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BRIEF COMMUNICATIONS

Interpolation of Intersections Generated by a Linear Functional

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Abstract. Let (X_0, X_1) be a Banach couple such that $X_0 \cap X_1$ is dense in X_0 and X_1. By $(X_0, X_1)_{\theta,q}$, $0 < \theta < 1$, $1 \leqslant q < \infty$, we denote the spaces of the real interpolation method. Let ψ be a nonzero linear functional defined on some linear space $M \subset X_0 + X_1$ and such that $\psi \in (X_0 \cap X_1)^*$, and let $N = \operatorname{Ker} \psi$. We examine conditions under which the natural formula

$$(X_0 \cap N, X_1 \cap N)_{\theta,q} = (X_0, X_1)_{\theta,q} \cap N$$

is valid. In particular, the results obtained here imply those due to Ivanov and Kalton on the comparison of the interpolation spaces $(X_0, X_1)_{\theta,q}$ and $(N_0, X_1)_{\theta,q}$, where $\psi \in X_0^*$ and $N_0 = \operatorname{Ker} \psi$. By way of application, we consider a problem, posed by Krugljak, Maligranda, and Persson, on the interpolation of intersections generated by an integral functional defined on weighted L_p-spaces.

Key words: Banach space, interpolation space, subspace, Banach couple, subcouple, $\mathcal{K}$-functional, real interpolation method, weighted space.

1. Introduction. Let (X_0, X_1) be a Banach couple, i.e., a pair of Banach spaces linearly and continuously embedded in a Hausdorff topological vector space $\mathcal{T}$. Every linear subspace N of $\mathcal{T}$ generates a normed (in general, non-Banach) couple $(X_0 \cap N, X_1 \cap N)$, where the norm on $X_i \cap N$ is just obtained by restriction from X_i $(i = 0, 1)$.

The problem on the interpolation of intersections is the problem of finding conditions on the triple (X_0, X_1, N) and the parameters $\theta \in (0, 1)$ and $q \in [1, \infty]$ of the real interpolation method under which the natural formula

$$(X_0 \cap N, X_1 \cap N)_{\theta,q} = (X_0, X_1)_{\theta,q} \cap N \tag{1}$$

is valid (in the sense that the spaces coincide as sets and the norms are equivalent). The definition of the spaces $(X_0, X_1)_{\theta,q}$ will be given below.

The important special case of this problem in which (X_0, X_1) is a couple $(L_p(w_0), L_p(w_1))$ of weighted L_p-spaces and N is the linear space of all functions $f\colon (0, \infty) \to \mathbb{R}$ such that

$$\int_0^\infty f(x)\,dx = 0 \tag{2}$$

was considered by Krugljak, Maligranda, and Persson [1], who also posed a more general problem: what conditions on $w_0(x)$, $w_1(x)$, $p_0, p_1 \in [1, \infty)$, $\theta \in (0, 1)$, and $q \in [1, \infty)$ guarantee that

$$(L_{p_0}(w_0) \cap N, L_{p_1}(w_1) \cap N)_{\theta,q} = (L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,q} \cap N, \tag{3}$$

where, in general, $p_0 \neq p_1$ and the space N is still defined by (2)? This question has become a starting point for the statement of the problem considered in this note.

Suppose that N is the kernel of a linear functional ψ defined on a linear space $M \subset \mathcal{T}$ and such that $\psi \in (X_0 \cap X_1)^*$. We define four dilation indices of the $\mathcal{K}$-functional of ψ in the Banach couple (X_0^*, X_1^*) of dual spaces and use these indices to find conditions necessary for formula (1) to be valid. For an appropriate choice of N, these conditions are also sufficient. As a consequence, we obtain the results due to Ivanov and Kalton [2] on the comparison of the interpolation spaces $(X_0, X_1)_{\theta,q}$ and $(N_0, X_1)_{\theta,q}$, where $\psi \in X_0^*$ and $N_0 = \operatorname{Ker} \psi$ with the norm inherited from X_0. At the same time, the proof of Theorem 1 is based on the idea [2] of reducing the general problem of interpolation of subspaces to a study of the shift operator in some weighted l_p sequence space. Under certain conditions, our results allow one to solve the problem on the interpolation of intersections of weighted L_p-spaces posed in [1] (see formula (3)). We discuss this at the end of the note.

In conclusion, we note that the problem on the interpolation of intersections, as well as the problem considered in [2], is an important (but nevertheless special) case of the general, rather complicated problem on the interpolation of subspaces stated as early as in the book [3] by Lions and Magenes (in particular, see [4–13]).

2. Definitions and notation. For a normed couple (X_0, X_1) and $t > 0$, the Peetre $\mathcal{K}$- and $\mathcal{J}$-functionals are defined by the formulas

$$\mathcal{K}(t, x; X_0, X_1) = \inf_{x=x_0+x_1, x_i \in X_i} (\|x_0\|_{X_0} + t\|x_1\|_{X_1}), \qquad x \in X_0 + X_1,$$
$$\mathcal{J}(t, x; X_0, X_1) = \max(\|x\|_{X_0}, t\|x\|_{X_1}), \qquad x \in X_0 \cap X_1.$$

If $0 < \theta < 1$ and $1 \leqslant q < \infty$, then the space $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ of the real interpolation method consists of all $x \in X_0 + X_1$ such that

$$\|x\|_{X_{\theta,q}} = \left\{ \int_0^\infty (t^{-\theta}\mathcal{K}(t, x : X_0, X_1))^q \, \frac{dt}{t} \right\}^{1/q} < \infty.$$

If $X_0 \cap X_1$ is dense in X_0 and X_1 and $\psi \in (X_0 \cap X_1)^*$, then we can consider the Banach couple (X_0^*, X_1^*) of dual spaces, and $\psi \in X_0^* + X_1^*$ [14, §3.7]. In what follows, an important role will be played by the $\mathcal{K}$-functional $k(t) = \mathcal{K}(t, \psi; X_0^*, X_1^*)$ and the functions

$$M(t) = \sup_{s>0} \frac{k(ts)}{k(s)}, \quad M_0(t) = \sup_{0<s\leqslant\min(1,1/t)} \frac{k(ts)}{k(s)}, \quad M_\infty(t) = \sup_{s\geqslant\max(1,1/t)} \frac{k(ts)}{k(s)}.$$

These functions are submultiplicative for $t > 0$, and therefore, there exist limits

$$\alpha = \lim_{t\to 0} \frac{\log_2 M(t)}{\log_2 t}, \qquad \beta = \lim_{t\to\infty} \frac{\log_2 M(t)}{\log_2 t},$$
$$\alpha_0 = \lim_{t\to 0} \frac{\log_2 M_0(t)}{\log_2 t}, \qquad \beta_\infty = \lim_{t\to\infty} \frac{\log_2 M_\infty(t)}{\log_2 t},$$

which are called the *dilation indices* of the function $k(t)$. It readily follows that $0 \leqslant \alpha \leqslant \alpha_0 \leqslant \beta \leqslant 1$ and $0 \leqslant \alpha \leqslant \beta_\infty \leqslant \beta \leqslant 1$. In particular, if $\psi \in X_0^*$, then $k(t) \leqslant C$ $(t > 0)$. Hence $\alpha = \beta_\infty = 0$. The remaining two indices α_0 and β were introduced and used by Ivanov and Kalton [2].

3. Main results. Suppose that (X_0, X_1) is a Banach couple such that $X_0 \cap X_1$ is dense in X_0 and X_1. Let $\psi \in (X_0 \cap X_1)^*$ be a nonzero functional, let $N = \operatorname{Ker}\psi$, and let N_i be the normed space N with norm inherited from X_i $(i = 0, 1)$.

Theorem 1. *If*

$$\theta \in (0, \alpha) \cup (\beta_\infty, \alpha_0) \cup (\beta, 1), \tag{4}$$

then the norms of the interpolation spaces $N_{\theta,q} = (N_0, N_1)_{\theta,q}$ and $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ are equivalent on N. Under the condition $\beta_\infty \leqslant \alpha_0$, the converse is also true: if the norms of the spaces $N_{\theta,q}$ and $X_{\theta,q}$ are equivalent on N, then (4) holds.

Moreover, if $\theta \in (0, \alpha) \cup (\beta, 1)$, then $N_{\theta,q}$ is dense in $X_{\theta,q}$; if $\theta \in (\beta_\infty, \alpha_0)$, then $N_{\theta,q}$ is dense in some subspace of $X_{\theta,q}$ of codimension 1.

Corollary 1. *Suppose that ψ is a linear functional defined on some linear space $M \supset X_0 \cap X_1$ and $N = \operatorname{Ker}\psi$. Let ψ_0 be the restriction of ψ to the intersection $X_0 \cap X_1$. Suppose that $\psi_0 \in (X_0 \cap X_1)^*$, $\psi_0 \neq 0$, and α, β, α_0, and β_∞ are the dilation indices of the function $k(t) = \mathcal{K}(t, \psi_0; X_0^*, X_1^*)$, $\beta_\infty \leqslant \alpha_0$. Then formula (1) implies condition (4).*

Theorem 2. *Suppose that $\psi_0 \in (X_0 \cap X_1)^*$, $\psi_0 \neq 0$. If $\theta \in (\beta_\infty, \alpha_0)$, then ψ_0 can be extended to a functional $\psi \in ((X_0, X_1)_{\theta,q})^*$. If $\theta \in (0, \alpha)$ (respectively, $\theta \in (\beta, 1)$), then ψ_0 can be extended to a* *functional* $\psi \in ((X_0, X_1)_{\theta,q} \cap X_1)^*$ *(respectively,* $\psi \in ((X_0, X_1)_{\theta,q} \cap X_0)^*$*). Moreover, if* $N = \operatorname{Ker}\psi$*, then formula* (1) *is valid in all three cases.*

Theorems 1 and 2 imply the following important assertion.

Corollary 2. *Let* $\psi \in (X_0 \cap X_1)^*$*,* $\psi \neq 0$*, and* $N = \operatorname{Ker}\psi$*. If* $\theta \in (0, \alpha) \cup (\beta, 1)$*, then the interpolation space* $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ *coincides with the set of all* $x \in X_0 + X_1$ *such that* $x = \sum_{k=-\infty}^{\infty} x_k$*,* $x_k \in N$*, and*

$$\left(\sum_{k=-\infty}^{\infty} (\mathcal{J}(2^k, x_k; X_0, X_1) 2^{-\theta k})^q\right)^{1/q} \leqslant C\|x\|_{X_{\theta,q}},$$

where the constant $C > 0$ *is independent of* x*. A similar assertion holds for* $\theta \in (\beta_\infty, \alpha_0)$ *if we replace the space* $X_{\theta,q}$ *by the intersection* $X_{\theta,q} \cap \operatorname{Ker}\tilde{\psi}$*, where the functional* $\tilde{\psi}$ *is an extension of* ψ *to the space* $X_{\theta,q}$*.*

From this, one can readily obtain the results [2] on the comparison of the spaces $(X_0, X_1)_{\theta,q}$ and $(N_0, X_1)_{\theta,q}$, where $N_0 = \operatorname{Ker}\psi$ and $\psi \in X_0^*$. Indeed, $\alpha = \beta_\infty = 0$ in this case. Moreover, since ψ is defined on the entire space X_0, we have $(X_0, X_1)_{\theta,q} \subset N_0 + X_1$. Hence we have, in view of Corollary 2 and the theorem on the equivalence of $\mathcal{K}$- and $\mathcal{J}$-interpolation methods [14, Theorem 3.3.1],

$$(N_0, X_1)_{\theta,q} = (X_0, X_1)_{\theta,q} \quad \text{if } \theta \in (\beta, 1)$$

and

$$(N_0, X_1)_{\theta,q} = (X_0, X_1)_{\theta,q} \cap \operatorname{Ker}\tilde{\psi} \quad \text{if } \theta \in (0, \alpha_0).$$

Furthermore, it follows from Theorem 1 that $(N_0, X_1)_{\theta,q}$ is not closed in $(X_0, X_1)_{\theta,q}$ if $\theta \in [\alpha_0, \beta]$.

4. Applications. Let us return to the problem related to Eq. (3). In this case, the space N defined by (2) is the kernel of the functional $\psi(f) = \int_0^\infty f(x)\,dx$ with domain M consisting of all measurable functions $f\colon (0, \infty) \to \mathbb{R}$ such that $\int_a^b |f(x)|\,dx < \infty$ for arbitrary $a, b \in (0, \infty)$ and there exists a limit $\lim_{a\to 0,\, b\to\infty} \int_a^b f(x)\,dx$.

For simplicity, we restrict our considerations to the case in which $1 \leqslant p_0 < p_1 < \infty$ and p_0, p_1, w_0, and w_1 satisfy regularity conditions similar to those in [1]. Namely, we assume that

$(*)$ $w_0(x)$ is an increasing function, $w_0(2x) \leqslant C_1 w_0(x)$ $(x > 0)$ and $\int_x^\infty w_0(s)^{1/(1-p_0)}\,ds \leqslant C_2 x w_0(x)^{1/(1-p_0)}$ $(x > 0)$ for $p_0 > 1$, and $w_1(x)$ is either a decreasing function with $w_1(x) \leqslant C_3 w_1(2x)$ $(x > 0)$ or an increasing function such that $w(x) = w_0(x)/w_1(x)$ increases and $\int_0^x w_1(s)^{1/(1-p_1)}\,ds \leqslant C_4 x w_1(x)^{1/(1-p_1)}$ $(x > 0)$.

Suppose that $r^{-1} = p_0^{-1} - p_1^{-1}$, $v(x) = w_0(x)^{p_1/(p_1-p_0)} w_1(x)^{-p_0/(p_1-p_0)}$, and $\gamma(t)$ is a solution of the equation $\int_0^x v(s)\,ds = t^{-r}$ for each $t > 0$.

Theorem 3. *Let* p_0*,* p_1*,* $w_0(x)$*, and* $w_1(x)$ *satisfy assumptions* $(*)$*, and let* α*,* β*,* α_0*, and* β_∞ *be the dilation indices of the function* $k(t) = \gamma(t)^{(p_0-1)/p_0} w_0(\gamma(t))^{-1/p_0}$*,* $\beta_\infty \leqslant \alpha_0$*. Then formula* (3) *is valid if and only if* (4) *holds.*

In particular, consider the case of power weight functions $w_0(x) = x^l$ and $w_1(x) = x^m$. Then conditions $(*)$ are satisfied if either (a) $l > p_0 - 1$ and $m \leqslant 0$ or (b) $l > p_0 - 1$, $l > m$, and $0 \leqslant m < p_1 - 1$. Since $k(t) \asymp t^{\theta_0}$, where $\theta_0 = \frac{p_1(l-p_0+1)}{p_1(l+1)-p_0(m+1)}$, it follows that $\alpha = \beta = \alpha_0 = \beta_\infty = \theta_0$, and we arrive at the following assertion.

Corollary 3. *Suppose that* p_0*,* p_1 $(1 \leqslant p_0 < p_1 < \infty)$*,* l*, and* m *satisfy condition* (a) *or* (b)*. Then one has*

$$(L_{p_0}(x^l) \cap N, L_{p_1}(x^m) \cap N)_{\theta,q} = (L_{p_0}(x^l), L_{p_1}(x^m))_{\theta,q} \cap N$$

(in the sense of equivalence of norms) if and only if $\theta \neq \frac{p_1(l-p_0+1)}{p_1(l+1)-p_0(m+1)}$*.*

For $p_0 = p_1 = q$, the last result was proved in a different way in [1, Corollary 1].

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