sym}(\mathcal{R}, M(\varphi)) \subset M(\overline{\varphi})$ where $\overline{\varphi}(t) = t/\Phi(t)$ and $\Phi(t)$ is given by the right-hand side of (15). It is easily verified that $\overline{\varphi}$ is the largest concave minorant of $\varphi(t) \log^{1/2}(e/t)$.

Remark 4.4. The space $\text{Sym}(\mathcal{R}, M(\varphi))$ need not be a Marcinkiewicz space: for $\varphi(t) = 1$, $M(\varphi) = L^1$ and $\text{Sym}(\mathcal{R}, L^1) = L \log^{1/2} L = \Lambda(t \log^{1/2}(e/t))$. In more generality, this occurs for every φ with $\log^{-1/2}(e/t)\varphi(t)/t$ decreasing and $\varphi' \notin L(\log \log L)$. The function $\varphi(t) = \log^{-1} \log(e/t)$ is an example of this situation.

Corollary 4.5. Let φ be a quasi-concave function with $\gamma_\varphi > 0$. Then $\text{Sym}(\mathcal{R}, M(\varphi))$ is a Marcinkiewicz space.

Proof. It follows from Theorem 4.1 since [15, Corollary 3, p. 57] implies that

$$\int_0^t \frac{\varphi(s)}{s} \log^{-1}(e/s) ds \leq \int_0^t \frac{\varphi(s)}{s} ds \leq C \varphi(t). \quad \square$$

Remark 4.6. The condition in Corollary 4.5 is not necessary. Consider $\varphi(t) = \log^{-1/p}(e/t)$, for $p \in (0, 2)$. Direct computation shows that $\gamma_\varphi = 0$ and $L_N \subset (M(\varphi))_0$. Since

$$\int_0^t \frac{\log^{-1/p}(e/s)}{s} \log^{-1}(e/s) ds \asymp \log^{-1/p}(e/t),$$

it follows from Theorem 4.1 that $\text{Sym}(\mathcal{R}, M(\varphi))$ is a Marcinkiewicz space. In particular, from (15) we have $\text{Sym}(\mathcal{R}, M(\varphi)) = M(\phi)$, for $\phi(t) = \log^{-(1/2)-(1/p)}(e/t)$.

Let ϕ be an Orlicz function. The exponential Orlicz space $\text{Exp } L^\phi$ is the Orlicz space associated to the Orlicz function $e^{\phi(t)} - 1$. The next result shows that, under certain conditions, for these spaces the Rademacher multiplier space $\Lambda(\mathcal{R}, X)$ is a r.i. space different from L^∞ . Note that the spaces $\text{Exp } L^\phi$ are “close” to L^∞ . For details on Orlicz spaces see [14].

Theorem 4.7. Let ϕ be an Orlicz function such that $\phi^{-1}(t)/t^{1/2}$ is increasing. Then $\Lambda(\mathcal{R}, \text{Exp } L^\phi) = \text{Exp } L^\Psi$, where Ψ is the Orlicz function satisfying $\Psi^{-1}(t) = \phi^{-1}(t)/t^{1/2}$.

Proof. The function Ψ^{-1} is quasi-concave: it is increasing by assumption and $\Psi^{-1}(t)/t = (\phi^{-1}(t)/t) \cdot t^{-1/2}$ is decreasing, since $\phi^{-1}(t)$ is quasi-concave. Thus, Ψ is (equivalent to) an Orlicz function. Since $\phi(t) \leq Ct^2$, for $t \geq 1$, we have $L_N \subset \text{Exp } L^\phi$ so, $\mathcal{R}(\text{Exp } L^\phi) \approx \ell^2$.

We first show that $\Lambda(\mathcal{R}, \text{Exp } L^\phi) \subset \text{Exp } L^\Psi$. From [9, Lemma 1], given $f \in L^p$, $1 \leq p < \infty$, there exists a Rademacher series $h_p = \sum b_k r_k$ with $\|(b_k)\|_{\ell^2} = 1$, such that $\|f\|_p \leq 3 p^{-1/2} \|f h_p\|_p$. Using the extrapolation result [1, Theorem 1]

(see [3, Proposition 1] for a detailed proof), since $\phi^{-1}(t) = \Psi^{-1}(t)/t^{-1/2}$, we have

$$\begin{aligned}
 \|f\|_{\text{Exp } L^\Psi} &= \sup_{1 \leq p < \infty} \frac{1}{\Psi^{-1}(p)} \|f\|_p \\
 &\leq 3 \sup_{1 \leq p < \infty} \frac{p^{-1/2}}{\Psi^{-1}(p)} \|fh_p\|_p \\
 &\leq 3 \sup_{1 \leq p < \infty} \frac{p^{-1/2}}{\Psi^{-1}(p)} \sup_{\|(a_k)\|_{\ell^2} \leq 1} \left\| f \sum a_k r_k \right\|_p \\
 &= 3 \sup_{\|(a_k)\|_{\ell^2} \leq 1} \sup_{1 \leq p < \infty} \frac{1}{\phi^{-1}(p)} \left\| f \sum a_k r_k \right\|_p \\
 &\asymp \sup_{\|(a_k)\|_{\ell^2} \leq 1} \left\| f \sum a_k r_k \right\|_{\text{Exp } L^\phi} \\
 &\asymp \|f\|_{\Lambda(\mathcal{R}, \text{Exp } L^\phi)}.
 \end{aligned}$$

The space $\text{Exp } L^\phi$ is a Marcinkiewicz space $M(\varphi)$ for $\varphi(t) = t\phi^{-1}(\log^{1/2}(e/t))$ [17]. Direct computation shows $\gamma_\varphi > 0$. Thus, Corollary 4.5 implies that $\text{Sym}(\text{Exp } L^\phi)$ is a Marcinkiewicz space. Since $t^{1/2}/\phi^{-1}(t)$ is decreasing, the fundamental function of $\text{Sym}(\text{Exp } L^\phi)$ is

$$\begin{aligned}
 \varphi_{\text{Sym}(\text{Exp } L^\phi)}(t) &= \|\chi_{[0,t]} \log^{1/2}(e/s)\|_{\text{Exp } L^\phi} \\
 &\asymp \sup_{0 < s \leq t} \frac{\log^{1/2}(e/s)}{\phi^{-1}(\log(e/s))} = \frac{\log^{1/2}(e/t)}{\phi^{-1}(\log(e/t))}.
 \end{aligned}$$

This coincides with the fundamental function of $\text{Exp } L^\Psi$ which is $\varphi_{\text{Exp } L^\Psi}(t) = 1/\Psi^{-1}(\log^{1/2}(e/t))$. Since $\text{Exp } L^\Psi$ is also a Marcinkiewicz space, we have $\text{Sym}(\text{Exp } L^\phi) = \text{Exp } L^\Psi$. Hence, $\Lambda(\mathcal{R}, \text{Exp } L^\phi) = \text{Exp } L^\Psi$. $\square$

Remark 4.8. The spaces $X = \text{Exp } L^p$ of functions of p th exponential integrability are Marcinkiewicz spaces for $\varphi(t) = t \log^{1/p}(e/t)$; Lorentz–Zygmund spaces with parameters $p = q = \infty$ and $\alpha = -1/p$; and exponential Orlicz spaces $\text{Exp } L^\phi$ for $\phi(t) = t^p$. In the case $0 < p < 2$, from Theorem 4.7 we have $\Lambda(\mathcal{R}, \text{Exp } L^p) = \text{Exp } L^\alpha$, for $\alpha = 2p/(2 - p)$. This result coincides with the one obtained in [10, Theorem 3].

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