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The Banach-Saks p-property

S.V. Astashkin · E.M. Semenov · F.A. Sukochev

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Abstract. We obtain general theorems which enable the calculation of the Banach-Saks index set for rearrangement invariant function spaces in terms of their Boyd indices. For many important classes of rearrangement invariant spaces, the index set is non-trivial if and only if the Boyd indices of the underlying space are non-trivial, although the latter equivalence does not hold in full generality. We also study Banach-Saks index sets defined for weakly null sequences of (identically distributed) independent random variables and their connections with the classical Banach-Saks index set.

1. Introduction

The classical Banach-Saks theorem [B], Ch.12, Theorem 2 states that every weakly null sequence $x_n \in L_p$ $(1 < p < \infty)$ contains a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\|\sum_{k=1}^{m} x_{n_k}\|_{L_p} \leq Cm^{\max(\frac{1}{p},\frac{1}{2})} \tag{1.1}$$

for every $m \in \mathbb{N}$. The exponent $\max(\frac{1}{p}, \frac{1}{2})$ is exact for every $p \in (1, \infty)$. For $p \in (2, \infty)$ the estimate (1.1) also follows from [KP]. This theorem leads to the following definitions.

Let X be a Banach space and $p \geq 1$. A bounded sequence $\{x_n\} \subset X$ is called a p-BS-sequence (BS-sequence) if there exists a subsequence $\{y_n\} \subset \{x_n\}$ such

S.V. Astashkin
Department of Mathematics, Samara State University, Samara, Russia
(e-mail: astashkn@ssu.samara.ru)

E.M. Semenov*
Department of Mathematics, Voronezh State University, Universitetskaya pl.1, Voronezh, 394006, Russia (e-mail: root@func.vsu.ru)

F.A. Sukochev**
School of Informatics and Engineering, Flinders University, Bedford Park, SA 5042, Australia
(e-mail: sukochev@infoeng.flinders.edu.au)

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that

$$C = \sup_{m \in N} m^{-\frac{1}{p}} \| \sum_{k=1}^{m} y_k \|_X < \infty \quad \left(\lim_{m \to \infty} \frac{1}{m} \| \sum_{k=1}^{m} y_k \|_X = 0 \right).$$

Following [J], [Ro], [Ra] we say that X has the p-BS-property (the BS-property) and we write $X \in BS(p)$ ($X \in BS$) if each weakly null sequence contains a p-BS-sequence (BS-sequence). Clearly, every Banach space has 1-BS-property. The set

$$\Gamma(X) = \{p : \ p \geq 1, \ X \in BS(p)\}$$

is said to be the index set of X, and is of the form $[1, \alpha]$, or $[1, \alpha)$ for some $\alpha = \alpha(X)$. This number $\alpha(X)$ is called the Banach-Saks index of X. Thus, it follows from (1.1) that $\Gamma(L_p) = [1, \min(p, 2)]$ for any $p \in (1, \infty)$ and it is established in [B], Ch. 12, Theorem 3 that $\Gamma(l_p) = [1, p]$ for $1 < p < \infty$ and $\Gamma(c_0) = \Gamma(l_1) = [1, \infty)$. A number of articles study the BS- and p-BS-properties in symmetric (or rearrangement invariant) function and sequence spaces ([Ra], [KO], [Ba], [DSS], [SS]). In particular, for (normed) Lorentz function spaces $L_{p,q}$, the set $\Gamma(L_{p,q})$ is completely described in [SS] for all pairs $p, q \in [1, \infty]$ and the description of all Orlicz and Lorentz spaces possessing the BS-property is obtained in [DSS].

In this article we study the index set of separable rearrangement invariant (=r.i.) spaces. It should be noted that any r.i. Banach space with the BS-property (and moreover, with the p-BS-property) is necessarily separable. Indeed, if a r.i. space X is non-separable, then it contains the universal space l_∞ as a subspace. Since the latter space contains a further subspace without the BS-property, and since the BS-property is inherited by subspaces, we see that the space X itself does not possess the BS-property. The main focus of this article is the description of the set $\Gamma(X)$ in terms of more familiar function space indices. In particular, our main results in Section 4 show that there is a non-trivial connection with the (so-called) Boyd indices of the underlying r.i. Banach function space. In fact, for the familiar classes of Orlicz, Lorentz and (separable parts of) Marcinkiewicz spaces, the index set is non-trivial (i.e. different from $\{1\}$) if and only if the Boyd indices of the underlying space are non-trivial (Corollaries 4.8- 4.10). On the other hand, the latter equivalence fails in the class of all r.i. spaces (Theorem 4.11). We also study the question for which r.i. spaces X, the set $\Gamma(X)$ coincides with its counterparts defined for all weakly null sequences of (identically distributed) independent random variables. This question has a precise answer in terms of Boyd indices (Theorem 4.4, Corollary 4.6 and Proposition 4.12). We also characterize (Theorems 4.3, 4.5 and remark following Proposition 4.12) the connection between the set $\Gamma(X)$ and its counterpart defined for all weakly null sequences of disjointly supported elements.

2. Definitions and preliminaries

A Banach space $(X, \|\cdot\|_X)$ of real-valued Lebesgue measurable functions (with identification λ-a.e.) on the interval J, where $J = [0, 1]$, or else $J = [0, \infty)$ will be called *rearrangement invariant* if

(i). X is an ideal lattice, that is, if $y \in X$, and if x is any measurable function on J with $0 \le |x| \le |y|$ then $x \in X$ and $\|x\|_X \le \|y\|_X$;
(ii). X is rearrangement invariant in the sense that if $y \in X$, and if x is any measurable function on J with $x^* = y^*$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

Here, λ denotes Lebesgue measure and x^* denotes the non-increasing, right-continuous rearrangement of x given by

$$x^*(t) = \inf\{\, s \ge 0 : \lambda(\{|\, x \,| > s\}) \le t \,\}, \quad t > 0.$$

For basic properties of rearrangement invariant spaces, we refer to the monographs [KPS], [LT]. We note that for any rearrangement invariant (=r.i.) space $X = X[0, \infty)$ (resp., $X = X[0, 1]$)

$$L_1 \cap L_\infty[0, \infty) \subseteq X \subseteq L_1 + L_\infty[0, \infty), \quad (\text{resp. } L_\infty[0, 1] \subseteq X \subseteq L_1[0, 1])$$

with continuous embeddings [KPS] (see definitions of the sum and the intersection of Banach spaces forming a Banach couple below in this section).

Let X be a r. i. space on $[0, 1]$. We shall also work with a r.i. space $X([0, 1] \times [0, 1])$ of measurable functions on the rectangle $[0, 1] \times [0, 1]$ given by

$$X([0, 1] \times [0, 1]) := \{f \in L_1([0, 1] \times [0, 1]) : f^* \in X\},$$
$$\|f\|_{X([0,1]\times[0,1])} := \|f^*\|_X.$$

Here, the decreasing rearrangement f^* is calculated with respect to product Lebesgue measure on the square $[0, 1] \times [0, 1]$. It is well known that there exists an isometric isomorphism between r.i. spaces X and $X([0, 1] \times [0, 1])$. If $x, y \in L_1[0, 1]$, then $(x \otimes y)(t, s) = x(t)y(s)$ belongs to $L_1([0, 1] \times [0, 1])$ and $\|x \otimes y\|_{L_p} = \|x\|_{L_p} \|y\|_{L_p}$ for every $x, y \in L_p,\ 1 \le p \le \infty$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\{\mathcal{F}_n\}_{n=0}^\infty$ be an increasing sequence of σ-algebras, $\mathcal{F}_n \subset \mathcal{F}$, $n \ge 0$. A sequence of random variables $\{d_n\}_{n=0}^\infty \subseteq L_1(\Omega, \mathcal{F}, P)$ is said to be a martingale difference sequence (with respect to $\{\mathcal{F}_n\}_{n=0}^\infty$), if d_n is $\mathcal{F}_n$-measurable and the mathematical expectation $\mathbb{E}(d_{n+1}|\mathcal{F}_n)$ vanishes for every $n \ge 0$.

The Köthe dual $X^\times$ of an r.i. space X on the interval J consists of all measurable functions y for which

$$\|y\|_{X^\times} := \sup\{\int_J |x(t)y(t)|dt :\ x \in X,\ \|x\|_X \le 1\} < \infty.$$

Basic properties of Köthe duality may be found in [LT] and [KPS] (where the Köthe dual is called the *associate* space). If X^* denotes the Banach dual of X, it is known that $X^\times \subset X^*$ and $X^\times = X^*$ if and only if the norm $\|\cdot\|_X$ is order-continuous, i.e. from $\{x_n\} \subseteq X$, $x_n \downarrow_n 0$, it follows that $\|x_n\|_X \to 0$. We note that the norm $\|\cdot\|_X$ of the rearrangement invariant space X on J is order-continuous if and only if X is separable. An r.i. space X on J is said to have the *Fatou property* if whenever $\{f_n\}_{n\geq 1} \subseteq X$ and f measurable on J satisfy $f_n \to f$ a.e. on J and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$. It is well known that the rearrangement invariant space X has the Fatou property if and only if the natural embedding of X into its Köthe bidual $X^{\times\times}$ is a surjective isometry. Such spaces are called *maximal*. In this paper, we shall only consider separable and/or maximal r.i. spaces X. In this case, the natural embedding $X \hookrightarrow X^{\times\times}$ is isometric. We denote by $(X)_0$ the closure of L_∞ (if $J = [0, 1]$) or $L_1 \bigcap L_\infty$ (if $J = [0, \infty)$) in X. The space $(X)_0$ is a r.i. subspace of X. We denote $L_0[0, \infty)$ the closure of $L_1[0, \infty) \cap L_\infty[0, \infty)$ in $L_1 + L_\infty[0, \infty)$.

Recall that for $0 < \tau < \infty$, the dilation operator σ_τ is defined by setting

$$x(t) = x(t/\tau), \quad \text{if} \quad J = [0, \infty)$$

and, when $J = [0, 1]$, by setting

$$\sigma_\tau x(t) = \begin{cases} x(t/\tau), \ 0 \leq t \leq \min(1, \tau) \\ 0, \ \min(1, \tau) < t \leq 1. \end{cases}$$

Operators σ_τ are bounded in every r.i. space X. The numbers α_X and β_X given by

$$\alpha_X := \lim_{\tau\to 0} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}, \quad \beta_X := \lim_{\tau\to\infty} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}$$

belong to the closed interval $[0, 1]$ and are called the Boyd indices of X (see [KPS]). The Boyd indices of a given r.i. space X are said to be non-trivial if $0 < \alpha_X \leq \beta_X < 1$.

Let us recall some classical examples of r.i. spaces on $[0, 1]$. Denote by Ω the set of all increasing concave functions on $[0, 1]$ with $\varphi(0) = \varphi(+0) = \lim\limits_{t\to 0} \frac{t}{\varphi(t)} = 0$. Each function $\varphi \in \Omega$ generates the Lorentz space $\Lambda(\varphi)$ endowed with the norm

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t) d\varphi(t),$$

and the Marcinkiewicz space $M(\varphi)$ endowed with the norm

$$\|x\|_{M(\varphi)} = \sup_{0<\tau\leq 1} \frac{1}{\varphi(\tau)} \int_0^\tau x^*(t) dt \tag{2.1}$$

The space $M(\varphi)$ is not separable, but the space

$$\left\{ x \in M(\varphi) : \lim_{\tau \to 0} \frac{1}{\varphi(\tau)} \int_0^\tau x^*(t)dt = 0 \right\}$$

endowed with the norm (2.1) is a separable r.i. space which, in fact, coincides with the space $(M(\varphi))_0$.

Let $M(t)$ be a convex function on $[0, \infty)$ such that $M(t) > 0$ for all $t > 0$ and such that

$$0 = M(0) = \lim_{t \to 0} \frac{M(t)}{t} = \lim_{t \to \infty} \frac{t}{M(t)}$$

Denote by L_M the Orlicz space on $[0, 1]$ (see e.g. [LT], [KPS]) endowed with the norm

$$\|x\|_{L_M} = \inf\{\lambda : \lambda > 0, \int_0^1 M(|x(t)|/\lambda)dt \le 1\}.$$

Throughout this paper, we denote by $\overrightarrow{X} = (X_0, X_1)$ a (compatible) Banach couple [KPS], [LT]. The sum $X_0 + X_1$ and the intersection $X_0 \cap X_1$ are equipped with the usual norms:

$$\|x\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\},$$

$$\|x\|_{X_0 \cap X_1} = \max\{\|x\|_{X_0},\ \|x\|_{X_1}\}.$$

The K-functional $K(t, x; \overrightarrow{X})$ is defined for $x \in X_0 + X_1$ and $t > 0$ by setting

$$\begin{aligned} K(t, x; \overrightarrow{X}) &= K(t, x; X_0, X_1) \\ &= \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} :\ x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}. \end{aligned}$$

It will be sometimes convenient to write $K(t, x) = K(t, x; \overrightarrow{X})$ for brevity, and this should cause no confusion in the sequel.

Let Φ be a Banach lattice over $((0, \infty), \frac{dt}{t})$ satisfying the condition $\min(1, t) \in \Phi$. Denote by $(X_0, X_1)^K_\Phi$ the set of all elements $x \in X_0 + X_1$ such that $K(t, x, X_0, X_1) \in \Phi$ endowed with the norm

$$\|x\|_{(X_0,X_1)^K_\Phi} = \|K(t, x)\|_\Phi.$$

It is well known that $(X_0, X_1)^K_\Phi$ is an interpolation functor (see e.g. [BK] 3.3.12). The latter means, in particular, that if $\overrightarrow{X} = (X_0, X_1)$ and $\overrightarrow{Y} = (Y_0, Y_1)$ are two Banach couples, then any bounded linear operator $A : X_0 + X_1 \to Y_0 + Y_1$ which maps X_i boundedly into Y_i, $(i = 0, 1)$ also maps $X = (X_0, X_1)^K_\Phi$ boundedly into $Y = (Y_0, Y_1)^K_\Phi$ and

$$\|A\|_{\mathcal{L}(X,Y)} \leq \max_{i=0,1} \|A\|_{\mathcal{L}(X_i,Y_i)}$$

(here, the symbol $\mathcal{L}(E, F)$ denotes the set of all linear bounded operators from a Banach space E to a Banach space F). This interpolation method is called the K-method and the lattice Φ is called the parameter of the K-method.

For a given Banach couple $\overrightarrow{X} = (X_0, X_1)$, denote by $\mathrm{Int}(X_0, X_1)$ the set of all interpolation spaces with respect to $\overrightarrow{X}$. For every $X \in \mathrm{Int}(X_0, X_1)$, we have, in particular, $X_0 \cap X_1 \subseteq X \subseteq X_0 + X_1$.

A couple of Banach spaces $\overrightarrow{X} = (X_0, X_1)$ is said to be a K-couple, if there exists a constant $C > 0$, such that for any $x, y \in X_0 + X_1$ with $K(t, x) \leq K(t, y)$, $t \in (0, \infty)$, there exists a bounded operator $A : X_0 + X_1 \to X_0 + X_1$ such that $x = Ay$ and such that $A \in \mathcal{L}(X_0, X_0), \mathcal{L}(X_1, X_1)$ with $\max_{i=0,1} \|A\|_{\mathcal{L}(X_i,X_i)} \leq C$.

If X is an r.i. space on $[0, 1]$ and $\alpha_X > \frac{1}{2}$ (respectively, $\beta_X < \frac{1}{2}$) then X is an interpolation space for the pair (L_1, L_2) (respectively, (L_2, L_∞)) [AM].

We denote by $\chi_e(t)$ the characteristic function of a measurable set $e \subset [0, \infty)$ and by $supp(f)$ the support of a measurable function f.

3. Auxiliary results

For every $n \in \mathbb{N}$ we consider a sublinear operator acting from $L_1(0, \infty) + L_\infty(0, \infty)$ into $L_1(0, 1)$ given by

$$B_n x(t) = \left(\sum_{k=1}^{n} x^2(t + k - 1) \right)^{\frac{1}{2}}, \ 0 < t < 1.$$

Lemma 3.1. *B_n maps $L_2(0, \infty)$ boundedly into $L_2(0, 1)$ and $(L_1 + L_2)(0, \infty)$ into $L_1(0, 1)$; in both cases $\|B_n\| = 1$, $\forall n \in \mathbb{N}$.*

Proof. Since for every $x \in L_2(0, \infty)$

$$\|B_n x\|_{L_1(0,1)} \leq \|B_n x\|_{L_2(0,1)} = \|x\|_{L_2(0,\infty)},$$

and for every $x \in L_1(0, \infty)$

$$\|B_n x\|_{L_1(0,1)} \leq \|x\|_{L_1(0,\infty)},$$

we get

$$\|B_n x\|_{L_1(0,1)} \leq \|x\|_{(L_1+L_2)(0,\infty)}. \quad \square$$

Lemma 3.2. *The equality*

$$(X_0 + X_1, X_1)^K_\Phi = (X_0, X_1)^K_\Phi + X_1$$

holds for any Banach couple $\overrightarrow{X} = (X_0, X_1)$ and any parameter Φ of the K-method.

The proof of this lemma is based on the formula

$$K(t, x_1; X_0 + X_1, X_1) = \begin{cases} K(t, x; X_0, X_1), & t \le 1 \\ K(1, x; X_0, X_1), & t > 1, \end{cases}$$

we omit further details.

Let X be an r.i. space. Following [LT] 2.f, we denote by $\mathcal{X}$ the r.i. space on $[0, \infty)$ consisting of all functions $x \in (L_1 + L_\infty)[0, \infty)$ for which $x^*\chi_{(0,1)} \in X$ and $x^*\chi_{(1,\infty)} \in L_2$ endowed with the following quasi-norm

$$\|x\|_{\mathcal{X}} = \|x^*\chi_{(0,1)}\|_X + \|x^*\chi_{(1,\infty)}\|_{L_2}.$$

It is known that $\|\cdot\|_{\mathcal{X}}$ is equivalent to a norm. Therefore there exists a constant $C > 0$ such that

$$\|x + y\|_{\mathcal{X}} \le C(\|x\|_{\mathcal{X}} + \|y\|_{\mathcal{X}})$$

for any $x, y \in \mathcal{X}$. The following lemma shows that the correspondence $X \to \mathcal{X}$ is compatible with the operation of taking the Köthe dual.

Lemma 3.3. *If X is a separable r.i. space on* $[0, 1]$ *and Y is its Köthe dual $X^\times$, then $\mathcal{Y} = \mathcal{X}^\times$.*

The proof is routine and therefore is omitted.

For any given sequence $\{x_n\}_{n\ge1} \subset X$, we set $\overline{x_n}(t) = x_n(t-n)$ if $t \in (n, n+1)$ and $\overline{x_n}(t) = 0$ if $t \notin (n, n+1)$, $n \in \mathbb{N}$. Clearly, $\{\overline{x_n}\}_{n\ge1}$ is a sequence of pairwise disjoint elements from $\mathcal{X}$ such that $x_n^* = \overline{x_n}^*$ for all $n \in \mathbb{N}$.

Lemma 3.4. *If X is a separable r.i. space on* $[0, 1]$ *and $\{x_n\}_{n\ge1} \subseteq X$ is a weakly null sequence in X, then $\{\overline{x_n}\}_{n\ge1}$ is a weakly null sequence in $\mathcal{X}$.*

Proof. Suppose $g \in \mathcal{X}^\times$. Then we can split $g = g\chi_A + h$ where $\lambda(A) = 1$, $f = g\chi_A \in X^\times(A)$ and $h \in (L_\infty \cap L_2)[0, \infty)$. Then, there exists a sequence $\{\tilde{h}_n\}_{n\ge1} \subseteq L_\infty(0, 1)$ such that

$$\int_0^\infty \overline{x}_n h = \int_0^1 x_n \tilde{h}_n, \quad n \ge 1$$

and $\sum \|\tilde{h}_n\|_2^2 < \infty$, $\sup \|\tilde{h}_n\|_\infty < \infty$. Since $(x_n)_{n=1}^\infty$ is weakly compact in L_1, given $\epsilon > 0$ there exists a decomposition $x_n = y_n + z_n$ where $\|z_n\|_2 \le M(\epsilon)$ and $\|y_n\|_1 \le \epsilon$. It is easy to check that

$$\int_0^1 x_n \tilde{h}_n \to 0.$$

It remains to check $\int_0^\infty f\overline{x}_n \to 0$. We let $f_n(t) = f(n+t)$ for $0 \le t \le 1$ and note that

$$\int_0^\infty f\overline{x}_n = \int_0^1 f_n x_n.$$

Now $f_n x_n \to 0$ a.e. since the sum of the measures of the supports of the f_n's is at most one. If we do not have convergence to 0 then we can find a subsequence (n_k) and disjoint sets (B_k) so that

$$\left|\int_{B_k} f_{n_k} x_{n_k}\right| \ge \epsilon > 0 \qquad k = 1, 2, \dots .$$

Let $F = \sum_k f_{n_k}\chi_{B_k}$ so that $F \in X^\times$ and $Fx_n \to 0$ weakly in L_1. We then contradict the standard criteria for weak compactness in L_1. □

Lemma 3.5. *If X a separable r.i. space on* $[0, 1]$ *and $X \in \mathrm{Int}(L_1, L_2)$, then there exists a constant $C > 0$ such that*

$$\|\sum_{i=1}^n u_i\|_X \le C\|\sum_{i=1}^n \overline{u_i}\|_{\mathcal{X}}$$

for any sequence of martingale differences $\{u_i\}_{i=1}^n \subset X$ and any integer $n \in \mathbb{N}$.

Proof. Since (L_1, L_2) is a K-couple [BK], 4.4.38, it follows from [BK], 3.3.20 that there exists a parameter Φ of K-method such that

$$(L_1, L_2)_\Phi^K = X.$$

We claim that

$$\mathcal{X} = \overline{X} + L_2[0, \infty), \tag{3.1}$$

where $\overline{X} := \overline{X}[0, \infty) = (L_1[0, \infty), L_2[0, \infty))_\Phi^K$. To see the validity of our claim, first note that by the assumption $L_2[0, 1] \subseteq X$, and this easily implies that $L_2[0, \infty) \subseteq \mathcal{X}$. To see that $\overline{X} \subseteq \mathcal{X}$, fix an element $x \in \overline{X}$ and observe that for every $t \in [0, 1]$, we have

$$K(t, x^*\chi_{[0,1]}; L_1[0, \infty), L_2[0, \infty)) = K(t, x^*\chi_{[0,1]}; L_1[0, 1], L_2[0, 1]).$$

It follows that $x^*\chi_{[0,1]} \in X$. Since $x \in (L_1 + L_2)[0, \infty)$, we easily see that $x^*\chi_{[1,\infty]} \in L_2[0, \infty)$. Thus, $x \in \mathcal{X}$, and we have established that $\overline{X} + L_2[0, \infty) \subseteq \mathcal{X}$. Since the inverse embedding trivially follows from the definition of $\mathcal{X}$, our claim is proved.

Since the K-method interpolates sublinear operators [BK], Corollary 3.3.12, it follows from (3.1) and Lemmas 3.1 and 3.2 that the operator B_n acts from $\mathcal{X}$ into X for any $n \in \mathbb{N}$ and

$$\sup_{n\in\mathbb{N}} \|B_n\|_{\mathcal{L}(\mathcal{X},X)} = C_1 < \infty.$$

In other words, for any sequence $\{x_i\}_{i=1}^n \subseteq X$ we have

$$\| \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} \|_X \leq C_1 \| \sum_{i=1}^n \overline{x_i} \|_{\mathcal{X}} \tag{3.2}$$

Since, by the assumption, $X \in \mathrm{Int}(L_1, L_2)$ we have $\alpha_X \geq \frac{1}{2}$ (see e.g. [LT2], Proposition 2.b.13). Therefore, it follows from [JS1], Theorem 3 that

$$\| \sum_{i=1}^n u_i \|_X \leq C_2 \| \left(\sum u_i^2 \right)^{\frac{1}{2}} \|_X$$

for any sequence of martingale differences $\{u_i\}_{i=1}^n \subseteq X$ and some constant $C_2 > 0$. Combining this fact with (3.2), we obtain the assertion of Lemma 3.5. □

The following lemma establishes a variant of (so-called) subsequence splitting property for separable r.i. spaces which do not necessarily have the Fatou property. For maximal r.i. spaces a subsequence splitting property was earlier established in [SS] Theorem 3.2 and [DSS] Proposition 3.2.

Lemma 3.6. *If X is a separable r.i. space on $[0, 1]$ or $[0, \infty)$ and $\{x_n\}_{n\geq 1} \subseteq X$, $\|x_n\|_X \leq 1,\ n \in \mathbb{N}$, then*

(i). *There exist a function $u \in X^{\times\times}$, $\|u\|_{X^{\times\times}} \leq 1$, a subsequence $\{y_n\}_{n\geq 1} \subset \{x_n\}_{n\geq 1}$ and sequences $\{u_n\}_{n\geq 1}, \{v_n\}_{n\geq 1}, \{w_n\}_{n\geq 1} \subseteq X$ such that $y_n = u_n + v_n + w_n,\ n \in \mathbb{N}$ and*
1) $u_n^* \leq u, \quad n \in \mathbb{N}$;
2) v_n's are pairwise disjoint and $\sup_n \|v_n\|_X \leq 2$;
3) $\lim_{n\to\infty} \|w_n\|_X = 0$.

(ii). *If, in addition, the sequence $\{x_n\}_{n\geq 1}$ is weakly null and $X^\times \subseteq L_0[0, \infty)$ (in particular, if X is a r.i. space on $(0, 1)$), then the sequences $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ may be chosen to be weakly null as well.*

The proof is omitted since it is very similar to previous results which are quoted.

Let $n \in N$, $\omega_1, \omega_2, ...\omega_n$ be a sequence of measure preserving transformations of $[0, 1]$ and

$$A_n x(t) = \left(\sum_{k=1}^n x^2(\omega_k(t)) \right)^{\frac{1}{2}}$$

Lemma 3.7. *Let $1 < p \leq 2$, X be an r.i. space on $[0, 1]$ and $0 < \alpha_X \leq \beta_X < \frac{1}{p}$. Then*

$$\|A_n x\|_X \leq C n^{\frac{1}{p}} \|x\|_X \tag{3.3}$$

where C does not depend on $x \in X, n \in N, \omega_1, \omega_2, ...\omega_n$.

Proof. The estimate

$$\|A_n x\|_{L_q} \le C_q n^{\max(\frac{1}{2},\frac{1}{q})}\|x\|_{L_q}$$

was proved in Lemma 5.3 [SS] for any $q \in [1,\infty]$. Applying the interpolation theorem 2.b.11 [LT] and remark after it to the pair $q = p$ and $q = \infty$ and quasi-linear operator A_n we get (3.3). □

We denote by $\{r_n\}_{n=1}^{\infty}$ the usual Rademacher system on [0, 1] defined by setting

$$r_n(t) = \operatorname{sgn}\ \sin(2^n \pi t), \quad t \in [0,1].$$

Lemma 3.8. *If $\{x_n\}_{n\ge1}$ is a weakly null sequence in a separable r.i. space X on* [0, 1]*, then any sequence $\{y_n\}_{n\ge1}$, where $y_n = z_n \otimes r_n$, $z_n^* = x_n^*$, $n \ge 1$ is weakly null in $X([0,1]\times[0,1])$.*

Proof. The set $\{y_n\}_{n\ge1}$ is relatively weakly compact for every r.i. space X on [0, 1] (see [DScS], Proposition 2.1 (v)). If $X \ne L_1$, then $(X^\times)_0$ is a separable r.i. space such that $(X^\times)_0^* = X^{\times\times}$, therefore, we may say that the set $\{y_n\}_{n\ge1}$ is relatively $\sigma(X^{\times\times}([0,1]\times[0,1]), (X^\times)_0([0,1]\times[0,1]))$-compact, and furthermore there exists a subsequence $\{u_n\}_{n\ge1} \subset \{y_n\}_{n\ge1}$ and $u \in X^{\times\times}([0,1]\times[0,1])$ such that

$$\sigma(X^{\times\times}([0,1]\times[0,1]), (X^\times)_0([0,1]\times[0,1])) - \lim_{n\to\infty} u_n = u.$$

The same argument as in Lemma 3.4 [SS] now shows that $u = 0$. This completes the proof in the case when $X \ne L_1$. The case $X = L_1$ is dealt with in a similar fashion: the existence of a weakly convergent subsequence $\{u_n\}_{n\ge1} \subset \{y_n\}_{n\ge1}$ converging to some element $u \in L_1[0,1]\times[0,1])$ is then guaranteed by the Eberlein-Šmulian theorem and by the weak sequential completeness of L_1-spaces. □

4. Index sets

In this section we present the main results of this article which concern the description of the index set $\Gamma(X)$ of a separable r.i. space X and its connection with the Boyd indices α_X, β_X of X. Along with the set $\Gamma(X)$, we consider also the following three index sets. We denote by $\Gamma_{\mathrm{d}}(X)$ the set of all positive real p such that, for every weakly null sequence $\{x_n\}_{n\ge1} \subseteq X$ of pairwise disjoint elements, there exists a subsequence $\{y_n\}_{n\ge1}$ such that

$$\sup_{m\in\mathbb{N}} m^{-\frac{1}{p}} \left\| \sum_{k=1}^{m} y_k \right\|_X < \infty.$$

Similarly, if in the preceding definition we replace weakly null sequences of pairwise disjoint elements by weakly null sequences of independent random variables (respectively, by weakly null sequences of independent identically distributed random variables), we obtain the sets $\Gamma_{\mathrm{i}}(X)$ (respectively, $\Gamma_{\mathrm{iid}}(X)$).

Lemma 4.1. *If X is a separable r.i. space on* $[0, 1]$, *then*

(i). $\Gamma(X) \subseteq \Gamma_{\mathrm{i}}(X) \subseteq \Gamma_{\mathrm{iid}}(X) \subseteq [1, 2]$.

(ii). $\Gamma_{\mathrm{i}}(X) \subseteq \Gamma_{\mathrm{d}}(X)$.

Proof. (i). The embeddings $\Gamma(X) \subseteq \Gamma_{\mathrm{i}}(X) \subseteq \Gamma_{\mathrm{iid}}(X)$ are obvious. The embedding $\Gamma_{\mathrm{iid}}(X) \subseteq [1, 2]$ follows from the proof of [SS], Lemma 4.1.

(ii). Let $\{x_i\}_{i\geq 1}$ be a normalized weakly null sequence of pairwise disjointly supported functions from X. It follows from Lemma 3.8 that there exists a weakly null sequence of independent random variables $z_i(t) \subseteq X$ such that $z_i^* = x_i^*$ for every $n \in \mathbb{N}$. By [JS2] Lemma 3, there exists a constant $C > 0$ such that

$$\|\sum_{i\in I} x_i\|_X \leq C\|\sum_{i\in I} z_i\|_X$$

for every finite subset $I \subset \mathbb{N}$. If $p \in \Gamma_{\mathrm{i}}(X)$, then there exists a subsequence $\{i_k\} \subset \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} n^{-\frac{1}{p}} \|\sum_{k=1}^{n} z_{i_k}\|_X < \infty$$

Hence

$$\sup_{n\in\mathbb{N}} n^{-\frac{1}{p}} \|\sum_{k=1}^{n} x_{i_k}\|_X < \infty$$

This proves that $p \in \Gamma_{\mathrm{d}}(X)$. □

Theorem 4.2. *If X is a separable r.i. space and $p \in (1, 2]$, then*

(i). $p \in \Gamma(X) \Longrightarrow \alpha_X > 0$;

(ii). $p \in \Gamma_{\mathrm{i}}(X) \Longrightarrow \beta_X \leq \frac{1}{p}$.

Proof. (i). Let $p \in (1, 2]$ belong to $\Gamma(X)$ and suppose, on the contrary, that $\alpha_X = 0$. It follows from [LT], 2.b.7 that for every $k \in \mathbb{N}$, there exists a sequence $\{x_i\}_{i=1}^{2^k}$ of pairwise disjoint equidistributed random variables from X such that for any scalars $\{a_i\}_{i=1}^{2^k}$

$$\max_{1\leq i\leq 2^k} |a_i| \leq \|\sum_{i=1}^{2^k} a_i x_i\|_X \leq \frac{3}{2} \max_{1\leq i\leq 2^k} |a_i| \tag{4.1}$$

For every $k \in \mathbb{N}$, we consider the sequence $\{y_j^{(k)}\}_{j=1}^{\infty} \subseteq X([0, 1] \times [0, 1])$ given by

$$y_j^{(k)}(s, t) = \sum_{i=1}^{2^k} r_{2^{j+k}+i}(s) x_i(t), \ \ j = 1, 2, \ldots \quad .$$

The right hand side of (4.1) implies that, for all $k, j \in \mathbb{N}$, we have $\|y_j^{(k)}\|_{X([0,1]\times[0,1])} \leq \frac{3}{2}$. It follows from [SS] Lemma 3.4 that $\{y_j^{(k)}\}_{j=1}^{\infty}$ is a weakly null sequence in

$X([0,1]\times[0,1])$. We claim that for any subsequence $\{y_{j_s}^{(k)}\}_{s\geq 1}\subseteq\{y_j^{(k)}\}_{j\geq 1}$ the inequality

$$\left\|\sum_{s=1}^{2k} y_{j_s}^{(k)}\right\|_{X([0,1]\times[0,1])} \geq \frac{k(e-1)}{e}, \tag{4.2}$$

holds for all sufficiently large $k\in\mathbb{N}$. Suppose for a moment that (4.2) is already established. To see that (4.2) leads to the contradiction with the assumption $p\in\Gamma(X)$, we note first that this assumption together with [SS] Lemma 4.2 yields the existence of a constant $C>0$, such that any weakly null sequence $\{y_i\}_{i=1}^{\infty}\subseteq X$ with $\|y_i\|_X\leq\frac{3}{2}$ contains a subsequence $\{y_{i_s}\}_{s=1}^{\infty}\subseteq\{y_i\}_{i=1}^{\infty}$ for which the inequality

$$\sup_{k\in\mathbb{N}} k^{-\frac{1}{p}}\left\|\sum_{s=1}^{k} y_{i_s}\right\|_X\leq C$$

holds. Taking into account the facts that the r.i. spaces X and $X([0,1]\times[0,1])$ are isometric and that (4.1) holds for an arbitrary $k\in\mathbb{N}$, we arrive at the contradiction with the inequality above. Thus, to complete the proof of the first part of Theorem 4.2, it is sufficient to verify (4.2). It is clear from the definition of the sequences $\{y_j^{(k)}\}_{j=1}^{\infty}$, $k\in\mathbb{N}$ that it is sufficient to make such a verification only for the case $j_s=s,\ s\geq 1$. We set for brevity

$$z_{2k}(s,t)=\sum_{j=1}^{2k} y_j^{(k)}(s,t)=\sum_{j=1}^{2k}\sum_{i=1}^{2^k} r_{2^{j+k}+i}(s)x_i(t)=\sum_{i=1}^{2^k}\left(\sum_{n\in Q_i} r_n(s)\right)x_i(t)$$

where $Q_i=\{2^{j+k}+i\}_{j=1}^{2k}$, $i=1,\dots,2^k$. Obviously $|Q_i|=2k$ and since $2^{k+j}+1>2^k+2^{k+j-1}$, $(j\geq 1)$, we have $Q_i\bigcap Q_l=\emptyset$, for any $i,l=1,\dots,2^k$ such that $i\neq l$. It is well known, that

$$\lambda\left\{s\in[0,1]:\left|\sum_{n\in Q} r_n(s)\right|\geq k\right\}\geq 2^{-k}$$

for any $Q\subseteq\mathbb{N},\ |Q|=2k$. Since the system $\left\{\sum_{n\in Q_i} r_n\right\}_{i=1}^{2^k}$ consists of independent random variables, we get

$$\begin{aligned}\lambda&\left\{s:\max_{1\leq i\leq 2^k}\left|\sum_{n\in Q_i} r_n(s)\right|<k\right\}\\&=\prod_{1\leq i\leq 2^k}\left(1-\lambda\left\{s:\left|\sum_{n\in Q_i} r_n(s)\right|\geq k\right\}\right)\leq\left(1-2^{-k}\right)^{2^k},\end{aligned}$$

in particular, for all k's

$$\lambda\left\{s : \max_{1\le i\le 2^k}\left|\sum_{n\in Q_i} r_n(s)\right| \ge k\right\} > 1-\frac{1}{e}. \tag{4.3}$$

We now set

$$E_j := \left\{s\in[0,1] : \max_{1\le i\le 2^k}\left|\sum_{n\in Q_i} r_n(s)\right| = \left|\sum_{n\in Q_j} r_n(s)\right|\right\}, \quad j=1,2,\ldots,2^k.$$

Without loss of generality, may assume that E_j's are chosen to be pairwise disjoint and that $\lambda\left(\cup_{j=1}^{2^k} E_j\right) \ge 1-\frac{1}{e}$ (see (4.3)). Let us also set for brevity $F_i := supp(x_i)$, and note that $F_i \cap F_j = \emptyset$ for all $i\ne j, i, j=1,2,\ldots,2^k$. Note further, that it follows from the definition of sets E_i's and from (4.3) that for every $i_0=1,2,\ldots,2^k$ and every point $(s,t)\in E_{i_0}\times F_{i_0}$, we have

$$|z_{2k}(s,t)| = \left|\sum_{i=1}^{2^k}\left(\sum_{n\in Q_i} r_n(s)\right)\chi_{E_{i_0}}(s)x_i(t)\chi_{F_{i_0}}(t)\right| \ge k\left|\chi_{E_{i_0}}(s)x_i(t)\chi_{F_{i_0}}(t)\right|.$$

Since $(E_i\times F_i)\cap(E_j\times F_j)=\emptyset$ for all $i\ne j, i, j=1,2,\ldots,2^k$, we infer from the above that

$$\begin{aligned}|z_{2k}(s,t)| &\ge \left|z_{2k}(s,t)\chi_{\cup_{i=1}^{2^k} E_i\times F_i}(t,s)\right| \\ &\ge k\left|\sum_{i=1}^{2^k}\chi_{E_i}(s)x_i(t)\chi_{F_i}(t)\right|, \quad \forall (t,s)\in[0,1]\times[0,1]\end{aligned}$$

and further (recalling the definition of sets F_i's) that

$$\|z_{2k}\|_{X([0,1]\times[0,1])} \ge k\left\|\sum_{i=1}^{2^k}\chi_{E_i}(s)x_i(t)\right\|_{X([0,1]\times[0,1])}.$$

Since the functions x_i's are equidistributed, then again via (4.3) we get

$$\lambda\left\{(s,t) : \left|\sum_{i=1}^{2^k}\chi_{E_i}(s)x_i(t)\right| > \tau\right\} =$$

$$= \sum_{i=1}^{2^k}\lambda\{|x_i(t)|>\tau\}\lambda(E_i) = \lambda\{|x_1(t)|>\tau\}\sum_{i=1}^{2^k}\lambda(E_i) > \lambda\{|x_1(t)|>\tau\}\cdot\tfrac{e-1}{e}$$

for all $\tau>0$. Since $\|x_i\|_X\ge 1$ (see (4.1)), we obtain (4.2)

$$\|z_{2k}\|_{X([0,1]\times[0,1])} \ge k\|\sigma_{(e-1)e^{-1}}x_1\|_X \ge \frac{k(e-1)}{e}\|x_1\|_X \ge \frac{k(e-1)}{e}.$$

This completes the proof of the inequality $\alpha_X > 0$.

(ii). Let $p \in (1,2]$ belong to $\Gamma_{\mathrm{i}}(X)$. Fix a positive integer n, an element $0 \le x \in X$ with $\|x\|_X = 1$ and $supp(x) \subset (0, \frac{1}{n})$. Let $\{y_k\}_{k=1}^n$ be a sequence of equidistributed copies of x such that $supp(y_k) \subset (\frac{k-1}{n}, \frac{k}{n})$. Furthermore, let $\{x_k\}_{k=1}^\infty \subseteq X$ be a weakly null sequence of independent random variables such that $|x_k|$ is equimeasurable with $x(t)$ for each $k \in \mathbb{N}$ (we can construct such a sequence via [SS] Lemma 3.4). By [JS2] Lemma 3, we have

$$\lambda\left\{t : \sum_{k=1}^n y_k(t) > \tau\right\} \le 2\lambda\left\{t : \sum_{k=1}^n x_k(t) > \tau\right\}$$

for every $\tau > 0$. Hence

$$\|\sigma_n x\|_X = \left\|\sum_{k=1}^n y_k\right\|_X \le 2\left\|\sum_{k=1}^n x_k\right\|_X$$

Observe that

$$\left\|\sum_{k=1}^n x_k\right\|_X = \left\|\sum_{k=1}^n x_{i_k}\right\|_X$$

for any sequence $\{i_k\}_{k\ge1} \subseteq \mathbb{N}$. Using this observation together with the assumption that $p \in \Gamma_{\mathrm{i}}(X)$, we get

$$\|\sum_{k=1}^n x_k\|_X \le Cn^{\frac{1}{p}}$$

for some $C > 0$ and every $n \in \mathbb{N}$. Hence,

$$\|\sigma_n x\|_X \le 2Cn^{\frac{1}{p}}, \quad \forall n \in \mathbb{N}$$

It follows from the definition of the dilation operators σ_τ that the norm of the operator σ_n, $n \in \mathbb{N}$ in X is attained for some $x \in X$ such that $\lambda\{supp(x)\} \le \frac{1}{n}$. Therefore

$$\|\sigma_n\|_{X\to X} \le 2Cn^{\frac{1}{p}}$$

for every $n \in N$. This means that $\beta_X \le \frac{1}{p}$. □

Theorem 4.2 gives the best possible estimate of the Boyd indices of a r.i. space X in the terms of the set $\Gamma(X)$. Indeed, if $X = L_p(1 < p \le 2)$, then $\beta_X = \frac{1}{p}$, if $X = L_p(2 \le p < \infty)$, then $\alpha_X = \frac{1}{p}$.

Theorem 4.3. *Let X be a separable r.i. space on $[0, 1]$ such that $\alpha_X > \frac{1}{2}$ and let $p \in (1, 2]$. The following conditions are equivalent*

(i). $p \in \Gamma(X)$;

(ii). $p \in \Gamma_{\mathrm{d}}(\mathcal{X})$;
(iii). $p \in \Gamma(\mathcal{X})$.

Proof. If $p \in \Gamma(X)$, then Lemma 4.1(i) and Theorem 4.2(ii) show that $\beta_X \leq \frac{1}{p} < 1$, in particular, the Boyd indices of X are non-trivial. By [LT] 2.f.1, this implies that the Banach spaces X and $\mathcal{X}$ are isomorphic. Consequently, $p \in \Gamma(\mathcal{X})$. This proves the implication $(i) \Longrightarrow (iii)$. The implication $(iii) \Longrightarrow (ii)$ is trivial. Therefore, to complete the proof it is sufficient to establish the implication $(ii) \Longrightarrow (i)$. To this end, assume that $p \in \Gamma_{\mathrm{d}}(\mathcal{X})$ and that $\{x_i\} \subseteq X$ is a weakly null sequence such that $\|x_i\|_X \leq 1$, $i \in \mathbb{N}$. It follows from the proof of Theorem 4.3 in [SS], that there exists a subsequence $\{x_{j_i}\} \subset \{x_i\}$, admitting the decomposition $x_{j_i} = u_i + v_i$, where $\{u_i\}$ is a weakly null sequence of martingale differences and $\|v_i\| \leq 2^{-i}$, $i \in \mathbb{N}$. It follows from Lemma 3.4, that the sequence $\{\overline{u}_i\}_{i=1}^{\infty}$ is weakly null in $\mathcal{X}$. Since $p \in \Gamma_{\mathrm{d}}(\mathcal{X})$, there exists a subsequence $\{\overline{u}_{i_s}\} \subset \{\overline{u}_i\}$ such that

$$\|\sum_{s=1}^{n} \overline{u}_{i_s}\|_{\mathcal{X}} \leq C_1 n^{\frac{1}{p}}$$

for some $C_1 > 0$ and every $n \in \mathbb{N}$. By the assumption $\frac{1}{2} < \alpha_X \leq \beta_X \leq 1$, we may use Lemma 3.5, which yields (for some $C > 0$)

$$\|\sum_{s=1}^{n} u_{i_s}\|_X \leq C\|\sum_{s=1}^{n} \overline{u}_{i_s}\|_{\mathcal{X}} \leq CC_1 n^{\frac{1}{p}}, \quad \forall n \in \mathbb{N}.$$

Hence

$$\begin{aligned}\|\sum_{s=1}^{n} x_{i_s}\|_X &\leq \|\sum_{s=1}^{n} u_{i_s}\|_X + \|\sum_{s=1}^{n} v_{i_s}\|_X \\ &\leq CC_1 n^{\frac{1}{p}} + 1 \leq (CC_1 + 1) n^{\frac{1}{p}}, \quad \forall n \in \mathbb{N}.\end{aligned}$$

In other words, $p \in \Gamma(X)$ and this completes the proof of Theorem 4.3. $\square$.

An inspection of the proof of Theorem 4.3 shows that each of the assumptions (i), (ii), (iii) above is equivalent to

(iv). for every weakly null sequence $\{x_n\} \subset \mathcal{X}$ of pairwise disjoint elements with $\lambda\{supp(x_n)\} \leq 1$ for every $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\sup_{m \in \mathbb{N}} m^{-\frac{1}{p}} \left\| \sum_{k=1}^{m} x_{n_k} \right\|_{\mathcal{X}} < \infty \tag{4.4}$$

Theorem 4.4. *If X is a separable r.i. space on $[0, 1]$ with $\alpha_X > \frac{1}{2}$, then $\Gamma(X) = \Gamma_{\mathrm{i}}(X)$.*

Proof. By Lemma 4.1(i), Theorem 4.3 and remark thereafter, it is sufficient to prove that condition (iv) above holds for any $p \in \Gamma_{\rm i}(X)$. Fix an arbitrary sequence $\{x_n\}_{n\geq 1} \subset \mathcal{X}$ satisfying the assumption in (iv). By Lemma 3.8 there exists a weakly null sequence of independent random variables $\{y_n\} \subseteq X$ such that $|y_n|$ is equidistributed with $|x_n|$ and $\int\limits_0^1 y_n(t)dt = 0$ for every $n \in \mathbb{N}$. The assumption $p \in \Gamma_{\rm i}(X)$ implies that

$$\sup_{m\in\mathbb{N}} m^{-\frac{1}{p}} \| \sum_{k=1}^{m} y_{n_k} \|_X < \infty \tag{4.5}$$

for some subsequence $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$. By [JS2] Lemma 3, there exists a constant $C > 0$, which depends on the space X only such that

$$\| \sum_{k=1}^{m} x_{n_k} \|_{\mathcal{X}} \leq C \| \sum_{k=1}^{m} y_{n_k} \|_X$$

for every $m \in \mathbb{N}$. It follows from this estimate and (4.5) that (4.4) is satisfied. This proves that $p \in \Gamma(X)$ and completes the proof of Theorem 4.4. □

Our next result describes the index set $\Gamma(X)$ of r.i. spaces X with non-trivial Boyd indices via its index set for disjointly supported sequences $\Gamma_{\rm d}(X)$.

Theorem 4.5. *If X is a separable r.i. space on* $[0, 1]$ *with* $0 < \alpha_X \leq \beta_X < \frac{1}{p}$, *where* $p \in \Gamma_{\rm d}(X)$, $1 < p \leq 2$, *then* $p \in \Gamma(X)$.

Proof. Thanks to Lemma 4.1, it is sufficient to show that $p \in \Gamma(X)$, whenever $p \in \Gamma_{\rm d}(X) \bigcap [1, 2]$. Let $\{x_k\}_{k\geq 1} \subseteq X$ be a weakly null sequence with $\|x_k\|_X \leq 1$, $k \geq 1$. Using Lemma 3.6, we select a subsequence $\{y_k\} \subset \{x_k\}$, a weakly null sequence $\{u_k\}_{k\geq 1} \subseteq X$, a weakly null sequence $\{v_k\}_{k\geq 1} \subseteq X$ of pairwise disjoint functions, a sequence $\{w_k\}_{k\geq 1} \subseteq X$ with $\sum_{k=1}^{\infty} \|w_k\|_X \leq 1$ and a function $u \in X^{\times\times}$ with $\|u\|_{X^{\times\times}} \leq 1$ such that

$$y_k = u_k + v_k + w_k, \quad u_k^* \leq u, \quad k \in \mathbb{N}. \tag{4.6}$$

Thanks to the assumption on p, we may also assume that the sequence $\{v_k\}_{k\geq 1}$ in (4.6) satisfies the inequality

$$\left\| \sum_{k=1}^{m} v_k \right\|_X \leq C_1 m^{\frac{1}{p}} \tag{4.7}$$

for some constant $C_1 > 0$ and every $m \in \mathbb{N}$. The proof will be complete if we show a similar estimate for the sequence $\{u_k\}_{k\geq 1}$ in (4.6) (a similar estimate of this sort for the sequence $\{w_k\}_{k\geq 1}$ in (4.6) is trivially satisfied). To this end, using [SS] Theorem 4.3 (and its proof), we may (and shall) assume that the sequence $\{u_k\}_{k\geq 1}$ in (4.6) admits the splitting

$$u_k = a_k + b_k, \quad k \in \mathbb{N} \tag{4.8}$$

where $\{a_k\}_{k\geq 1} \subseteq X$ is a sequence of martingale differences and $\|b_k\|_X \leq 2^{-k}$ for each $k \in \mathbb{N}$. In other words, the decomposition (4.6) may be rewritten as

$$y_k = a_k + v_k + (b_k + w_k), \quad k \in \mathbb{N}. \tag{4.9}$$

We claim that there exists a constant $C_4 > 0$ such that

$$\left\| \sum_{k=1}^{m} a_k \right\|_X \leq C_4 m^{\frac{1}{p}}, \quad \forall m \in \mathbb{N}. \tag{4.10}$$

Again, since a similar estimate of this sort for the sequence $\{b_k + w_k\}_{k\geq 1}$ in (4.9) is trivially satisfied, the estimates (4.7) and (4.10) suffice to complete the proof of Theorem 4.5. To prove (4.10), we first shall apply [JS1] Theorem 3 to the sequence of martingale differences $\{a_k\}_{k\geq 1}$ (we may do so, thanks to the assumption $\alpha_X > 0$). By that theorem, there exists a constant $C_2 > 0$, such that

$$\left\| \sum_{k=1}^{m} a_k \right\|_X \leq C_2 \left\| \left(\sum_{k=1}^{m} a_k^2 \right)^{\frac{1}{2}} \right\|_X, \quad \forall m \in \mathbb{N}$$

Next, we observe that the assumption on the sequence $\{b_k\}_{k\geq 1}$ in (4.8) imply that $\left\| \left(\sum_{k=1}^{m} b_k^2 \right)^{\frac{1}{2}} \right\|_X \leq 1$, and so $\left\| \left(\sum_{k=1}^{m} a_k^2 \right)^{\frac{1}{2}} \right\|_X \leq \left\| \left(\sum_{k=1}^{m} u_k^2 \right)^{\frac{1}{2}} \right\|_X + 1$. Thus,

$$\left\| \sum_{k=1}^{m} a_k \right\|_X \leq C_2 \left(\left\| \left(\sum_{k=1}^{m} u_k^2 \right)^{\frac{1}{2}} \right\|_X + 1 \right), \quad \forall m \in \mathbb{N}. \tag{4.11}$$

Now, to estimate $\left\| \left(\sum_{k=1}^{m} u_k^2 \right)^{\frac{1}{2}} \right\|_X$ from above, we first note that for any given integer $m \in \mathbb{N}$, there exists a sequence $\widetilde{u}_1, \widetilde{u}_2, ..., \widetilde{u}_m \subseteq X$ such that the functions $\widetilde{u}_i^* = \widetilde{u}_j^*$ for all $1 \leq i, j \leq m$ and $u_k^* \leq \widetilde{u}_k^* \leq u^*$ for all $k = 1, 2, ...m$. Clearly, it follows from the assumption $\|u\|_{X^{\times\times}} \leq 1$ and the inequality $\widetilde{u}_1^* \leq u^*$ that $\|\widetilde{u}_1\|_X \leq 1$. Thus, to estimate $\left\| \left(\sum_{k=1}^{m} u_k^2 \right)^{\frac{1}{2}} \right\|_X$ from above, we may assume without loss of generality, that $u_i^* = u_j^*$ for all $1 \leq i, j \leq m$ and $\|u_1\|_X \leq 1$. By Lemma 3.7, the assumption $0 < \alpha_X \leq \beta_X < \frac{1}{p}$, $1 < p \leq 2$ guarantees that there exists a constant $C_3 > 0$ depending on X only, such that

$$\left\| \left(\sum_{k=1}^{m} u_k^2 \right)^{\frac{1}{2}} \right\|_X \leq C_3 m^{\frac{1}{p}} \quad \forall m \in \mathbb{N}. \tag{4.12}$$

Combining (4.11) and (4.12), we deduce (4.10) with $C_4 = C_2(C_3 + 1)$. This completes the proof of Theorem 4.5. □.

Corollary 4.6. *If X is a separable r.i. space on* $[0, 1]$ *with* $0 < \alpha_X \leq \beta_X < \frac{1}{2}$, *then*

$$\Gamma(X) = \Gamma_{\rm i}(X) = \Gamma_{\rm d}(X) \bigcap [1, 2]$$

Since $1 \in \Gamma(X)$ (respectively, $1 \in \Gamma_{\rm d}(X)$) for any r.i. space X, we shall say that $\Gamma(X)$ (respectively, $\Gamma_{\rm d}(X)$) is non-trivial if $\Gamma(X) \neq \{1\}$ (respectively, $\Gamma_{\rm d}(X) \neq 1$). Using Theorems 4.2 and 4.5 we get

Corollary 4.7. *Let X be a separable r.i. space on* $[0, 1]$. *The set* $\Gamma(X)$ *is non-trivial if and only if the set* $\Gamma_{\rm d}(X)$ *and the Boyd indices of X are non-trivial.*

Now we shall specialize the result of Corollary 4.7 to the key classes of r.i. spaces.

Corollary 4.8. *Let $\varphi \in \Omega$ and $\Lambda(\varphi)$ be the corresponding Lorentz space. The set* $\Gamma(\Lambda(\varphi))$ *is non-trivial if and only if the Boyd indices of* $\Lambda(\varphi)$ *are non-trivial. In this case*

$$[1, min\left(\frac{1}{\beta_{\Lambda(\varphi)}}, 2\right)) \subset \Gamma(\Lambda(\varphi)) \subset [1, min\left(\frac{1}{\beta_{\Lambda(\varphi)}}, 2\right)]$$

Proof. If the set $\Gamma(\Lambda(\varphi))$ is non-trivial, then the fact that the Boyd indices of $\Lambda(\varphi)$ are non-trivial follows from Theorem 4.2. On the other hand, assume that $0 < \alpha_{\Lambda(\varphi)}$ and $p < \frac{1}{\beta_{\Lambda(\varphi)}}$ for some $p \in (1, 2]$ (or, equivalently, that $0 < \alpha_{\Lambda(\varphi)} \leq \beta_{\Lambda(\varphi)} < \frac{1}{p}$). Any pairwise disjoint sequence $x_i \in \Lambda(\varphi)$ such that $0 < \inf_i \|x_i\|_{\Lambda(\varphi)} \leq \sup_i \|x_i\|_{\Lambda(\varphi)} < \infty$ contains a subsequence equivalent to the standard basis of l_1 [CD]. Weak convergence in l_1 coincides with the convergence in norm, and so any weakly null sequence of pairwise disjoint elements from $\Lambda(\varphi)$ contains a subsequence which converges in norm. Hence $\Gamma_{\rm d}(\Lambda(\varphi)) = [1, \infty)$. We are now in a position to apply Theorem 4.5, which yields $p \in \Gamma(\Lambda(\varphi))$. Finally, if $p \in (1, \infty)$ belongs to $\Gamma(\Lambda(\varphi))$, then $p \leq 2$ (Lemma 4.1), and, by Theorem 4.2, $\beta_{\Lambda(\varphi)} \leq \frac{1}{p}$. Hence $p \leq \min(\frac{1}{\beta_{\Lambda(\varphi)}}, 2]$. □

Both cases $\frac{1}{\beta_{\Lambda(\varphi)}} \in \Gamma(\Lambda(\varphi))$ and $\frac{1}{\beta_{\Lambda(\varphi)}} \notin \Gamma(\Lambda(\varphi))$ are possible. Indeed, if $\varphi_\lambda(t) = t^\lambda$, $0 < \lambda < 1$, then $\beta_{\Lambda(\varphi_\lambda)} = \lambda$ by [SS], Theorem 5.9

$$\Gamma(\Lambda(\varphi_\lambda)) = \begin{cases} [1,2], & \lambda < \frac{1}{2} \\ [1,2), & \lambda = \frac{1}{2} \\ [1, \frac{1}{\lambda}], & \lambda > \frac{1}{2} \end{cases}$$

Note that an embedding of a Lorentz space $\Lambda(\varphi)$ into the scale of L_p-spaces does not imply the non-triviality of $\Gamma(\Lambda(\varphi))$. Indeed, for any pair $1 < p_1 <$

$p_2 < \infty$, there exists a Lorentz space $\Lambda(\varphi)$ such that $L_{p_2} \subset \Lambda(\varphi) \subset L_{p_1}$ and such that the Boyd indices of $\Lambda(\varphi)$ are trivial. Our assertion then follows from Corollary 4.8.

A bounded disjointly supported sequence from (the separable part of) any Marcinkiewicz space $M_0(\varphi),\ \varphi \in \Omega$ contains a subsequence equivalent to the standard unit basis of c_0 [CD]. Therefore, $\Gamma_{\mathrm{d}}(M_0(\varphi)) = [1, \infty)$ for each $\varphi \in \Omega$. Reasoning as in the proof of Corollary 4.8 we obtain the following result.

Corollary 4.9. *Let $\varphi \in \Omega$ and $M(\varphi)$ be the corresponding Marcinkiewicz space. The set $\Gamma(M_0(\varphi))$ is non-trivial if and only if the Boyd indices of $M_0(\varphi)$ are non-trivial.*

It is well known that the Boyd indices of $\Lambda(\varphi)$ and/or $M_0(\varphi)$ are non-trivial if and only if

$$1 < \liminf_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} \leq \limsup_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < 2.$$

Thus, Corollaries 4.8 and 4.9 show that $\Gamma(\Lambda(\varphi))$ is non-trivial if and only if $\Gamma(M_0(\varphi))$ is non-trivial. However, there is a substantial difference in the behavior of the Banach-Saks property in the spaces $\Lambda(\varphi)$ and $M_0(\varphi)$. Indeed, it is proved in [DSS] that

(i) $\Lambda(\varphi)$ has the BS-property for any $\varphi \in \Omega$,

(ii) there exist functions $\varphi \in \Omega$ such that $M_0(\varphi)$ does not have the BS-property.

We now specialize our results to the case of Orlicz spaces.

Corollary 4.10. *Let L_M be an Orlicz space on* $[0, 1]$. *The set $\Gamma(L_M)$ is non-trivial if and only if the Boyd indices of L_M are non-trivial.*

Proof. It is well known that the Boyd indices of L_M are non-trivial if and only if L_M is p-convex and q-concave for some $1 < p \leq q < \infty$ [LT], 2.b.5 and 1.f.7. Applying further [LT] 1.f.3, we infer that in this case the space L_M is of type p. Using [Ra], Theorem 1 we get $p \in \Gamma(L_M)$. The converse assertion follows immediately from Theorem 4.3. □

It is well known that the Boyd indices of an Orlicz space L_M are non-trivial if and only if the space L_M is reflexive. We also mention that it follows from Theorem 5.5 (ii) [DSS] that the separable part $(L_M)_0$ of any non-separable Orlicz space L_M fails the Banach-Saks property.

The results contained in Corollaries 4.8 - 4.10 do not extend to the set of all separable r.i. spaces.

Theorem 4.11. *There exists a separable r.i. space X on* $[0, 1]$ *with non-trivial Boyd indices, such that $\Gamma_{\mathrm{d}}(X) = \Gamma(X) = \{1\}$.*

Proof. The space U_1 of Pelczynski, which is universal for all unconditional basis sequences, has a representation as a separable r.i. space X on [0, 1] (see [LT], 2.g.5). Since U_1 has an unconditional basis, its Boyd indices are non-trivial [LT], 2.c.6. A. Baernstein has constructed a reflexive Banach space with unconditional basis and without the BS-property [Ba]. The latter space is contained in U_1 as a subspace. Therefore, U_1 also does not possess the BS-property. Hence $\Gamma(X)$ is trivial. The fact that the set $\Gamma_{\mathrm{d}}(X)$ is also trivial follows now from Corollary 4.7. □.

Finally, we shall show that the result of Theorem 4.4 (and that of Corollary 4.6) is sharp.

Proposition 4.12. *If* $X = \Lambda(\varphi_\lambda)$, *with* $\lambda = \frac{1}{2}$, *then* $\alpha_X = \beta_X = \frac{1}{2}$, *but* $\Gamma_{\mathrm{i}}(X) \neq \Gamma(X)$.

Proof. The first assertion is well known. Since, by [SS] Theorem 5.9, $\Gamma(X) = [1, 2)$, the second assertion will be established as soon as we show that $2 \in \Gamma_{\mathrm{i}}(X)$. Let $\{x_k\}_{k\geq 1} \subseteq X$ be a weakly null sequence of independent random variables with $\|x_k\|_X \leq 1$, $k \geq 1$. We shall first make an additional assumption that the sequence $\{x_k\}_{k\geq 1}$ consists of symmetric random variables. The latter assumption lets us to apply [JS2] Theorem 1, which asserts that there exists a scalar $C > 0$ such that

$$\left\| \sum_{k=1}^{m} x_k \right\|_X \leq C \left\| \sum_{k=1}^{m} \overline{x_k} \right\|_{\mathcal{X}}, \quad m \in \mathbb{N}. \tag{4.13}$$

Applying Lemma 3.6, [DSS] Proposition 3.2 and [SS] Lemma 3.1 to the sequence $\{x_k\}_{k\geq 1} \subseteq X$, we infer that there exists a subsequence $\{y_k\}_{k\geq 1} \subseteq \{x_k\}_{k\geq 1}$, a weakly null sequence $\{u_k\}_{k\geq 1} \subseteq X$, such that $u_k^* = u_1^*$, $k \geq 1$, $\|u_1\|_X \leq 1$, a weakly null sequence $\{v_k\}_{k\geq 1} \subseteq X$, $\|v_k\|_X \leq 2$, $k \geq 1$ of pairwise disjoint elements, and a sequence $\{w_k\}_{k\geq 1} \subseteq X$ with $\sum_{k=1}^{\infty} \|w_k\|_X \leq 1$ such that

$$y_k = u_k + v_k + w_k, \quad k \in \mathbb{N}. \tag{4.14}$$

Repeating the same argument as in the proof of Corollary 4.8, we may further assume that $\|v_k\|_X \to 0$. In particular, (again passing to a subsequence if necessary), we may assume that $\sum_{k=1}^{\infty} \|v_k\|_X \leq 1$. Passing now to the sequence $\{\overline{y_k}\}_{k\geq 1} \subseteq \mathcal{X}$ we get the following decomposition

$$\overline{y_k} = \overline{u_k} + \overline{v_k} + \overline{w_k}, \quad k \in \mathbb{N}. \tag{4.15}$$

It is clear that $\sum_{k=1}^{\infty} \|\overline{v_k}\|_{\mathcal{X}} \leq 1$ and $\sum_{k=1}^{\infty} \|\overline{w_k}\|_{\mathcal{X}} \leq 1$. Thus, taking into account (4.13)- (4.15), to prove the assertion of Proposition 4.12 for the sequence $\{x_k\}_{k\geq 1}$ we need only to show that there exists a constant $C_1 > 0$ such that

$$\left\| \sum_{k=1}^{m} \overline{u_k} \right\|_{\mathcal{X}} \leq C_1 m^{\frac{1}{2}}, \quad m \in \mathbb{N}. \tag{4.16}$$

To see that (4.16) indeed holds, note that $(\sum_{k=1}^{m} u_k)^* = \sigma_m(u_1^*)$ and that a simple direct computation yields $\|\sigma_m\|_{\mathcal{L}(X,X)} = \|\sigma_m\|_{\mathcal{L}(L_2,L_2)} = m^{\frac{1}{2}}$. Thus, $\|\sigma_m\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \leq 2m^{\frac{1}{2}}$ and this yields (4.16) with $C_1 = 2$.

Let $\{x_k\}_{k\geq 1} \subseteq X$ be an arbitrary weakly null sequence of independent random variables. Set for brevity, $c_k := \int_0^1 x_k(t)dt$ and assume without loss of generality that $\sum_{k=1}^{\infty} |c_k| \leq 1$. Consider

$$h_k = (x_k - c_k) - (x_k' - c_k), \quad k \geq 1$$

where $\{x_k'\}_{k\geq 1}$ is a sequence of independent copies of x_k's. Clearly, $\{x_k - c_k\}_{k\geq 1}$ and $\{x_k' - c_k\}_{k\geq 1}$ are sequences of independent mean zero random variables, and so $\{h_k\}_{k\geq 1}$ is a sequence of independent symmetric random variables. Obviously, the latter sequence is weakly null in X and therefore, it follows from the first part of the proof that there exists a subsequence $\{k_i\}_{i\geq 1} \subseteq \mathbb{N}$ and a constant $C_2 > 0$ such that

$$\left\| \sum_{i=1}^{m} h_{k_i} \right\|_X \leq C_2 m^{\frac{1}{2}}, \quad m \in \mathbb{N}. \tag{4.17}$$

Since the elements $\sum_{i=1}^{m} x_k - c_k$ and $\sum_{i=1}^{m} x_k' - c_k$ are independent mean zero random variables, it follows from [Br] Proposition 11, page 6 that there exists a constant $C_3 > 0$ such that

$$\left\| \sum_{i=1}^{m} x_{k_i} - c_{k_i} \right\|_X \leq C_3 \left\| \sum_{i=1}^{m} h_{k_i} \right\|_X, \quad m \in \mathbb{N}. \tag{4.18}$$

Combining (4.17) and (4.18) with the condition $\sum_{k=1}^{\infty} |c_n| \leq 1$, we see that there exists a constant $C_4 > 0$ such that

$$\left\| \sum_{i=1}^{m} x_{k_i} \right\|_X \leq C_4 m^{\frac{1}{2}}, \quad m \in \mathbb{N}. \quad \square$$

The space $X = \Lambda(\varphi_\lambda)$ also serves as an example showing that the condition $\alpha_X > \frac{1}{2}$ in Theorems 4.5 and 4.3 is sharp as well. Indeed, it immediately follows from the proof above that $2 \in \Gamma_{\mathrm{d}}(X)$, and this shows that the assumption $\alpha_X > \frac{1}{2}$ in Theorems 4.5 can not be relaxed. Furthermore, the argument above can be modified to yield that $2 \in \Gamma_{\mathrm{d}}(\mathcal{X})$, with the same effect for Theorem 4.3. We omit further details.

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