

INTERPOLATION OF INTERSECTIONS BY THE REAL METHOD

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ABSTRACT. Let (X_0, X_1) be a Banach couple with $X_0 \cap X_1$ dense both in X_0 and in X_1 , and let $(X_0, X_1)_{\theta, q}$ ($0 < \theta < 1$, $1 \leq q < \infty$) denote the real interpolation spaces. Suppose ψ is a linear functional defined on some linear subspace $M \subset X_0 + X_1$ and satisfying $\psi \in (X_0 \cap X_1)^*$, $\psi \neq 0$. Conditions are considered that ensure the natural identity

$$(X_0 \cap \text{Ker}\psi, X_1 \cap \text{Ker}\psi)_{\theta, q} = (X_0, X_1)_{\theta, q} \cap \text{Ker}\psi.$$

The results obtained provide a solution for the problem posed by N. Kruglyak, L. Maligranda, and L.-E. Persson and pertaining to interpolation of couples of intersections that are generated by an integral functional in a weighted L_p -scale. Furthermore, an expression is found for the $\mathcal{K}$ -functional on a couple of intersections corresponding to a linear functional, and some other related questions are treated.

INTRODUCTION

Let (X_0, X_1) be a normed couple, i.e., a couple of normed spaces embedded linearly and continuously in a Hausdorff linear topological space $\mathcal{T}$. For $x \in X_0 + X_1$ and $t > 0$, we define the Peetre $\mathcal{K}$ -functional

$$\mathcal{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

If $0 < \theta < 1$ and $1 \leq q < \infty$, the real interpolation space $X_{\theta, q} = (X_0, X_1)_{\theta, q}$ consists of all $x \in X_0 + X_1$ with

$$\|x\|_{\theta, q} = \left\{ \int_0^\infty (t^{-\theta} \mathcal{K}(t, x; X_0, X_1))^q \frac{dt}{t} \right\}^{1/q} < \infty.$$

Next, if (X_0, X_1) is a Banach couple, then each linear subspace $N \subset \mathcal{T}$ determines the normed (in general, non-Banach) couple $(X_0 \cap N, X_1 \cap N)$ of intersections. The norm on $X_i \cap N$ is simply the restriction of the norm of X_i ($i = 0, 1$).

Now, it is natural to look for conditions ensuring the natural identity

$$(0.1) \quad (X_0 \cap N, X_1 \cap N)_{\theta, q} = (X_0, X_1)_{\theta, q} \cap N$$

(with equivalence of the norms).

This formula is true if, for instance, (X_0, X_1) is a couple of Banach function lattices and N is a linear space of functions (on the same measure space) with the lattice property: if $g \in N$ and $|f| \leq |g|$, then $f \in N$ (see [1, Remark 2]). On the other hand, if we take the couple $(L_1[0, 1], L_\infty[0, 1])$ for (X_0, X_1) and the closed linear span $[r_n]$ of the Rademacher system in $L_1[0, 1]$ for N , then (0.1) fails. Indeed, by the Khinchin inequality, $L_p \cap [r_n] \approx l_2$ for $1 \leq p < \infty$, and $L_\infty \cap [r_n] \approx l_1$. Thus, calculating the interpolation

2000 *Mathematics Subject Classification*. Primary 46B70.

Key words and phrases. Interpolation space, real interpolation method, $\mathcal{K}$ -functional, dilation indices of a function, spaces of measurable functions, weighted spaces.

spaces in (0.1) for arbitrary $0 < \theta < 1$ and $q = \frac{2}{1+\theta}$ (see [2, Theorem 5.2.1]), we arrive at $(X_0 \cap N, X_1 \cap N)_{\theta, q} \approx l_q$, and $(X_0, X_1)_{\theta, q} \cap N \approx l_2$.

In this paper we are interested in the case where N is the kernel of a linear functional defined on some linear subspace $D \subset \mathcal{T}$. An important particular case of this problem was treated in [1]. Namely, in that paper certain conditions on $\theta \in (0, 1)$, $p \in [1, \infty)$, and on weight functions $w_0(x)$ and $w_1(x)$ were found ensuring the formula

$$(0.2) \quad (L_p(w_0) \cap N, L_p(w_1) \cap N)_{\theta, p} = (L_p(w_0), L_p(w_1))_{\theta, p} \cap N,$$

where $L_p(w)$ is the weighted L_p -space on $(0, \infty)$ with the usual norm, and N is the space of all functions $f : (0, \infty) \rightarrow \mathbb{R}$ satisfying

$$(0.3) \quad \int_0^\infty f(x) dx = 0.$$

It turned out that the validity of (0.2) is closely related to the possibility to “interpolate” certain Hardy-type integral inequalities. In the same paper [1], the following more general question was asked: Under what conditions on $w_0(x)$, $w_1(x)$, $p_0, p_1 \in [1, \infty)$, $\theta \in (0, 1)$, and $q \in [1, \infty]$ do we have the identity

$$(0.4) \quad (L_{p_0}(w_0) \cap N, L_{p_1}(w_1) \cap N)_{\theta, q} = (L_{p_0}(w_0), L_{p_1}(w_1))_{\theta, q} \cap N?$$

(Generally speaking, here $p_0 \neq p_1$, but N is still determined by (3).)

It is well known (see [2, Chapter 5]) that the standard equivalent expressions for the $\mathcal{K}$ -functional on the couple $(L_{p_0}(w_0), L_{p_1}(w_1))$ differ substantially in the cases of $p_0 = p_1$ and of $p_0 \neq p_1$; as a result, the latter question is highly more complicated than that treated in [1]. This was the origin for the following general problem.

Suppose (X_0, X_1) is a Banach couple with $X_i \subset \mathcal{T}$ ($i = 0, 1$) and such that $X_0 \cap X_1$ is dense both in X_0 and in X_1 . Next, let ψ be a linear functional defined on some linear space $D \subset \mathcal{T}$, let ψ belong to $(X_0 \cap X_1)^*$, and let $\psi(x) \neq 0$ for some $x \in X_0 \cap X_1$. We shall see that, for the validity of the formula

$$(0.5) \quad (X_0 \cap \text{Ker } \psi, X_1 \cap \text{Ker } \psi)_{\theta, q} = (X_0, X_1)_{\theta, q} \cap \text{Ker } \psi,$$

of primary importance are four indices α, β, α_0 , and β_∞ , related to the Peetre $\mathcal{K}$ -functional $\mathcal{K}(t, \psi; X_0^*, X_1^*)$ in the couple (X_0^*, X_1^*) of dual spaces. These indices satisfy the inequalities

$$0 \leq \alpha \leq \min(\alpha_0, \beta_\infty) \leq \max(\alpha_0, \beta_\infty) \leq \beta \leq 1.$$

It will be shown that, under the additional condition $\beta_\infty \leq \alpha_0$, the norms of the interpolation spaces $X_{\theta, q} = (X_0, X_1)_{\theta, q}$ and $N_{\theta, q} = (N_0, N_1)_{\theta, q}$ are equivalent on $N := X_0 \cap X_1 \cap \text{Ker } \psi$ if and only if $\theta \in (0, \alpha) \cup (\beta_\infty, \alpha_0) \cup (\beta, 1)$ (see Theorem 1). Here N_i is the space N endowed with the norm of X_i ($i = 0, 1$). So, the latter condition is necessary for (0.5) to be true. Furthermore, we shall show that if ψ is defined on $X_0 \cap X_1$, $\psi \in (X_0 \cap X_1)^*$, then for every $\theta \in (0, \alpha) \cup (\beta_\infty, \alpha_0) \cup (\beta, 1)$ the functional ψ admits extension by continuity to some space $\widetilde{M}_\theta \supset X_0 \cap X_1$ such that (0.5) is fulfilled for the kernel of this extension (see Theorem 2).

In the proof of Theorem 1, an approach developed by Ivanov and Kalton in [3] is used. The first part of [3] was devoted to comparison of the interpolation spaces $(X_0, X_1)_{\theta, q}$ and $(Y_0, X_1)_{\theta, q}$ ($0 < \theta < 1, 1 \leq q < \infty$), where $Y_0 = \text{Ker } \psi$, $\psi \in X_0^*$. The essence of that approach is in reduction of the general interpolation problem for subspaces to the study of the shift operator in a certain weighted l_p -space. At the same time, the results of [3] are consequences of those obtained here (see Corollary 3 and Remark 2).

It should be noted that interpolation of intersections and the problem studied in [3] are only partial (though important) cases of the general subspace interpolation problem formulated as early as in the monograph [4] by Lions and Magenes. Various aspects of

this general problem have been treated in [5]–[14] (the author is very far from claiming the completeness of this citation list).

Under certain conditions, the results of the first part of the paper allow us to answer the question from [1] mentioned above. In this setting, N is the kernel of the integral functional $\psi(f) = \int_0^\infty f(x) dx$. Not entering in the details, we only signalize that the answer depends essentially on p_0 and p_1 and has the simplest form in the case of power-type weights w_1 and w_2 . Specifically (see Corollary 8), if $1 \leq p_0 \leq p_1 < \infty$ and either (a) $l > p_0 - 1$ and $m \leq 0$, or (b) $l > p_0 - 1$, $l > m$, and $0 \leq m < p_1 - 1$, then (0.4) is fulfilled with $w_0(x) = x^l$ and $w_1(x) = x^m$ if and only if $\theta \neq \frac{p_1(l-p_0+1)}{p_1l-p_0m+p_1-p_0}$.

For general weights, the situation is different. For instance, it is shown (see Corollary 7) that for every numbers $0 \leq a \leq b \leq c \leq d \leq 1$ there exist weights $w_0(x)$ and $w_1(x)$ such that, for $p_0 = p_1 = 1$ and $q \in [1, \infty)$, relation (0.4) is true if and only if $\theta \in (0, a) \cup (b, c) \cup (d, 1)$. Thus, the values of θ for which the natural interpolation identity (0.1) fails may fill one or two arbitrary subsegments of $(0, 1)$ (cf. [1, Corollary 2, Remark 8]).

Surely, a most natural approach to the subspace interpolation problem is via calculation of the $\mathcal{K}$ -functional for the couple of subspaces in question; see [1, 8, 12]. This problem proves to be interesting in itself and is treated in §3. In Theorem 3 an expression for $\mathcal{K}(t, x; X_0 \cap N, X_1 \cap N)$ is found under the condition that the functional ψ in question is defined on the set $\{y_0(t) \in X_0 : y \in X_0 \cap N + X_1 \cap N, t > 0\}$, where $y_0(t)$ is the first summand in an optimal decomposition $y = y_0(t) + y_1(t)$, $y_i(t) \in X_i$ ($i = 0, 1$), i.e., in a decomposition for which

$$\frac{1}{2}(\|y_0(t)\|_{X_0} + t\|y_1(t)\|_{X_1}) \leq \mathcal{K}(t, y; X_0, X_1) \leq \|y_0(t)\|_{X_0} + t\|y_1(t)\|_{X_1} \quad (t > 0).$$

Note that this condition is fulfilled in the case of weighted L_p -spaces that was treated in [1]. Therefore, the theorem proved in the present paper extends the expressions for $\mathcal{K}(t, f; L_p(w_0) \cap N, L_p(w_1) \cap N)$ obtained in [1] by direct inspection. In conclusion, we mention that a similar result for the couple (Y_0, X_1) , $Y_0 = \text{Ker } \psi$, $\psi \in X_0^*$, was obtained by Wallsten [8] (see also [12, Theorem 2.1]).

Definitions and notation. Let (X_0, X_1) be an arbitrary normed couple. The definitions of the Peetre $\mathcal{K}$ -functional $\mathcal{K}(t, x; X_0, X_1)$ ($x \in X_0 + X_1$, $t > 0$) and of the real interpolation spaces $X_{\theta, q} = (X_0, X_1)_{\theta, q}$ ($0 < \theta < 1$, $1 \leq q < \infty$) were given at the beginning of the paper. Analyzing the proof of the equivalence theorem [2, §3], we see that it remains true for normed (rather than merely for Banach) couples. Thus, the space $X_{\theta, q}$ (with an equivalent norm) can be defined in terms of the $\mathcal{J}$ -functional

$$\mathcal{J}(t, x; X_0, X_1) = \max\{\|x\|_{X_0}, t\|x\|_{X_1}\} \quad (x \in X_0 \cap X_1).$$

Specifically, $X_{\theta, q}$ consists of all $x \in X_0 + X_1$ representable in the form

$$x = \sum_{k \in \mathbb{Z}} 2^{\theta k} x_k \quad (\text{convergence in } X_0 + X_1)$$

with the norm

$$\inf \left\{ \sum_{k \in \mathbb{Z}} (\mathcal{J}(2^k, x_k; X_0, X_1))^q \right\}^{1/q},$$

where the infimum is taken over all representation indicated above.

Next, it will be tacitly assumed that $X_0 \cap X_1$ is dense in X_0 and in X_1 . Therefore, we may consider the Banach couple (X_0^*, X_1^*) of conjugate spaces, and if $\psi \in (X_0 \cap X_1)^*$,

then $\psi \in X_0^* + X_1^*$ (see [2, §3.7]). The function $k(t) = \mathcal{K}(t, \psi; X_0^*, X_1^*)$ will play a crucial role in what follows; we have

$$(0.6) \quad k(t) = \sup\{|\psi(x)| : x \in X_0 \cap X_1, \mathcal{J}(t^{-1}, x; X_0, X_1) \leq 1\}$$

(see [2, §3.7]).

We introduce the functions

$$M(t) = \sup_{s>0} \frac{k(ts)}{k(s)}, \quad M_0(t) = \sup_{0<s \leq \min(1, 1/t)} \frac{k(ts)}{k(s)}, \quad M_\infty(t) = \sup_{s \geq \max(1, 1/t)} \frac{k(ts)}{k(s)},$$

which are semimultiplicative for $t > 0$, and so the following numbers are well defined:

$$\begin{aligned} \alpha &= \lim_{t \rightarrow 0} \frac{\log_2 M(t)}{\log_2 t}, & \beta &= \lim_{t \rightarrow \infty} \frac{\log_2 M(t)}{\log_2 t}, \\ \alpha_0 &= \lim_{t \rightarrow 0} \frac{\log_2 M_0(t)}{\log_2 t}, & \beta_\infty &= \lim_{t \rightarrow \infty} \frac{\log_2 M_\infty(t)}{\log_2 t}. \end{aligned}$$

These numbers are called the *dilation indices* of $k(t)$. It is easily seen that

$$0 \leq \alpha \leq \min(\alpha_0, \beta_\infty) \leq \max(\alpha_0, \beta_\infty) \leq \beta \leq 1.$$

Now, putting $\mu_n = (k(2^{-n}))^{-1}$ ($n \in \mathbb{Z}$), we see that $\mu_n > 0$, and, since $k(t)$ is monotone increasing and concave on $(0, \infty)$, also

$$(0.7) \quad \mu_n \leq \mu_{n+1} \leq 2\mu_n \quad (n \in \mathbb{Z}).$$

Furthermore,

$$(0.8) \quad \begin{aligned} \alpha &= - \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 \sup_{k \in \mathbb{Z}} \frac{\mu_k}{\mu_{n+k}}, & \beta &= \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 \sup_{k \in \mathbb{Z}} \frac{\mu_k}{\mu_{k-n}}, \\ \alpha_0 &= - \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 \sup_{k \geq 0} \frac{\mu_k}{\mu_{n+k}}, & \beta_\infty &= \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 \sup_{k \leq 0} \frac{\mu_k}{\mu_{k-n}}. \end{aligned}$$

Let e_k ($k \in \mathbb{Z}$) denote the standard unit vectors in a sequence space, and let $l_q(\mu)$ be the space of two-sided numerical sequences $a = (a_k)_{k=-\infty}^\infty$ with the norm

$$\|a\|_{q, \mu} = \left\{ \sum_{k \in \mathbb{Z}} |a_k|^q \mu_k^q \right\}^{1/q}.$$

In what follows, we shall study the shift operator $S(a_k) = (a_{k-1})$, its inverse S^{-1} , and also the operators $T_\theta = S - 2^\theta I$ ($0 < \theta < 1$), where I is the identity mapping. By (0.7), both S and S^{-1} are bounded on $l_q(\mu)$, and $\|S\| \leq 2$, $\|S^{-1}\| \leq 1$. Next, by P_+ and P_- we denote the projections

$$P_+(a_k) = \sum_{k=1}^\infty a_k e_k, \quad P_-(a_k) = \sum_{k=-1}^{-\infty} a_k e_k,$$

and $r(T)$ stands for the spectral radius of an operator T on $l_q(\mu)$.

Finally, writing $F_1 \asymp F_2$ we mean that there exist two constants $c > 0$, $C > 0$ with $cF_1 \leq F_2 \leq CF_1$. Usually, c and C do not depend on all (or some) of the arguments of the functions F_1 and F_2 .

§1. AUXILIARY RESULTS

Lemma 1. *We have $r(S) = 2^\beta$, $r(S^{-1}) = 2^{-\alpha}$, $r(SP_-) = 2^{\beta_\infty}$, and $r(S^{-1}P_+) = 2^{-\alpha_0}$.*

Proof. Since $S^n(a_k) = (a_{k-n})$ and $S^{-n}(a_k) = (a_{k+n})$, we have

$$\|S^n\|_{l_q(\mu) \rightarrow l_q(\mu)} = \sup_{k \in \mathbb{Z}} \frac{\mu_k}{\mu_{k-n}}, \quad \|S^{-n}\|_{l_q(\mu) \rightarrow l_q(\mu)} = \sup_{k \in \mathbb{Z}} \frac{\mu_k}{\mu_{k+n}}.$$

Thus, by (0.8), we see that

$$r(S) = \lim_{n \rightarrow \infty} \left(\sup_{k \in \mathbb{Z}} \frac{\mu_k}{\mu_{k-n}} \right)^{1/n} = 2^\beta,$$

$$r(S^{-1}) = \lim_{n \rightarrow \infty} \left(\sup_{k \in \mathbb{Z}} \frac{\mu_k}{\mu_{k+n}} \right)^{1/n} = 2^{-\alpha}.$$

Similarly,

$$(SP_-)^n(a_k) = \sum_{k=-n}^{-\infty} a_k e_{k+n}, \quad (S^{-1}P_+)^n(a_k) = \sum_{k=n}^{\infty} a_k e_{k-n}, \quad n = 1, 2, \dots,$$

and, therefore,

$$\|(SP_-)^n\|_{l_q(\mu) \rightarrow l_q(\mu)} = \sup_{k \leq 0} \frac{\mu_k}{\mu_{k-n}}, \quad \|(S^{-1}P_+)^n\|_{l_q(\mu) \rightarrow l_q(\mu)} = \sup_{k \geq 0} \frac{\mu_k}{\mu_{k+n}}.$$

This results in

$$r(SP_-) = \lim_{n \rightarrow \infty} \left(\sup_{k \leq 0} \frac{\mu_k}{\mu_{k-n}} \right)^{1/n} = 2^{\beta_\infty},$$

$$r(S^{-1}P_+) = \lim_{n \rightarrow \infty} \left(\sup_{k \geq 0} \frac{\mu_k}{\mu_{k+n}} \right)^{1/n} = 2^{-\alpha_0}. \quad \square$$

The next key statement about properties of $T_\theta = S - 2^\theta I$ (I is the identity) is similar to Lemma 2.2 in [3]. However, there is an important point of difference. In [3] the functional ψ belonged to X^* , which resulted in $\inf_{n \in \mathbb{Z}} \mu_n > 0$. The latter condition implies that T_θ is injective for every $\theta \in (0, 1)$. In the present more general setting, where $\psi \in (X_0 \cap X_1)^*$, the operator T_θ may fail to be injective because $\inf_{n \in \mathbb{Z}} \mu_n$ may happen to be zero.

Lemma 2. *Suppose $0 < \theta < 1$ and $1 \leq q < \infty$. Then T_θ is an isomorphic embedding of $l_q(\mu)$ into $l_q(\mu)$ if and only if $\theta \in (0, \alpha) \cup (\beta_\infty, \alpha_0) \cup (\beta, 1)$.*

Moreover, if $\theta \in (0, \alpha) \cup (\beta, 1)$, then $\text{Im } T_\theta = l_q(\mu)$; if $\theta \in (\beta_\infty, \alpha_0)$, then $\text{Im } T_\theta$ is the closed subspace of codimension 1 in $l_q(\mu)$ consisting of all $(a_k)_{k \in \mathbb{Z}} \in l_q(\mu)$ with

$$(1.1) \quad \sum_{k \in \mathbb{Z}} 2^{k\theta} a_k = 0.$$

Proof. First, if $\theta \in (\beta, 1)$, then Lemma 1 implies the inequality $2^\theta > r(S)$, whence T_θ is an automorphism of $l_q(\mu)$ onto $l_q(\mu)$. The same is true for $\theta \in (0, \alpha)$ because then $r(S^{-1}) < 2^{-\theta}$ (again by Lemma 1) and

$$(1.2) \quad T_\theta = 2^\theta S(2^{-\theta} I - S^{-1}).$$

We show that neither T_β nor T_α is an isomorphism of $l_q(\mu)$ onto $l_q(\mu)$. Indeed, for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ the operators $Q_\lambda(a_i) = (\lambda^i a_i)$ and $Q'_\lambda(a_i) = (\lambda^{i+1} a_i)$ are isometries of $l_q(\mu)$. Since $S - \lambda 2^\beta I = Q_\lambda^{-1} T_\beta Q'_\lambda$, the supposition that T_β is an isomorphism onto $l_q(\mu)$ would imply that all operators $S - \lambda 2^\beta I$ ($\lambda \in \mathbb{C}$, $|\lambda| = 1$) have the same property.

But this contradicts the relation $r(S) = 2^\beta$. Similarly, the supposition that T_α is an isomorphism onto $l_q(\mu)$ would imply the same property for $S^{-1} - \lambda 2^{-\alpha} I$ (apply (1.2) with $\theta = \alpha$ and take the relation $S^{-1} - \lambda 2^{-\alpha} I = Q'_\lambda(S^{-1} - 2^{-\alpha} I)Q_\lambda^{-1}$ into account). This contradicts the fact that $r(S^{-1}) = 2^{-\alpha}$.

Now we determine the possible form of $\text{Im } T_\theta$. Since

$$T_\theta \left(\sum_{i=0}^{n-1} 2^{(n-1-i)\theta} e_i \right) = e_n - 2^{\theta n} e_0,$$

we have $e_n - 2^{\theta n} e_0 \in \text{Im } T_\theta$ for all $n \in \mathbb{Z}$. Suppose f_θ is a linear functional vanishing at every $b \in \text{Im } T_\theta$. Then $f_\theta(e_n) = 2^{\theta n} C$. Thus, we may assume that f_θ corresponds to the sequence $(2^{\theta n})_{n \in \mathbb{Z}}$.

Clearly, the condition $f_\theta \in l_q(\mu)^*$ is equivalent to the condition

$$(1.3) \quad \sum_{k \in \mathbb{Z}} 2^{kq'\theta} \mu_k^{-q'} < \infty, \quad \text{where } \frac{1}{q'} + \frac{1}{q} = 1$$

(with the usual modification if $q' = \infty$). If f_θ is a bounded functional, the Hahn–Banach theorem and the preceding arguments imply

$$(1.4) \quad \overline{\text{Im } T_\theta} = \text{Ker } f_\theta.$$

On the other hand, if $f_\theta \notin l_q(\mu)^*$, then

$$(1.5) \quad \overline{\text{Im } T_\theta} = l_q(\mu).$$

We show that if $\theta \in (\beta_\infty, \alpha_0)$, then the functional f_θ is bounded and $\text{Im } T_\theta$ is closed. Let $\beta_\infty + \varepsilon < \theta < \alpha_0 - \varepsilon$ for some $\varepsilon > 0$. By (0.8), we have

$$\sup_{k \geq 0} \frac{\mu_k}{\mu_{k+n}} \leq C_1 2^{-n(\alpha_0 - \varepsilon)}, \quad \sup_{k \leq 0} \frac{\mu_k}{\mu_{k-n}} \leq C_2 2^{n(\beta_\infty + \varepsilon)} \quad (n = 1, 2, \dots),$$

whence

$$(1.6) \quad \mu_n^{-1} \leq \mu_0^{-1} C_1 2^{-n(\alpha_0 - \varepsilon)}, \quad \mu_{-n}^{-1} \leq \mu_0^{-1} C_2 2^{n(\beta_\infty + \varepsilon)} \quad (n = 1, 2, \dots).$$

Therefore,

$$\sum_{n \in \mathbb{Z}} 2^{n\theta} \mu_n^{-1} \leq \mu_0^{-1} C_1 \sum_{n=0}^{\infty} 2^{n(\theta - \alpha_0 + \varepsilon)} + \mu_0^{-1} C_2 \sum_{n=1}^{\infty} 2^{-n(\theta - \beta_\infty - \varepsilon)} < \infty,$$

i.e., (1.3) is true for every $1 \leq q < \infty$, yielding $f_\theta \in l_q(\mu)^*$. In order to prove that $\text{Im } T_\theta$ is closed, we represent $l_q(\mu)$ as follows:

$$l_q(\mu) = E_- + E_0 + E_+,$$

where $E_- = [\{e_n\}_{n \leq -1}]_{q, \mu}$, $E_0 = [\{e_0\}]_{q, \mu}$, and $E_+ = [\{e_n\}_{n \geq 1}]_{q, \mu}$ (here $[F]_{q, \mu}$ is the closed linear span of a set F in the space $l_q(\mu)$). Since

$$\text{Im } T_\theta = T_\theta(E_-) + T_\theta(E_0) + T_\theta(E_+),$$

$T_\theta(E_-) \cap T_\theta(E_+) = \{0\}$, and the space $T_\theta(E_0)$ is one-dimensional, we see that it suffices to prove that $T_\theta(E_-)$ and $T_\theta(E_+)$ are closed.

Since $\beta_\infty < \theta < \alpha_0$, and $r(SP_-) = 2^{\beta_\infty}$, $r(S^{-1}P_+) = 2^{-\alpha_0}$ by Lemma 1, it follows that the operators $SP_- - 2^\theta I$ and $S^{-1}P_+ - 2^{-\theta} I$ map $l_q(\mu)$ onto $l_q(\mu)$ isomorphically. Therefore, the image $T_\theta(E_-) = (SP_- - 2^\theta I)(E_-)$ of the closed set E_- is also closed. Similarly, $T_\theta(E_+)$ is closed because S is an isomorphism of $l_q(\mu)$ and $T_\theta(E_+) = -2^\theta S(S^{-1}P_+ - 2^{-\theta} I)(E_+)$.

Furthermore, it is easily seen that T_θ is injective for $\theta \in (0, \alpha_0) \cup (\beta_\infty, 1)$. Indeed, the relation $T_\theta a = 0$ means that $a_{n-1} = 2^\theta a_n$, i.e., $a_n = 2^{-n\theta} a_0$ ($n \in \mathbb{Z}$). Since $a \in l_q(\mu)$, we must have

$$(1.7) \quad |a_0|^q \sum_{k \in \mathbb{Z}} 2^{-kq\theta} \mu_k^q < \infty.$$

If, for instance, $\theta < \alpha_0$, then the first inequality in (1.6) with $0 < \varepsilon < \alpha_0 - \theta$ shows that (1.7) is true only if $a_0 = 0$, i.e., $a = 0$. But if $\theta > \beta_\infty$, we can argue similarly by using the second inequality in (1.6) with $0 < \varepsilon < \theta - \beta_\infty$.

Thus, for $\beta_\infty < \theta < \alpha_0$ the operator T_θ maps $l_q(\mu)$ isomorphically onto the subspace of codimension 1 consisting of all $(a_k) \in l_q(\mu)$ that satisfy (1.1).

Next, suppose $\theta \notin (0, \alpha] \cup [\beta, 1)$ and $T_\theta : l_q(\mu) \rightarrow \text{Im } T_\theta$ is an isomorphism. Then there exists $c > 0$ such that

$$(1.8) \quad \|T_\theta a\|_{l_q(\mu)} \geq c \|a\|_{l_q(\mu)}$$

for all $a \in l_q(\mu)$.

For arbitrary (for the moment) $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, put $a = (I + 2^{-\theta}S + \dots + 2^{-n\theta}S^n)^2 e_k$. Direct inspection shows that $a \geq n2^{-n\theta}e_{k+n}$, whence

$$(1.9) \quad \|a\|_{l_q(\mu)} \geq n2^{-n\theta} \mu_{k+n}.$$

We estimate the quantity $\|T_\theta^2 a\|_{l_q(\mu)}$ from above. First,

$$\begin{aligned} T_\theta^2 (I + 2^{-\theta}S + \dots + 2^{-n\theta}S^n)^2 \\ = 2^\theta (S - 2^\theta I)(2^{-(n+1)\theta}S^{n+1} - I)(I + 2^{-\theta}S + \dots + 2^{-n\theta}S^n) \\ = 2^{2\theta}I - 2^{1-(n-1)\theta}S^{n+1} + 2^{-2n\theta}S^{2n+2}. \end{aligned}$$

Consequently,

$$T_\theta^2 a = 2^{2\theta}e_k - 2^{1-(n-1)\theta}e_{n+k+1} + 2^{-2n\theta}e_{2n+k+2},$$

and inequalities (0.7) imply that

$$\begin{aligned} \|T_\theta^2 a\|_{l_q(\mu)} &= (2^{2\theta}\mu_k^q - 2^q 2^{-(n-1)q\theta}\mu_{n+k+1}^q + 2^{-2qn\theta}\mu_{2n+k+2}^q)^{1/q} \\ &\leq 4\mu_k + 8 \cdot 2^{-n\theta}\mu_{n+k} + 4 \cdot 2^{-2n\theta}\mu_{2n+k}. \end{aligned}$$

Therefore,

$$\|T_\theta^2 a\|_{l_q(\mu)} - 8 \cdot 2^{-n\theta}\mu_{n+k} \leq 4(\mu_k + 2^{-2n\theta}\mu_{2n+k}) \leq 8 \max(\mu_k, 2^{-2n\theta}\mu_{2n+k}).$$

Now we choose n with $nc^2 > 16$. Then (1.8) and (1.9) imply yield

$$\|T_\theta^2 a\|_{l_q(\mu)} \geq c^2 n 2^{-n\theta} \mu_{n+k},$$

whence

$$2^{-n\theta}\mu_{n+k} < \max(\mu_k, 2^{-2n\theta}\mu_{2n+k}),$$

or, equivalently,

$$(1.10) \quad \nu_{n+k} < \max(\nu_k, \nu_{2n+k}) \quad (k \in \mathbb{Z})$$

with $\nu_n = 2^{-n\theta}\mu_n$.

Since $\theta < \beta$ by assumption, for the number n chosen above we can find $k \in \mathbb{Z}$ with $\mu_{n+k} > 2^{n\theta}\mu_k$, i.e., $\nu_{k+n} > \nu_k$. Therefore, $\nu_{k+2n} > \nu_{k+n}$ by (1.10). Substituting $k+n$ for k in (1.10) and arguing in the same way, we obtain $\nu_{k+3n} > \nu_{k+2n}$, and so on. Thus, the sequence $(\nu_{k+rn})_{r=0}^\infty$ is monotone increasing.

Let $j \geq k$ and $m \geq n$. We find $0 \leq r_1 \leq r_2$ with

$$k + (r_1 - 1)n \leq j \leq k + r_1 n, \quad k + r_2 n \leq j + m \leq k + (r_2 + 1)n.$$

Then

$$\frac{\mu_{j+m}}{\mu_j} \geq \frac{\mu_{k+r_2n}}{\mu_{k+r_1n}} = \frac{2^{r_2n\theta}\nu_{k+r_2n}}{2^{r_1n\theta}\nu_{k+r_1n}} \geq 2^{n(r_2-r_1)\theta}.$$

Since $m - 2n \leq (r_2 - r_1)n$, we see that

$$\frac{\mu_{j+m}}{\mu_j} \geq 2^{(m-2n)\theta} \quad (j \geq k, m \geq n),$$

or

$$\sup_{j \geq k} \frac{\mu_j}{\mu_{j+m}} \leq C 2^{-m\theta} \quad (m \geq n).$$

But by (0.7), for $k > 0$ and $0 \leq j \leq k$ we have

$$\frac{\mu_j}{\mu_{j+m}} \leq \frac{\mu_k}{\mu_{k+m}} \frac{\mu_{k+m}}{\mu_{j+m}} \leq 2^k \frac{\mu_k}{\mu_{k+m}},$$

implying

$$\sup_{j \geq 0} \frac{\mu_j}{\mu_{j+m}} \leq C' 2^{-m\theta} \quad (m \geq n).$$

Taking (0.8) into account, we obtain

$$(1.11) \quad \theta \leq \alpha_0.$$

The inequality

$$(1.12) \quad \theta \geq \beta_\infty$$

is proved similarly. Indeed, for the element $b = (I + 2^{-\theta}S + \dots + 2^{-n\theta}S^n)^2 e_{k-2n}$ we have

$$(1.13) \quad \|b\|_{l_q(\mu)} \geq n 2^{-n\theta} \mu_{k-n}.$$

Since

$$T_\theta^2 b = 2^{2\theta} e_{k-2n} - 2^{1-(n-1)\theta} e_{k-n+1} + 2^{-2n\theta} e_{k+2},$$

by (0.7) we have

$$\begin{aligned} \|T_\theta^2 b\|_{l_q(\mu)} &\leq 2^{2\theta} \mu_{k-2n} + 4 \cdot 2^{-n\theta} \mu_{k-n+1} + 2^{-2n\theta} \mu_{k+2} \\ &\leq 4\mu_{k-2n} + 8 \cdot 2^{-n\theta} \mu_{k-n} + 4 \cdot 2^{-2n\theta} \mu_k. \end{aligned}$$

Again, if $nc^2 > 16$, then (1.8) and (1.13) imply

$$2^{-n\theta} \mu_{k-n} < \max(\mu_{k-2n}, 2^{-2n\theta} \mu_k),$$

so that

$$(1.14) \quad \nu_{k-n} < \max(\nu_k, \nu_{k-2n}) \quad (k \in \mathbb{Z}),$$

where $\nu_n = 2^{-n\theta} \mu_n$, as before. If $\theta > \alpha$, then for n chosen above there exists $k \in \mathbb{Z}$ with $\mu_{k-n} > 2^{-n\theta} \mu_k$. Consequently, $\nu_k < \nu_{k-n}$. Applying (1.14), we obtain $\nu_{k-n} < \nu_{k-2n}$. Repeating the arguments, we show that the sequence $(\nu_{k-rn})_{r=0}^\infty$ is monotone increasing.

If $j \geq -k$ and $m \geq n$, there exist $0 \leq r_1 \leq r_2$ with

$$k - (r_1 + 1)n \leq -j \leq k - r_1 n, \quad k - r_2 n \leq -j - m \leq k - (r_2 - 1)n.$$

Then

$$\frac{\mu_{-j}}{\mu_{-m-j}} \leq \frac{\mu_{k-r_1n}}{\mu_{k-r_2n}} = \frac{2^{-r_1n\theta}\nu_{k-r_1n}}{2^{-r_2n\theta}\nu_{k-r_2n}} \leq 2^{n(r_2-r_1)\theta},$$

and since $(r_2 - r_1)n \leq m + 2n$, we obtain

$$\sup_{j \geq -k} \frac{\mu_{-j}}{\mu_{-j-m}} \leq C 2^{m\theta} \quad (m \geq n).$$

As before, from this we deduce the estimate

$$\sup_{j \geq 0} \frac{\mu_{-j}}{\mu_{-j-m}} \leq C' 2^{m\theta} \quad (m \geq n).$$

This implies (1.12) by (0.8).

Summarizing, we see that the operator $T_\theta : l_q(\mu) \rightarrow \text{Im } T_\theta$ is an isomorphism for $\theta \in (0, \alpha) \cup (\beta_\infty, \alpha_0) \cup (\beta, 1)$, and it is not an isomorphism for $\theta \in (\alpha, \beta_\infty) \cup (\alpha_0, \beta)$ by (1.11) and (1.12). Furthermore, the operator T_θ is Fredholm with index 0 for $\theta \in (0, \alpha) \cup (\beta, 1)$ and it is Fredholm with index -1 for $\theta \in (\beta_\infty, \alpha_0)$. Since the set of all Fredholm operators with fixed index is open, and for $\theta = \alpha$ and $\theta = \beta$ the operator T_θ is not an isomorphism onto $l_q(\mu)$, the lemma is proved by (1.4) and (1.5). $\square$

Remark 1. Lemma 2 and its proof show that, under the condition $\beta_\infty \leq \alpha_0$, the operator T_θ is an isomorphic embedding of $l_q(\mu)$ into $l_q(\mu)$ if and only if the image of T_θ is closed. Indeed, since T_θ is injective for $\theta \in (0, \alpha_0) \cup (\beta_\infty, 1)$, it suffices to consider the case where $\beta_\infty = \alpha_0$ and show that T_{α_0} must be injective if its image is closed. But otherwise T_θ is a Fredholm operator of constant index in some small neighborhood of α_0 ; since it is injective for $\theta \neq \alpha_0$, it is an isomorphic embedding of $l_q(\mu)$ into $l_q(\mu)$. However, its index can only be equal to zero, i.e., $T_\theta(l_q(\mu)) = l_q(\mu)$. In particular, the index of T_{α_0} is also equal to zero, whence $T_{\alpha_0}(l_q(\mu))$ is a closed subspace of codimension 1 in $l_q(\mu)$. But this implies (1.3), consequently, (1.7) may only be fulfilled if $a = 0$, that is, T_{α_0} is injective, a contradiction.

The following simple statement is contained in fact in [15, Lemma 3.2]. We include it for completeness.

Lemma 3. *For every Banach couple (Y_0, Y_1) , every $y \in Y_0 + Y_1$, and $t \in (0, 1]$ we have*

$$\frac{1}{4}\mathcal{K}(t, y; Y_0, Y_1) \leq \mathcal{K}(t, y; Y_0 + Y_1, Y_1) \leq \mathcal{K}(t, y; Y_0, Y_1).$$

Proof. It suffices to verify the left-hand inequality. By the definition of the $\mathcal{K}$ -functional, for every $t > 0$ there is a representation $y = u^t + v^t$ with $u^t \in Y_0 + Y_1$, $v^t \in Y_1$, and

$$\mathcal{K}(t, y; Y_0 + Y_1, Y_1) \geq \frac{1}{2}(\|u^t\|_{Y_0 + Y_1} + t\|v^t\|_{Y_1}).$$

For a similar reason, $u^t = u_0^t + u_1^t$, where $u_0^t \in Y_0$, $u_1^t \in Y_1$ and

$$\|u^t\|_{Y_0 + Y_1} \geq \frac{1}{2}(\|u_0^t\|_{Y_0} + \|u_1^t\|_{Y_1}).$$

Since $t \leq 1$, it follows that

$$\begin{aligned} \mathcal{K}(t, y; Y_0 + Y_1, Y_1) &\geq \frac{1}{4}(\|u_0^t\|_{Y_0} + t\|u_1^t + v^t\|_{Y_1}) \\ &\geq \frac{1}{4}\mathcal{K}(t, y; Y_0, Y_1). \end{aligned} \quad \square$$

Lemma 4. *For every Banach couple (X_0, X_1) and arbitrary $0 < \theta < 1$ and $1 \leq q \leq \infty$ we have*

$$(1.15) \quad (X_0 \cap X_1, X_1)_{\theta, q} = (X_0, X_1)_{\theta, q} \cap X_1$$

and

$$(1.16) \quad (X_0, X_0 \cap X_1)_{\theta, q} = (X_0, X_1)_{\theta, q} \cap X_0.$$

Proof. Since $(A_0, A_1)_{\theta, q} = (A_1, A_0)_{1-\theta, q}$, it suffices to verify (1.15). The embedding

$$(X_0 \cap X_1, X_1)_{\theta, q} \subset (X_0, X_1)_{\theta, q} \cap X_1$$

is obvious. Thus, the lemma will be proved if we show that

$$(1.17) \quad \mathcal{K}(t, x; X_0 \cap X_1, X_1) \leq 2(\mathcal{K}(t, x; X_0, X_1) + \min(1, t)\|x\|_{X_1})$$

for all $x \in X_1$ and $t > 0$.

If $0 < t \leq 1$, then $\mathcal{K}(t, x; X_0 \cap X_1, X_1) = t\|x\|_{X_1}$, and (1.17) is fulfilled. Now, let $t > 1$. For $x \in X_1$ and $\varepsilon > 0$, we find a decomposition $x = x_0 + x_1$, $x_i \in X_i$ ($i = 0, 1$), such that

$$\mathcal{K}(2t, x; X_0, X_1) \geq (1 - \varepsilon)(\|x_0\|_{X_0} + 2t\|x_1\|_{X_1}).$$

Then $x_0 \in X_0 \cap X_1$. Since the $\mathcal{K}$ -functional is concave in t , the preceding inequality implies

$$\begin{aligned} \mathcal{K}(t, x; X_0 \cap X_1, X_1) &\leq \max(\|x_0\|_{X_0}, \|x_0\|_{X_1}) + t\|x_1\|_{X_1} \\ &\leq \|x_0\|_{X_0} + (t + 1)\|x_1\|_{X_1} + \|x\|_{X_1} \\ &\leq (1 - \varepsilon)^{-1}\mathcal{K}(2t, x; X_0, X_1) + \|x\|_{X_1} \\ &\leq \frac{2}{1 - \varepsilon}(\mathcal{K}(t, x; X_0, X_1) + \|x\|_{X_1}). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (1.17) follows. $\square$

A statement close to the above was obtained in [7].

§2. REAL INTERPOLATION FOR COUPLES OF INTERSECTIONS

In what follows, (X_0, X_1) is a Banach couple with $X_0 \cap X_1$ dense both in X_0 and in X_1 , and $X_{\theta, q} = (X_0, X_1)_{\theta, q}$ ($0 < \theta < 1$, $1 \leq q < \infty$). Let ψ be a linear functional with $\psi \in (X_0 \cap X_1)^*$, and let α , β , α_0 , and β_∞ be the dilation indices of the function $k(t) = \mathcal{K}(t, \psi, X_0^*, X_1^*)$.

We denote $N = \text{Ker } \psi$ and $N_{\theta, q} = (N_0, N_1)_{\theta, q}$, where N_i is the space N endowed with the norm of X_i ($i = 0, 1$). Clearly, $N_{\theta, q} \subset X_{\theta, q}$ and $\|x\|_{X_{\theta, q}} \leq \|x\|_{N_{\theta, q}}$ ($x \in N_{\theta, q}$).

Theorem 1. *If*

$$(2.1) \quad \theta \in (0, \alpha) \cup (\beta_\infty, \alpha_0) \cup (\beta, 1),$$

then the norms of the spaces $N_{\theta, q}$ and $X_{\theta, q}$ are equivalent on N . Under the condition $\beta_\infty \leq \alpha_0$, the converse is also true: if the norms of $N_{\theta, q}$ and $X_{\theta, q}$ are equivalent on N , then (2.1) is fulfilled.

Moreover, if $\theta \in (0, \alpha) \cup (\beta, 1)$, then $N_{\theta, q}$ is dense in $X_{\theta, q}$, and if $\theta \in (\beta_\infty, \alpha_0)$, then $N_{\theta, q}$ is dense in some subspace of codimension 1 in $X_{\theta, q}$.

Proof. First, suppose (2.1) is fulfilled. In the previous notation, by Lemma 2 we have

$$(2.2) \quad \|a\|_{l_q(\mu)} \leq B\|T_\theta a\|_{l_q(\mu)}, \quad a \in l_q(\mu),$$

with some $B > 0$. Furthermore, if a) $\theta \in (0, \alpha) \cup (\beta, 1)$, then $\text{Im } T_\theta = l_q(\mu)$, and if b) $\theta \in (\beta_\infty, \alpha_0)$, then $\text{Im } T_\theta = \text{Ker } f_\theta$, where the functional $f_\theta \in l_q(\mu)^*$ is defined by

$$f_\theta(a) = \sum_{n \in \mathbb{Z}} 2^{n\theta} a_n, \quad a = (a_n) \in l_q(\mu).$$

In case b) condition (1.3) is fulfilled, i.e.,

$$\sum_{k \in \mathbb{Z}} 2^{kq'\theta} \mathcal{K}(2^{-k}, \psi)^{q'} = C_0^{q'} < \infty, \quad \frac{1}{q'} + \frac{1}{q} = 1$$

(here and below $\mathcal{K}(t, \psi) := \mathcal{K}(t, \psi, X_0^*, X_1^*)$, $\mathcal{J}(t, x) := \mathcal{J}(t, x, X_0, X_1)$); therefore, ψ can be extended up to a functional $\tilde{\psi} \in X_{\theta, q}^*$ [16, Theorem 3.7.2].

Now, let $x \in X_{\theta,q}$ with $\|x\|_{X_{\theta,q}} = 1$. Then there is a sequence $\{x_k\} \subset X_0 \cap X_1$ such that

$$(2.3) \quad x = \sum_{k \in \mathbb{Z}} 2^{k\theta} x_k, \quad \|(\mathcal{J}(2^k, x_k))_k\|_q \leq 2.$$

Since $\psi \in (X_0 \cap X_1)^*$ by assumption, formula (0.6) implies

$$|\psi(x_k)| \leq \mathcal{K}(2^{-k}, \psi) \mathcal{J}(2^k, x_k) = \mu_k^{-1} \mathcal{J}(2^k, x_k) \quad (k \in \mathbb{Z}).$$

By (2.3), it follows that

$$(2.4) \quad \left(\sum_{k \in \mathbb{Z}} |\psi(x_k)|^q \mu_k^q \right)^{1/q} \leq \|(\mathcal{J}(2^k, x_k))_k\|_q \leq 2,$$

i.e., $(\psi(x_k)) \in l_q(\mu)$.

In case b) we assume additionally that $x \in \text{Ker } \tilde{\psi}$. Since the series in (2.3) converges in $X_{\theta,q}$ and $\tilde{\psi} \in X_{\theta,q}^*$, we have $\sum_{k \in \mathbb{Z}} 2^{\theta k} \psi(x_k) = 0$. Now, Lemma 2 shows that $(\psi(x_k)) \in \text{Im } T_\theta$ both in case a) and in case b). Thus, there exists a sequence $a \in l_q(\mu)$ such that $T_\theta a = (\psi(x_k))$ and, moreover, $\|a\|_{l_q(\mu)} \leq 2B$ by (2.2) and (2.4).

Using (0.6) once again, we find a sequence $\{u'_n\} \in X_0 \cap X_1$ such that $\mathcal{J}(2^n, u'_n) \leq 2$ and $\psi(u'_n) = \mu_n^{-1}$ ($n \in \mathbb{Z}$). Putting $u_n = a_n \mu_n u'_n$, we see that $u_n \in X_0 \cap X_1$, $\mathcal{J}(2^n, u_n) \leq 2|a_n| \mu_n$, and $\psi(u_n) = a_n$ ($n \in \mathbb{Z}$). Consequently,

$$(2.5) \quad \|(\mathcal{J}(2^n, u_n))_n\|_q \leq 2\|a\|_{l_q(\mu)} \leq 4B.$$

In particular, it follows immediately that the series $\sum_{k \in \mathbb{Z}} 2^{\theta k} u_k$ converges in $X_0 + X_1$. Indeed,

$$\begin{aligned} \left\| \sum_{n \geq N} 2^{\theta n} u_n \right\|_{X_1} &\leq \sum_{n \geq N} 2^{(\theta-1)n} 2^n \|u_n\|_{X_1} \\ &\leq \left(\sum_{n \geq N} (2^n \|u_n\|_{X_1})^q \right)^{1/q} \left(\sum_{n \geq N} 2^{n(\theta-1)q'} \right)^{1/q'} \\ &\leq C \left(\sum_{n \geq N} (\mathcal{J}(2^n, u_n))^q \right)^{1/q} \rightarrow 0 \quad \text{as } N \rightarrow \infty, \end{aligned}$$

and similarly

$$\begin{aligned} \left\| \sum_{n \leq -N} 2^{\theta n} u_n \right\|_{X_0} &\leq \sum_{n \leq -N} 2^{\theta n} \|u_n\|_{X_0} \\ &\leq C' \left(\sum_{n \leq -N} (\mathcal{J}(2^n, u_n))^q \right)^{1/q} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Therefore, putting $v_n = u_{n-1} - 2^\theta u_n$, we obtain $\sum_{k \in \mathbb{Z}} 2^{\theta k} v_k = 0$, whence

$$(2.6) \quad x = \sum_{k \in \mathbb{Z}} 2^{\theta k} y_k$$

with $y_k = x_k - v_k$, by (2.3). Furthermore,

$$\psi(y_k) = \psi(x_k) - \psi(u_{k-1}) + 2^\theta \psi(u_k) = (T_\theta a)_k - a_{k-1} + 2^\theta a_k = 0,$$

i.e., $y_k \in N$ ($k \in \mathbb{Z}$). Since the series in (2.6) converges in $X_{\theta,q}$, the second part of the theorem is proved.

Next, by (2.3) and (2.5) we obtain

$$\begin{aligned}
\|(\mathcal{J}(2^n, y_n))_n\|_q &\leq \|(\mathcal{J}(2^n, x_n))_n\|_q + \|(\mathcal{J}(2^n, v_n))_n\|_q \\
&\leq 2 + \left(\sum_{k \in \mathbb{Z}} (\max(\|u_{k-1}\|_{X_0}, 2^k \|u_{k-1}\|_{X_1})^q) \right)^{1/q} \\
&\quad + 2 \left(\sum_{k \in \mathbb{Z}} (\max(\|u_k\|_{X_0}, 2^k \|u_k\|_{X_1})^q) \right)^{1/q} \\
&\leq 2 + 16B.
\end{aligned}$$

Consequently, for $x \in N$ formula (2.6) implies $\|x\|_{N_{\theta,q}} \leq 16B + 2$. Thus, the norms of $X_{\theta,q}$ and $N_{\theta,q}$ are equivalent on N .

We prove the converse under the condition $\beta_\infty \leq \alpha_0$. By assumption, there exists $D > 0$ such that

$$(2.7) \quad \|x\|_{N_{\theta,q}} \leq D \|x\|_{X_{\theta,q}}$$

for all $x \in N$. By Lemma 2, it suffices to prove that the operator $T_\theta : l_q(\mu) \rightarrow \text{Im } T_\theta$, $T_\theta = S - 2^\theta I$, is an isomorphism.

Again, we distinguish two cases: a) the functional ψ is not bounded on $X_0 \cap X_1$ in the norm of $X_{\theta,q}$; b) ψ is bounded on $X_0 \cap X_1$ in the norm of $X_{\theta,q}$. In the second case, we let $\tilde{\psi} \in X_{\theta,q}^*$ denote the extension of ψ .

So, let $a = (a_n) \in l_q(\mu)$, $\|a\|_{l_q(\mu)} \leq 1$. As before, there exists a sequence $\{x_n\} \subset X_0 \cap X_1$ ($n \in \mathbb{Z}$) with

$$\mathcal{J}(2^n, x_n) \leq 2|a_n|\mu_n, \quad \psi(x_n) = a_n \quad (n \in \mathbb{Z}).$$

Therefore,

$$\|(\mathcal{J}(2^n, x_n))_n\|_q \leq 2\|a\|_{l_q(\mu)} \leq 2,$$

whence the series

$$(2.8) \quad x = \sum_{k \in \mathbb{Z}} 2^{\theta k} x_k$$

converges in $X_0 + X_1$, $x \in X_{\theta,q}$, and $\|x\|_{X_{\theta,q}} \leq 2$. Moreover, in case b) we have $|\tilde{\psi}(x)| \leq C\|x\|_{X_{\theta,q}} \leq 2C$. Since the series in (2.8) converges in norm in $X_{\theta,q}$, we are led to the conclusion that

$$|f_\theta(a)| = \left| \sum_{k \in \mathbb{Z}} 2^{\theta k} a_k \right| = \left| \sum_{k \in \mathbb{Z}} 2^{\theta k} \psi(x_k) \right| = |\tilde{\psi}(x)| \leq 2C$$

for every $a \in l_q(\mu)$ with $\|a\|_{l_q(\mu)} \leq 1$, i.e., $f_\theta \in l_q(\mu)^*$. By the facts proved above, $f_\theta(a) = 0$ in this case, i.e., $x \in X_{\theta,q} \cap \tilde{N}$, where $\tilde{N} := \text{Ker } \tilde{\psi}$.

In the next lemma, $Z_{\theta,q}$ is the closure of N in $X_{\theta,q}$.

Lemma 5. *We have $Z_{\theta,q} = X_{\theta,q}$ in case a) and $Z_{\theta,q} = X_{\theta,q} \cap \tilde{N}$ in case b).*

Proof. If the functional ψ is not bounded on $X_0 \cap X_1$ in the norm of $X_{\theta,q}$, then N is dense in $X_0 \cap X_1$, and, consequently, also in $X_{\theta,q}$ (relative to the same norm), see [2, Theorem 3.4.2]. Therefore, $Z_{\theta,q} = X_{\theta,q}$.

In case b), if $x \in Z_{\theta,q}$, then $\tilde{\psi}(x) = 0$, and, consequently, $Z_{\theta,q} \subset X_{\theta,q} \cap \tilde{N}$. To prove the reverse inclusion, take $x \in X_{\theta,q}$ with $\tilde{\psi}(x) = 0$. There exists a sequence $\{x_n\} \subset X_0 \cap X_1$ such that $x_n \rightarrow x$ in $X_{\theta,q}$. Since $\psi \in (X_0 \cap X_1)^*$, we have $x_n = y_n + \lambda_n z$, where $y_n \in N$, $z \in X_0 \cap X_1$, $\psi(z) = 1$, and $\lambda_n \in \mathbb{R}$. Then $\psi(x_n) = \lambda_n \psi(z) = \lambda_n \rightarrow \tilde{\psi}(x) = 0$. Therefore, $y_n \rightarrow x$ in $X_{\theta,q}$, whence $x \in Z_{\theta,q}$. $\square$

We continue the proof of the theorem. We remind the reader that the vector x defined by (2.8) with $\psi(x_k) = a_k$ belongs to $X_{\theta,q}$ in case a) and to $X_{\theta,q} \cap \tilde{N}$ in case b). In both cases, Lemma 5 implies the existence of a sequence $\{z_n\} \subset N$ such that

$$(2.9) \quad z_n \rightarrow x \text{ in } X_{\theta,q}, \quad \|z_{n+1} - z_n\|_{X_{\theta,q}} \leq 2^{-n}, \quad n = 1, 2, \dots$$

It follows that

$$(2.10) \quad x = \sum_{n=0}^{\infty} (z_{n+1} - z_n), \quad z_0 = 0,$$

where the series converges absolutely in $X_{\theta,q}$. Since $\|z_{n+1} - z_n\|_{N_{\theta,q}} \leq 2^{-n}D$ by (2.7), we see that for every $n = 0, 1, 2, \dots$ there exists a sequence $\{z_{n,i}\}_{i \in \mathbb{Z}} \subset N$ with

$$(2.11) \quad z_{n+1} - z_n = \sum_{i \in \mathbb{Z}} 2^{i\theta} z_{n,i}, \quad \|(\mathcal{J}(2^i, z_{n,i}))_i\|_q \leq 2^{1-n}D.$$

Consequently,

$$\begin{aligned} \sum_{n=0}^{\infty} \|z_{n,i}\|_{X_0 \cap X_1} &\leq \max(1, 2^{-i}) \sum_{n=0}^{\infty} \mathcal{J}(2^i, z_{n,i}) \\ &\leq \max(1, 2^{-i}) \sum_{n=0}^{\infty} \|(\mathcal{J}(2^i, z_{n,i}))_i\|_q \\ &\leq 4D \max(1, 2^{-i}), \end{aligned}$$

and the series

$$(2.12) \quad y_i = \sum_{n=0}^{\infty} z_{n,i}$$

converges absolutely in $X_0 \cap X_1$ for every $i \in \mathbb{Z}$. But $z_{n,i} \in N$ and N is the kernel of the functional $\psi \in (X_0 \cap X_1)^*$; therefore, $y_i \in N$, $i \in \mathbb{Z}$. Moreover, (2.11) and the Minkowski inequality show that

$$\begin{aligned} \|(\mathcal{J}(2^i, y_i))_i\|_q &\leq \left\{ \sum_{i \in \mathbb{Z}} \left(\sum_{n=0}^{\infty} \mathcal{J}(2^i, z_{n,i}) \right)^q \right\}^{1/q} \\ &\leq \sum_{n=0}^{\infty} \left(\sum_{i \in \mathbb{Z}} (\mathcal{J}(2^i, z_{n,i}))^q \right)^{1/q} \leq 4D. \end{aligned}$$

Therefore, the series $\sum_{i \in \mathbb{Z}} 2^{\theta i} y_i$ converges in $X_0 + X_1$ to a vector $y \in X_{\theta,q}$. We show that $y = x$. By (2.10)–(2.12), it suffices to prove that the double series

$$\sum_{n=0}^{\infty} \sum_{i \in \mathbb{Z}} 2^{\theta i} z_{n,i}$$

converges absolutely in $X_0 + X_1$. To do this, we apply the Hölder inequality and (2.12), obtaining

$$\begin{aligned} \sum_{i=0}^{\infty} 2^{\theta i} \|z_{n,i}\|_{X_1} &\leq \sum_{i=0}^{\infty} \mathcal{J}(2^i, z_{n,i}) 2^{i(\theta-1)} \\ &\leq \left(\sum_{i=0}^{\infty} 2^{i(\theta-1)q'} \right)^{1/q'} \|(\mathcal{J}(2^i, z_{n,i}))_i\|_q \leq C_{\theta,q} 2^{1-n}D \end{aligned}$$

and

$$\sum_{i=-\infty}^{-1} 2^{\theta i} \|z_{n,i}\|_{X_0} \leq \left(\sum_{i=-\infty}^{-1} 2^{i\theta q'} \right)^{1/q'} \|(\mathcal{J}(2^i, z_{n,i}))_i\|_q \leq C'_{\theta,q} 2^{1-n} D,$$

whence

$$\sum_{n=0}^{\infty} \sum_{i \in \mathbb{Z}} 2^{\theta i} \|z_{n,i}\|_{X_0+X_1} \leq 4D \max(C_{\theta,q}, C'_{\theta,q}).$$

Thus, along with (2.8), both in case a) and in case b) we have a representation of the form

$$(2.13) \quad x = \sum_{n \in \mathbb{Z}} 2^{\theta n} y_n, \quad \text{where } y_n \in N \text{ and } \|(\mathcal{J}(2^n, y_n))_n\|_q \leq 4D.$$

Next, as in [3], consider the vectors

$$u_n = x_n - y_n \text{ and } v_n = \sum_{k=n+1}^{\infty} 2^{(k-n-1)\theta} u_k.$$

Then

$$(2.14) \quad \psi(u_n) = \psi(x_n) = a_n, \quad n \in \mathbb{Z},$$

and from (2.8) and (2.13) it follows that

$$(2.15) \quad \sum_{n \in \mathbb{Z}} 2^{\theta n} u_n = 0, \quad \|(\mathcal{J}(2^n, u_n))_n\|_q \leq 4D + 2.$$

Moreover,

$$2^n \|v_n\|_{X_1} \leq \sum_{k=n+1}^{\infty} 2^{(k-n-1)(\theta-1)} \|u_k\|_{X_1} 2^k,$$

and by (2.15) we have

$$\|v_n\|_{X_0} \leq \left\| \sum_{k=-\infty}^n 2^{(k-n-1)\theta} u_k \right\|_{X_0} \leq \sum_{k=-\infty}^n 2^{(k-n-1)\theta} \|u_k\|_{X_0}.$$

Now, the elementary inequality $\max(a, b) \leq a + b$ ($a, b > 0$) combined with the Minkowski inequality and (2.15) yields

$$\begin{aligned} & \|(\mathcal{J}(2^n, v_n))_n\|_q \\ & \leq \left\{ \sum_{n \in \mathbb{Z}} \left(\sum_{k=n+1}^{\infty} 2^{(k-n-1)(\theta-1)} \mathcal{J}(2^k, u_k) + \sum_{k=-\infty}^n 2^{(k-n-1)\theta} \mathcal{J}(2^k, u_k) \right)^q \right\}^{1/q} \\ & = \left\{ \sum_{n \in \mathbb{Z}} \left(\sum_{l=0}^{\infty} 2^{l(\theta-1)} \mathcal{J}(2^{l+n+1}, u_{l+n+1}) + \sum_{l=-\infty}^{-1} 2^{l\theta} \mathcal{J}(2^{l+n+1}, u_{l+n+1}) \right)^q \right\}^{1/q} \\ & \leq \left(\sum_{l=0}^{\infty} 2^{l(\theta-1)} + \sum_{l=-\infty}^{-1} 2^{l\theta} \right) \|(\mathcal{J}(2^n, u_n))_n\|_q \leq C_{\theta}(4D + 2). \end{aligned}$$

So, $v_n \in X_0 \cap X_1$, $v = \sum_{n \in \mathbb{Z}} 2^{\theta n} v_n \in X_{\theta,q}$, and $\|v\|_{X_{\theta,q}} \leq C_{\theta}(4D + 2)$. By (0.6), we have $\psi(v_n) \leq \mu_n^{-1} \mathcal{J}(2^n, v_n)$. Consequently, the element $b = (\psi(v_n))_n$ belongs to $l_q(\mu)$ and $\|b\|_{l_q(\mu)} \leq C_{\theta}(4D + 2)$. Furthermore, by (2.14) we have

$$\begin{aligned} (T_{\theta} b)_n &= b_{n-1} - 2^{\theta} b_n = \psi(v_{n-1}) - 2^{\theta} \psi(v_n) \\ &= \psi \left(\sum_{k=n}^{\infty} 2^{(k-n)\theta} u_k - \sum_{k=n+1}^{\infty} 2^{(k-n-1)\theta} u_k \right) = \psi(u_n) = a_n \end{aligned}$$

for all $n \in \mathbb{Z}$.

Thus, the operator $T_\theta : l_q(\mu) \rightarrow l_q(\mu)$ is closed. Moreover, $\text{Im } T_\theta = l_q(\mu)$ in case a) and $\text{Im } T_\theta = \text{Ker } f_\theta$ in case b). Since $\beta_\infty \leq \alpha_0$ by assumption, Remark 1 implies that T_θ is an isomorphic embedding, and the theorem is proved. $\square$

Now, we present some corollaries. The first of them is a direct consequence of the proof of Theorem 1.

Corollary 1. *The following statements are equivalent:*

- 1) *the norms of $X_{\theta,q}$ and $N_{\theta,q}$ are equivalent on N , and ψ extends up to a functional $\tilde{\psi} \in X_{\theta,q}^*$;*
- 2) *$\theta \in (\beta_\infty, \alpha_0)$.*

Corollary 2. *Suppose a linear functional φ is defined on some linear space $D \supset X_0 \cap X_1$. Let ψ be the restriction of φ to $X_0 \cap X_1$, and suppose that $\psi \in (X_0 \cap X_1)^*$, $\psi \neq 0$. Let α, β, α_0 , and β_∞ be the dilation indices of the function $k(t) = \mathcal{K}(t, \psi; X_0^*, X_1^*)$. We assume that $\beta_\infty \leq \alpha_0$. If*

$$(2.16) \quad (X_0 \cap \text{Ker } \varphi, X_1 \cap \text{Ker } \varphi)_{\theta,q} = (X_0, X_1)_{\theta,q} \cap \text{Ker } \varphi,$$

then (2.1) is true.

Proof. Relations (2.16) imply, in particular, that the norms of $(X_0, X_1)_{\theta,q}$ and $(X_0 \cap \text{Ker } \varphi, X_1 \cap \text{Ker } \varphi)_{\theta,q}$ are equivalent on $N := \text{Ker } \psi$. On the other hand, the norms of $(X_0 \cap \text{Ker } \varphi, X_1 \cap \text{Ker } \varphi)_{\theta,q}$ and $(N_0, N_1)_{\theta,q}$ coincide on N . Therefore, the norms of $(X_0, X_1)_{\theta,q}$ and $(N_0, N_1)_{\theta,q}$ are equivalent on N , and Theorem 1 applies. $\square$

The last corollary is also a direct consequence of the proof of Theorem 1.

Corollary 3. *Suppose $\psi \in (X_0 \cap X_1)^*$, $\psi \neq 0$, and let $N = \text{Ker } \psi$. If $\theta \in (0, \alpha) \cup (\beta, 1)$, then for every $x \in X_{\theta,q}$ there exists a representation*

$$x = \sum_{k=-\infty}^{\infty} 2^{\theta k} x_k, \quad x_k \in N,$$

with

$$\|(\mathcal{J}(2^k, x_k; X_0, X_1))_k\|_{l_q} \leq C \|x\|_{X_{\theta,q}},$$

where the constant $C > 0$ does not depend on x . A similar statement is true also for $\theta \in (\beta_\infty, \alpha_0)$ if we replace $X_{\theta,q}$ with the intersection $X_{\theta,q} \cap \tilde{N}$, where $\tilde{N}$ is the kernel of the extension $\tilde{\psi}$ of ψ to $X_{\theta,q}$.

Remark. Corollary 3 readily implies the results of [3] that guarantee the identity of the spaces $(X_0, X_1)_{\theta,q}$ and $(N_0, X_1)_{\theta,q}$. In that situation, $\psi \in X_0^*$ and $N_0 := \text{Ker } \psi$ is endowed with the norm of X_0 . First, we observe that presently $k(t) := \mathcal{K}(t, \psi; X_0^*, X_1^*) \leq C$ ($t > 0$), whence $\alpha = \beta_\infty = 0$. Next, $(X_0, X_1)_{\theta,q} \subset N_0 + X_1$ because, if $x \in X_{\theta,q}$ and $x = x_0 + x_1$ with $x_i \in X_i$, then also $x = y_0 + y_1$, where $y_0 = x_0 - \psi(x_0)z \in N_0$, and $y_1 = x_1 + \psi(x_0)z \in X_1$; here z is an element of $X_0 \cap X_1$ satisfying $\psi(z) = 1$. By Corollary 3 and the equivalence of the $\mathcal{K}$ - and $\mathcal{J}$ -methods (see [2, Theorem 3.3.1]), we see that

$$(N_0, X_1)_{\theta,q} = (X_0, X_1)_{\theta,q} \quad \text{if } \theta \in (\beta, 1),$$

and

$$(N_0, X_1)_{\theta,q} = (X_0, X_1)_{\theta,q} \cap \tilde{N} \quad \text{if } \theta \in (0, \alpha_0),$$

where $\tilde{N}$ is the kernel of the extension $\tilde{\psi}$ of ψ to $(X_0, X_1)_{\theta,q}$.

Theorem 1 implies in addition that $(N_0, X_1)_{\theta,q}$ is not closed in $(X_0, X_1)_{\theta,q}$ for $\theta \in [\alpha_0, \beta]$.

Theorem 2. Suppose $\psi \in (X_0 \cap X_1)^*$, $\psi \neq 0$. If $\theta \in (\beta_\infty, \alpha_0)$, then ψ extends by continuity up to a functional $\tilde{\psi}_\theta \in ((X_0, X_1)_{\theta,q})^*$. If $\theta \in (0, \alpha)$ (respectively, $\theta \in (\beta, 1)$), then ψ extends by continuity up to a functional $\tilde{\psi}_\theta \in ((X_0, X_1)_{\theta,q} \cap X_1)^*$ (respectively, up to a functional $\tilde{\psi}_\theta \in ((X_0, X_1)_{\theta,q} \cap X_0)^*$). Moreover, if $\tilde{N}_\theta := \text{Ker } \tilde{\psi}_\theta$, then in the three cases we have

$$(2.17) \quad (X_0 \cap \tilde{N}_\theta, X_1 \cap \tilde{N}_\theta)_{\theta,q} = (X_0, X_1)_{\theta,q} \cap \tilde{N}_\theta.$$

Proof. If $\theta \in (\beta_\infty, \alpha_0)$, then ψ extends up to $\tilde{\psi}_\theta \in ((X_0, X_1)_{\theta,q})^*$ by Corollary 1. We prove (2.17) in this case.

For $\tilde{N} := \tilde{N}_\theta$, the inclusion

$$(X_0 \cap \tilde{N}, X_1 \cap \tilde{N})_{\theta,q} \subset (X_0, X_1)_{\theta,q} \cap \tilde{N}$$

is obvious. So, by Corollary 3, it suffices to prove the inclusion

$$(2.18) \quad X_{\theta,q} \cap \tilde{N} \subset X_0 \cap \tilde{N} + X_1 \cap \tilde{N}.$$

First, we observe that

$$(2.19) \quad X_{\theta,q} \subset X_{\theta,q} \cap X_0 + X_{\theta,q} \cap X_1.$$

Indeed, for every $x \in X_{\theta,q}$ there exists a sequence $\{x_n\} \subset X_0 \cap X_1$ such that

$$x = \sum_{k=-\infty}^{\infty} 2^{\theta k} x_k \quad (\text{convergence in } X_0 + X_1)$$

and $(\mathcal{J}(2^k, x_k))_k \in l_q$. As before, it is easily seen that

$$\sum_{n=-\infty}^{-1} 2^{\theta n} x_n = x_0 \in X_0 \quad (\text{convergence in } X_0)$$

and

$$\sum_{n=0}^{\infty} 2^{\theta n} x_n = x_1 \in X_1 \quad (\text{convergence in } X_1).$$

At the same time, the $\mathcal{J}$ -method definition of $X_{\theta,q}$ yields $x_i \in X_{\theta,q}$ ($i = 0, 1$), and (2.19) follows from the relation $x = x_0 + x_1$.

Passing to the proof of (2.18), we take $z \in X_0 \cap X_1$ with $\psi(z) = 1$. If $x \in X_{\theta,q} \cap \tilde{N}$ and $x = x_0 + x_1$, $x_i \in X_{\theta,q} \cap X_i$ ($i = 0, 1$), we put

$$(2.20) \quad y_0 = x_0 + \tilde{\psi}_\theta(x_1)z, \quad y_1 = x_1 - \tilde{\psi}_\theta(x_1)z.$$

Then $y_i \in X_i \cap \tilde{N}$ ($i = 0, 1$), and (2.18) follows.

Now, let $\theta \in (0, \alpha) \cup (\beta, 1)$. If $\theta \in (0, \alpha)$, we use Lemmas 3 and 4. For this, consider the Banach couple $(X_0 \cap X_1, X_1)$ and the corresponding $\mathcal{K}$ -functional

$$k'(t) := \mathcal{K}(t, \psi; (X_0 \cap X_1)^*, X_1^*) = \mathcal{K}(t, \psi; X_0^* + X_1^*, X_1^*).$$

The dilation indices for the function $k'(t)$ will be denoted by $\alpha', \beta', \alpha'_0$, and β'_∞ . Since $\psi \in (X_0 \cap X_1)^*$, we have $k'(t) \leq C$ ($t > 0$) and, therefore, $\alpha' = \beta'_\infty = 0$. Applying Lemma 3 to the couple (X_0^*, X_1^*) , we obtain $k'(t) \asymp k(t) := \mathcal{K}(t, \psi; X_0^*, X_1^*)$ for $t \in (0, 1]$. So, the definition of the dilation indices implies $\alpha'_0 \geq \alpha$. Thus, $\theta \in (\beta'_\infty, \alpha'_0)$, and by the first statement of the theorem (which has already been proved) we see that ψ extends by continuity up to a functional $\tilde{\psi}_\theta \in ((X_0 \cap X_1, X_1)_{\theta,q})^*$. But the latter space coincides with $((X_0, X_1)_{\theta,q} \cap X_1)^*$ by Lemma 4.

The case where $\theta \in (\beta, 1)$ reduces to the preceding case. We put

$$k''(t) := \mathcal{K}(t, \psi; X_0^*, (X_0 \cap X_1)^*),$$

and denote by $\alpha'', \beta'', \alpha_0'',$ and β_∞'' the dilation indices for k'' . Then $k''(t) \leq Ct$ ($t > 0$) and, therefore, $\alpha_0'' = \beta'' = 1$. Next, we observe that

$$k(t) = tl\left(\frac{1}{t}\right), \quad k''(t) = tm\left(\frac{1}{t}\right),$$

where

$$l(t) := \mathcal{K}(t, \psi; X_1^*, X_0^*), \quad m(t) := \mathcal{K}(t, \psi; (X_0 \cap X_1)^*, X_0^*).$$

It is easily seen that

$$(2.21) \quad \beta + \alpha^l = 1, \quad \beta_\infty'' + \alpha_0^m = 1,$$

where α^l and α_0^m stand for dilation indices of $l(t)$ and $m(t)$, respectively. From the above it is clear that $\alpha_0^m \geq \alpha^l$. But then (2.21) immediately implies the inequality $\beta_\infty'' \leq \beta$.

Thus, if $\theta \in (\beta, 1)$, then $\theta \in (\beta_\infty'', \alpha_0'')$, and again Lemma 4 shows that ψ extends up to a functional $\tilde{\psi}_\theta \in ((X_0, X_1)_{\theta, q} \cap X_0)^*$.

It remains to prove (2.17) for $\theta \in (0, \alpha)$ or $\theta \in (\beta, 1)$. For definiteness, let $\theta \in (0, \alpha)$ (the other case is treated similarly). By Corollary 3, it again suffices to prove (2.18), in which $\tilde{N} := \tilde{N}_\theta$ is the kernel of the extension $\tilde{\psi}_\theta$ of ψ .

If $x \in X_{\theta, q} \cap \tilde{N}$, we use (2.19) to represent x in the form $x = x_0 + x_1$, where $x_i \in X_{\theta, q} \cap X_i$ ($i = 0, 1$). It follows that x_1 and, with it, x_0 belong to the domain of $\tilde{\psi}_\theta$. Consequently, defining y_0 and y_1 by (2.20), we obtain $x = y_0 + y_1$, where $y_i \in X_i \cap \tilde{N}$ ($i = 0, 1$). Thus, we arrive at (2.18) and, therefore, also at (2.17). $\square$

Corollary 4. *Suppose a linear functional φ is defined on a linear space D , $D \supset X_0 \cap X_1$, the restriction ψ of φ to $X_0 \cap X_1$ is a bounded functional on $X_0 \cap X_1$, and $\psi \neq 0$. For every θ satisfying (2.1), let $\tilde{\psi}_\theta$ denote a continuous extension of ψ with domain $\tilde{M}_\theta$ (equal either to $X_{\theta, q}$, or to $X_{\theta, q} \cap X_1$, or to $X_{\theta, q} \cap X_0$), as in Theorem 2. If $\tilde{M}_\theta \subset D$ and $\varphi(x) = \tilde{\psi}_\theta(x)$ for $x \in \tilde{M}_\theta$, then (2.16) is fulfilled.*

Proof. If $\theta \in (0, \alpha) \cup (\beta, 1)$, then, again by Corollary 3, it suffices to show that

$$(2.22) \quad X_{\theta, q} \cap \text{Ker } \varphi \subset X_0 \cap \text{Ker } \varphi + X_1 \cap \text{Ker } \varphi.$$

Let $\theta \in (0, \alpha)$ for definiteness. If $x \in X_{\theta, q} \cap \text{Ker } \varphi$, then (2.19) shows that $x = x_0 + x_1$, where $x_i \in X_{\theta, q} \cap X_i$ ($i = 0, 1$). Then $x_1 \in \tilde{M}_\theta := X_{\theta, q} \cap X_1$. Defining y_0 and y_1 by (2.20), we obtain $x = y_0 + y_1$, hence y_1 and, with it, y_0 belong to $\text{Ker } \varphi$. Thus, (2.22) is proved. The case of $\theta \in (\beta, 1)$ is treated similarly.

Now, let $\theta \in (\beta_\infty, \alpha_0)$. If $X_{\theta, q} \cap \text{Ker } \varphi = X_{\theta, q} \cap \tilde{N}_\theta$, where $\tilde{N}_\theta := \text{Ker } \tilde{\psi}_\theta$, then the result follows from Theorem 2 (or Corollary 3). Suppose $X_{\theta, q} \cap \text{Ker } \varphi \not\supset X_{\theta, q} \cap \tilde{N}_\theta$. Since $\tilde{N}_\theta$ is a subspace of codimension 1 in $X_{\theta, q}$, we have $X_{\theta, q} \subset \text{Ker } \varphi$, i.e., $\varphi(x) = 0$ for all $x \in X_{\theta, q}$. This contradicts the assumptions of the corollary. $\square$

§3. THE $\mathcal{K}$ -FUNCTIONAL ON COUPLES OF INTERSECTIONS

Let (X_0, X_1) be a Banach couple with $X_0 \cap X_1$ dense in X_0 and in X_1 , and let Y_0 be a closed subspace of codimension 1 in X_0 . In [8] a formula connecting $\mathcal{K}(t, x; Y_0, X_1)$ and $\mathcal{K}(t, x; X_0, X_1)$ was found (see also [12, Theorem 2.1]). We show that, under a certain additional assumption, a similar result holds true also for a normed couple of intersections generated by a linear functional.

Let ψ be a linear functional defined on a subspace $M \supset X_0 \cap X_1$, and let $N = \text{Ker } \psi$. Moreover, we assume that the restriction of ψ to $X_0 \cap X_1$ is a bounded nonzero functional on this space. We put $Y_i = X_i \cap N$ ($i = 0, 1$).

For every $y \in Y_0 + Y_1$ and every $t > 0$, there exists an “optimal” decomposition

$$(3.1) \quad y = y_0(t) + y_1(t), \quad y_i(t) \in X_i, \quad i = 0, 1,$$

with the property

$$(3.2) \quad \frac{1}{2}(\|y_0(t)\|_{X_0} + t\|y_1(t)\|_{X_1}) \leq \mathcal{K}(t, y; X_0, X_1) \leq \|y_0(t)\|_{X_0} + t\|y_1(t)\|_{X_1}.$$

Theorem 3. *Suppose $X_0 \cap X_1$ is dense both in X_0 and in X_1 , and for every $y \in Y_0 + Y_1$ an “optimal” decomposition (3.1) can be chosen in such a way that $y_0(t) \in M$ for all $t > 0$. Then*

$$\mathcal{K}(t, y; Y_0, Y_1) \asymp \mathcal{K}(t, y; X_0, X_1) + \frac{|\psi(y_0(t))|}{\mathcal{K}(t^{-1}, \psi; X_0^*, X_1^*)},$$

where the constants of equivalence do not depend on $y \in Y_0 + Y_1$ and $t > 0$.

Proof. Take $h \in X_0 \cap X_1$ with $\psi(h) \neq 0$. For $y \in Y_0 + Y_1$, consider a decomposition (3.1) satisfying the assumptions of the theorem. Putting

$$z_0(t) = y_0(t) - \frac{\psi(y_0(t))h}{\psi(h)}, \quad z_1(t) = y_1(t) + \frac{\psi(y_0(t))h}{\psi(h)},$$

we see that $z_i(t) \in Y_i$ and $y = z_0(t) + z_1(t)$. Consequently,

$$\begin{aligned} & \mathcal{K}(t, y; Y_0, Y_1) \\ & \leq \|z_0(t)\|_{X_0} + t\|z_1(t)\|_{X_1} \\ & \leq \|y_0(t)\|_{X_0} + t\|y_1(t)\|_{X_1} + |\psi(y_0(t))| \frac{\|h\|_{X_0} + t\|h\|_{X_1}}{|\psi(h)|} \\ & \leq 2\mathcal{K}(t, y; X_0, X_1) + 2|\psi(y_0(t))| \frac{\mathcal{J}(t, h; X_0, X_1)}{|\psi(h)|} \end{aligned}$$

by (3.2). Applying (0.6) and passing to the infimum over all $h \in X_0 \cap X_1$ with $\psi(h) \neq 0$ in the second term on the right, we obtain

$$\mathcal{K}(t, y; Y_0, Y_1) \leq 2\left(\mathcal{K}(t, y; X_0, X_1) + \frac{|\psi(y_0(t))|}{\mathcal{K}(t^{-1}, \psi; X_0^*, X_1^*)}\right).$$

To prove the reverse inequality, it suffices to show that

$$(3.3) \quad \frac{|\psi(y_0(t))|}{\mathcal{K}(t^{-1}, \psi; X_0^*, X_1^*)} \leq 4\mathcal{K}(t, y; Y_0, Y_1).$$

We may assume that $\psi(y_0(t)) \neq 0$ (otherwise there is nothing to prove). For $t > 0$, we find a decomposition $y = z_0(t) + z_1(t)$, $z_i(t) \in Y_i$, such that

$$(3.4) \quad \mathcal{K}(t, y; Y_0, Y_1) \geq \frac{1}{2}(\|z_0(t)\|_{X_0} + t\|z_1(t)\|_{X_1})$$

(an “optimal” decomposition of y in the couple (Y_0, Y_1)). Using it and also the decomposition (3.1), we define $h = y_0(t) - z_0(t) = z_1(t) - y_1(t)$. Then $h \in X_0 \cap X_1$, and the relation $z_0(t) \in Y_0 \subset N$ implies

$$\psi(h) = \psi(y_0(t)) - \psi(z_0(t)) = \psi(y_0(t)) \neq 0.$$

Therefore,

$$z_0(t) = y_0(t) - \frac{\psi(y_0(t))h}{\psi(h)}, \quad z_1(t) = y_1(t) + \frac{\psi(y_0(t))h}{\psi(h)},$$

and by (3.4), (3.2), and (0.6) we obtain

$$\begin{aligned}
\mathcal{K}(t, y; Y_0, Y_1) &\geq \frac{1}{2} \left(\left\| y_0(t) - \frac{\psi(y_0(t))}{\psi(h)} h \right\|_{X_0} + t \left\| y_1(t) + \frac{\psi(y_0(t))}{\psi(h)} h \right\|_{X_1} \right) \\
&\geq \frac{1}{2} \left(|\psi(y_0(t))| \frac{\|h\|_{X_0} + t\|h\|_{X_1}}{|\psi(h)|} - (\|y_0(t)\|_{X_0} + t\|y_1(t)\|_{X_1}) \right) \\
&\geq \frac{1}{2} |\psi(y_0(t))| \inf_{h \in X_0 \cap X_1, \psi(h) \neq 0} \frac{\mathcal{J}(t, \psi; X_0, X_1)}{|\psi(h)|} - \mathcal{K}(t, y; X_0, X_1) \\
&= \frac{1}{2} \frac{|\psi(y_0(t))|}{\mathcal{K}(t^{-1}, \psi; X_0^*, X_1^*)} - \mathcal{K}(t, y; X_0, X_1).
\end{aligned}$$

Now, (3.4) follows immediately, because $\mathcal{K}(t, y; X_0, X_1) \leq \mathcal{K}(t, y; Y_0, Y_1)$. $\square$

§4. INTERPOLATION OF INTERSECTIONS OF WEIGHTED L_p -SPACES GENERATED BY AN INTEGRAL FUNCTIONAL

As usual, for a positive measurable function $w(x)$ on $(0, \infty)$, we define $L_p(w)$ as the space of measurable functions $f : (0, \infty) \rightarrow \mathbb{R}$ with the norm

$$\|f\|_{w,p} = \left(\int_0^\infty |f(x)|^p w(x) dx \right)^{1/p} \quad \text{if } 1 \leq p < \infty,$$

and

$$\|f\|_{w,\infty} = \text{ess sup}_{x>0} (|f(x)|w(x)).$$

For an arbitrary measurable function $g : (0, \infty) \rightarrow \mathbb{R}$, consider the linear functional

$$\varphi_g(f) = \int_0^\infty f(x)g(x) dx$$

with the domain D_g consisting of all measurable functions $f : (0, \infty) \rightarrow \mathbb{R}$ such that $\int_a^b |f(x)g(x)| dx < \infty$ for all $0 < a < b < \infty$ and the following limit exists:

$$\lim_{a \rightarrow 0, b \rightarrow \infty} \int_a^b f(x)g(x) dx.$$

Basically, we shall be interested in the case where $g(x) \equiv 1$. Then we use the notations φ and D , respectively.

First, we find conditions ensuring the relation

$$(4.1) \quad (L_{p_0}(w_0) \cap \text{Ker } \varphi, L_{p_1}(w_1) \cap \text{Ker } \varphi)_{\theta,q} = (L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,q} \cap \text{Ker } \varphi.$$

Thus, $g(x) \equiv 1$, and we consider three cases distinguished in terms of p_0 and p_1 .

a) $1 \leq p_0 < p_1 < \infty$. We denote by ψ the restriction of the functional $\varphi(f) = \int_0^\infty f(x) dx$ to $L_{p_0}(w_0) \cap L_{p_1}(w_1)$. For every Banach couple (X_0, X_1) with $X_0 \cap X_1$ dense both in X_0 and in X_1 , we have $(X_0 \cap X_1)^* = X_0^* + X_1^*$ isometrically (see [2, §2.7]). Thus, if $\frac{1}{p'_i} + \frac{1}{p_i} = 1$ ($i = 0, 1$), then the condition $\psi \in (L_{p_0}(w_0) \cap L_{p_1}(w_1))^*$ is equivalent to the condition

$$1 \in L_{p'_0}(w_0^{-1/(p_0-1)}) + L_{p'_1}(w_1^{-1/(p_1-1)}) \quad (p_0 > 1)$$

or

$$1 \in L_\infty(w_0^{-1}) + L_{p'_1}(w_1^{-1/(p_1-1)}) \quad (p_0 = 1).$$

It its turn, this is equivalent to the finiteness of the function

$$k(t) = \mathcal{K}(t, 1; L_{p'_0}(w_0^{-1/(p_0-1)}), L_{p'_1}(w_1^{-1/(p_1-1)})) \quad (p_0 > 1)$$

or

$$k'(t) = \mathcal{K}(t, 1; L_\infty(w_0^{-1}), L_{p'_1}(w_1^{-1/(p_1-1)})) \quad (p_0 = 1)$$

for some (all) $t > 0$.

Assume first that $p_0 > 1$. After reduction to one weight function, we can write

$$k(t) = \mathcal{K}(t, u; L_{p'_0}(v), L_{p'_1}(v)),$$

where

$$(4.2) \quad u(x) = w_0(x)^{-\frac{p_1-1}{p_1-p_0}} w_1(x)^{\frac{p_0-1}{p_1-p_0}}, \quad v(x) = w_0(x)^{\frac{p_1}{p_1-p_0}} w_1(x)^{-\frac{p_0}{p_1-p_0}}.$$

Applying the Holmstedt formula [2], we obtain

$$\begin{aligned} k(t) &= t\mathcal{K}\left(\frac{1}{t}, u; L_{p'_1}(v), L_{p'_0}(v)\right) \\ &\asymp t\left(\int_0^{t^{-r}} u^*(s)^{p'_1} ds\right)^{1/p'_1} + \left(\int_{t^{-r}}^\infty u^*(s)^{p'_0} ds\right)^{1/p'_0}, \end{aligned}$$

where $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}$, and $u^*(s)$ is a nonincreasing rearrangement of $u(x)$ with respect to the measure $\nu(E) = \int_E v(x) dx$. Putting

$$(4.3) \quad F_t = \{x > 0 : u(x) > u^*(t^{-r})\}, \quad F_t^c = (0, \infty) \setminus F_t,$$

we have $k(t) \asymp k_1(t)$, where

$$k_1(t) = t\left(\int_{F_t} w_1(x)^{-1/(p_1-1)} dx\right)^{1/p'_1} + \left(\int_{F_t^c} w_0(x)^{-1/(p_0-1)} dx\right)^{1/p'_0}.$$

Much in the same way, it can be shown that for $p_0 = 1$ we have $k'(t) \asymp k'_1(t)$, where

$$k'_1(t) = t\left(\int_{F_t} w_1(x)^{-1/(p_1-1)} dx\right)^{1/p'_1}.$$

We also note that $r = p'_1$ and $u(x) = w_0^{-1}(x)$ in the latter case.

Theorem 4. *Suppose $1 \leq p_0 < p_1 < \infty$, and let functions $u(x)$ and $v(x)$ and sets F_t, F_t^c ($t > 0$) be defined by (4.2) and (4.3), respectively. Suppose the function*

$$k_1(t) = t\left(\int_{F_t} w_1(x)^{-1/(p_1-1)} dx\right)^{1/p'_1} + \left(\int_{F_t^c} w_0(x)^{-1/(p_0-1)} dx\right)^{1/p'_0}$$

(if $p_0 = 1$, the second term on the right should be suppressed) is finite for some (all) $t > 0$, and let α, β, α_0 , and β_∞ denote its dilation indices. If $\beta_\infty \leq \alpha_0$, $0 < \theta < 1$, and $1 \leq q < \infty$, then (4.1) is true if and only if (2.1) is true.

Proof. If (4.1) is fulfilled, then Corollary 2 is applicable by the assumptions of the theorem, and we obtain (2.1).

Conversely, let (2.1) be fulfilled. By the arguments preceding the theorem, the restriction ψ of φ to $L_{p_0}(w_0) \cap L_{p_1}(w_1)$ belongs to $(L_{p_0}(w_0) \cap L_{p_1}(w_1))^*$. Moreover, $\psi \neq 0$. Consequently, by Theorem 2, there is a continuous extension of ψ up to a functional $\tilde{\psi}_\theta \in \widetilde{M}_\theta^*$, where $\widetilde{M}_\theta$ is either $(L_{p_0}(w_0), L_{p_1}(w_1))_{\theta, q}$, or $(L_{p_0}(w_0), L_{p_1}(w_1))_{\theta, q} \cap L_{p_1}(w_1)$, or $(L_{p_0}(w_0), L_{p_1}(w_1))_{\theta, q} \cap L_{p_0}(w_0)$, depending on whether θ belongs to (β_∞, α_0) , $(0, \alpha)$, or $(\beta, 1)$.

We show that $\widetilde{M}_\theta \subset D$ for every $\theta \in (0, \alpha) \cup (\beta_\infty, \alpha_0) \cup (\beta, 1)$, and for $f \in \widetilde{M}_\theta$ we have

$$(4.4) \quad \tilde{\psi}_\theta(f) = \varphi(f) = \int_0^\infty f(x) dx.$$

First, $L_{p_0}(w_0) \cap L_{p_1}(w_1) \subset D$ by assumption, and for $f \in L_{p_0}(w_0) \cap L_{p_1}(w_1)$ we have the estimate

$$(4.5) \quad \int_0^\infty |f(x)| dx \leq C_\theta \|f\|_{\widetilde{M}_\theta}.$$

Now, let $f \in \widetilde{M}_\theta$, and let a sequence $\{f_n\} \subset L_{p_0}(w_0) \cap L_{p_1}(w_1)$ satisfy $\|f_n - f\|_{\widetilde{M}_\theta} \rightarrow 0$ as $n \rightarrow \infty$. Then $f_n \chi_{[0,b]} \rightarrow f \chi_{[0,b]}$ as $n \rightarrow \infty$ in measure for every $b > 0$ (here and below χ_F stands for the characteristic function of a set $F \subset \mathbb{R}$). By (4.5) we obtain

$$\int_0^b |f_n(x)| dx \leq C_\theta \|f_n\|_{\widetilde{M}_\theta},$$

and the Fatou lemma shows that

$$\int_0^\infty |f(x)| dx \leq C_\theta \lim_{n \rightarrow \infty} \|f_n\|_{\widetilde{M}_\theta} = C_\theta \|f\|_{\widetilde{M}_\theta},$$

i.e., $\widetilde{M}_\theta \subset L_1(0, \infty) \subset D$. Next, applying (4.5) once again, we see that

$$\int_0^\infty |f_n(x) - f_m(x)| dx \leq C_\theta \|f_n - f_m\|_{\widetilde{M}_\theta} \quad (n, m = 1, 2, \dots).$$

Consequently, the sequence $\{f_n\}$ has a limit in $L_1(0, \infty)$, which must coincide with f . Since

$$\int_0^\infty f_n(x) dx = \tilde{\psi}_\theta(f_n) \rightarrow \tilde{\psi}_\theta(f)$$

at the same time, we arrive at (4.4). Now, all conditions of Corollary 4 are satisfied. Applying this corollary, we obtain (4.1). $\square$

If the numbers p_0, p_1 and the weight functions $w_0(x)$ and $w_1(x)$ satisfy some additional conditions, we can obtain more specific results. First, let $p_0 > 1$, and assume that

(*) $w_0(x)$ is monotone increasing, $w_0(2x) \leq C_1 w_0(x)$ ($x > 0$), and

$$\int_x^\infty w_0(s)^{\frac{1}{1-p_0}} ds \leq C_2 x w_0(x)^{\frac{1}{1-p_0}}, \quad x > 0;$$

next, either $w_1(x)$ is monotone decreasing and $w_1(x) \leq C_3 w_1(2x)$ ($x > 0$), or both $w_1(x)$ and $w(x) = \frac{w_0(x)}{w_1(x)}$ are monotone increasing and

$$\int_0^x w_1(s)^{\frac{1}{1-p_1}} ds \leq C_4 x w_1(x)^{\frac{1}{1-p_1}} \quad (x > 0).$$

It is easily seen that, under condition (*), $u(x)$ is monotone decreasing and $v(x)$ is monotone increasing. In particular, it follows that $F_t = (0, s(t))$, where $s(t)$ is a solution of the equation

$$(4.6) \quad V(x) := \int_0^x v(s) ds = t^{-r}, \quad \frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}.$$

Moreover, it is easily seen that $V(x) \asymp x v(x)$. Therefore,

$$(4.7) \quad s(t) v(s(t)) \asymp t^{-r}.$$

Now, on one hand, we have

$$k_1(t) \geq \left(\int_{s(t)}^{2s(t)} w_0(x)^{-1/(p_0-1)} dx \right)^{1/p'_0} \geq C_1^{-1/p_0} s(t)^{1/p'_0} w_0(s(t))^{-1/p_0}$$

by (*). On the other hand, (*) and (4.7) imply

$$\begin{aligned} k_1(t) &\leq C(s(t)^{1/p'_0} w_0(s(t))^{-1/p_0} + t s(t)^{1/p'_1} w_1(s(t))^{-1/p_1}) \\ &= C s(t)^{1/p'_0} w_0(s(t))^{-1/p_0} (1 + t s(t)^{1/r} v(s(t))^{1/r}) \\ &\leq C' s(t)^{1/p'_0} w_0(s(t))^{-1/p_0}. \end{aligned}$$

Thus, under condition (*) we have the equivalence

$$k_1(t) \asymp s(t)^{1/p_0} w_0(s(t))^{-1/p_0}.$$

If $p_0 = 1$, a slightly less restrictive condition can be employed:

(**) $w_0(x)$ is monotone increasing and $w_0(2x) \leq C_1 w_0(x)$ ($x > 0$); next, either $w_1(x)$ is monotone decreasing and $w_1(x) \leq C_2 w_1(2x)$ ($x > 0$), or both $w_1(x)$ and $w(x) = \frac{w_0(x)}{w_1(x)}$ are monotone increasing and

$$\int_0^x w_1(s)^{\frac{1}{1-p_1}} ds \leq C_3 x w_1(x)^{\frac{1}{1-p_1}} \quad (x > 0).$$

Much as in the case where $p_0 > 1$, it can be shown that (**) implies

$$k'_1(t) \asymp w_0(s(t))^{-1}.$$

As a result, applying Theorem 4, we arrive at the following statement.

Corollary 5. *Suppose two numbers p_0, p_1 ($1 \leq p_0 < p_1 < \infty$) and two weight functions $w_0(x), w_1(x)$ satisfy (*) if $p_0 > 1$ and satisfy (**) if $p_0 = 1$. Let α, β, α_0 , and β_∞ be the dilation indices of the function $k_2(t) = s(t)^{\frac{p_0-1}{p_0}} w_0(s(t))^{-\frac{1}{p_0}}$, where $s(t)$ is the solution of equation (4.6). If $\beta_\infty \leq \alpha_0$, $0 < \theta < 1$, and $1 \leq q < \infty$, then (4.1) is equivalent to (2.1).*

b) $1 < p_0 = p_1 = p < \infty$. It is well known (see [2]) that in this case

$$\begin{aligned} k(t) &= \mathcal{K}(t, 1; L_{p'}(w_0^{-1/(p-1)}), L_{p'}(w_1^{-1/(p-1)})) \\ &\asymp \left(\int_0^\infty \min(w_0(x)^{-1/(p-1)}, t^{p'} w_1(x)^{-1/(p-1)}) dx \right)^{1/p'} \\ &= \left(t^{p'} \int_{F_t} w_1(x)^{-1/(p-1)} dx + \int_{F_t^c} w_0(x)^{-1/(p-1)} dx \right)^{1/p'}, \end{aligned}$$

where

$$(4.8) \quad F_t = \left\{ x > 0 : \frac{w_0(x)}{w_1(x)} < t^{-p} \right\}, \quad F_t^c = (0, \infty) \setminus F_t.$$

The following statement is proved much as Theorem 4.

Theorem 5. *Let $p \in (1, \infty)$. Suppose $w_0(x)$ and $w_1(x)$ are such that the function*

$$k_3(t) = \left(t^{p'} \int_{F_t} w_1(x)^{-1/(p-1)} dx + \int_{F_t^c} w_0(x)^{-1/(p-1)} dx \right)^{1/p'},$$

where F_t and F_t^c are defined by (4.8), is finite for some (all) $t > 0$. If α, β, α_0 , and β_∞ are the dilation indices of the function $k_3(t)$ and $\beta_\infty \leq \alpha_0$, then the relation

$$(4.9) \quad (L_p(w_0) \cap \text{Ker } \varphi, L_p(w_1) \cap \text{Ker } \varphi)_{\theta, q} = (L_p(w_0), L_p(w_1))_{\theta, q} \cap \text{Ker } \varphi$$

is equivalent to (2.1).

Suppose (*) is fulfilled for $p_0 = p_1 = p$, $w_0(x)$, and $w_1(x)$. Then $F_t = (0, w^{-1}(t^{-p}))$, where $w^{-1}(x)$ is the function inverse to $w(x) = w_0(x)/w_1(x)$. Therefore, arguing as we did before Corollary 5, we obtain yet another statement.

Corollary 6. *Suppose (*) is fulfilled for $p_0 = p_1 = p \in (1, \infty)$, $w_0(x)$, and $w_1(x)$. Let α, β, α_0 , and β_∞ be the dilation indices of the function*

$$k_4(t) = (w^{-1}(t^{-p}))^{(p-1)/p} (w_0(w^{-1}(t^{-p})))^{-1/p},$$

and let $\beta_\infty \leq \alpha_0$. Then (4.9) is equivalent to (2.1).

c) $p_0 = p_1 = 1$. In this case

$$k(t) = \mathcal{K}(t, 1; L_\infty(1/w_0), L_\infty(1/w_1)) = \sup_{x>0} \frac{1}{w_0(x)} \min \left(1, t \frac{w_0(x)}{w_1(x)} \right).$$

If the weight functions obey the condition

$w_0(x)$ is monotone increasing, $w_1(x)$ is monotone decreasing, and

$$(4.10) \quad \lim_{x \rightarrow 0} w(x) = \lim_{x \rightarrow \infty} \frac{1}{w(x)} = 0,$$

then

$$k(t) \asymp \frac{1}{w_0(w^{-1}(1/t))}.$$

As before, here $w(x) := \frac{w_0(x)}{w_1(x)}$, and $w^{-1}(x)$ is the function inverse to $w(x)$.

The following statement is proved much as Theorem 4.

Theorem 6. *Suppose $w_0(x)$ and $w_1(x)$ satisfy (4.10). Let α , β , α_0 , and β_∞ be the dilation indices of the function $k_5(t) = \frac{1}{w_0(w^{-1}(1/t))}$, and let $\beta_\infty \leq \alpha_0$. Then the relation*

$$(4.11) \quad (L_1(w_0) \cap \text{Ker } \varphi, L_1(w_1) \cap \text{Ker } \varphi)_{\theta, q} = (L_1(w_0), L_1(w_1))_{\theta, q} \cap \text{Ker } \varphi$$

is equivalent to (2.1).

From the preceding theorem we deduce an interesting consequence. Let a, b, c, d be four numbers satisfying $0 \leq a \leq \min(b, c) \leq \max(b, c) \leq d \leq 1$. Then there exists a concave function $\nu(x)$ on $(0, \infty)$ whose dilation indices α , β_∞ , α_0 , and β coincide with a , b , c , and d .

In order to exhibit such a function, we take a two-sided monotone increasing sequence $\{x_k\}_{k=-\infty}^\infty$ of positive numbers such that $x_0 = 1$ and $\lim_{|k| \rightarrow \infty} \frac{x_{k+1}}{x_k} = \infty$.

If $b > 0$, we define a function ν' in the following way:

$$\begin{aligned} \nu'(x) &= \left(\frac{x}{x_{2k}} \right)^a \nu'(x_{2k}), \quad x_{2k} \leq x \leq x_{2k+1}, \\ \nu'(x) &= \left(\frac{x}{x_{2k+1}} \right)^b \nu'(x_{2k+1}), \quad x_{2k+1} \leq x \leq x_{2k+2}, \end{aligned}$$

for $k = 0, 1, 2, \dots$, and

$$\begin{aligned} \nu'(x) &= \left(\frac{x}{x_{2k}} \right)^c \nu'(x_{2k}), \quad x_{2k-1} \leq x \leq x_{2k}, \\ \nu'(x) &= \left(\frac{x}{x_{2k-1}} \right)^d \nu'(x_{2k-1}), \quad x_{2k-2} \leq x \leq x_{2k-1}, \end{aligned}$$

for $k = 0, -1, -2, \dots$.

If $b = 0$ and $d > 1$, we define $\nu'(x)$ as before if $x \in (0, 1]$, and for $x \in [1, \infty)$ we put $\nu'(x) = \ln ex$.

Finally, if $d = 0$, we put $\nu'(x) = \frac{1}{\ln e/x}$ for $x \in (0, 1]$ and $\nu'(x) = \ln ex$ for $x \in [1, \infty]$.

It can easily be shown that $\nu'(x)$ is monotone increasing, $\nu'(x)/x$ is monotone decreasing, and the range of $\nu'(x)$ is $(0, \infty)$. There exists a monotone increasing concave function $\nu(x)$ such that $\nu'(x) \leq \nu(x) \leq 2\nu'(x)$ ($x > 0$); see [17, Theorem 2.1.1]. Clearly, the range of $\nu(x)$ also coincides with $(0, \infty)$. The definition of ν' readily implies that its dilation indices α , β_∞ , α_0 , and β (therefore, also those of ν) coincide with a, b, c , and d , respectively.

We denote $w_0(x) = x$ and $w_1(x) = x\nu^{-1}(1/x)$ (ν^{-1} is the inverse function to ν). Since these functions satisfy (4.10) and

$$\nu(x) = \frac{1}{w_0(w^{-1}(1/t))},$$

application of Theorem 6 leads to the following result.

Corollary 7. *For arbitrary numbers $0 \leq a \leq b \leq c \leq d \leq 1$, there exist weight functions $w_0(x)$ and $w_1(x)$ such that (4.11) is fulfilled if and only if $\theta \in (0, a) \cup (b, c) \cup (d, 1)$.*

Remark. The last statement shows that all situations indicated in Theorem 1 realize indeed, already in the framework of weighted L_1 -spaces. Partial results of the same nature were known earlier; see [1, Corollary 2].

The case of power-type weight functions leads to the most transparent results. If $w_0(x) = x^l$ and $w_1(x) = x^m$, it can easily be seen that (*) is fulfilled if a) $l > p_0 - 1$ and $m \leq 0$ or (b) $l > p_0 - 1$, $l > m$, and $0 \leq m < p_1 - 1$.

By (4.2) and (4.7), we have

$$v(x) = x^{\frac{p_1 l - p_0 m}{p_1 - p_0}}, \quad s(t) \asymp t^{-\frac{r(p_1 - p_0)}{p_1 l - p_0 m + p_1 - p_0}}, \quad k_2(t) \asymp t^{\theta_0},$$

where $\theta_0 = \frac{p_1(l - p_0 + 1)}{p_1(l + 1) - p_0(m + 1)}$. Consequently, $\alpha = \beta = \alpha_0 = \beta_\infty = \theta_0$, and the preceding results yield the following.

Corollary 8. *Suppose p_0, p_1 ($1 \leq p_0 \leq p_1 < \infty$), l , and m satisfy (a) or (b). Then*

$$(L_{p_0}(x^l) \cap \text{Ker } \varphi, L_{p_1}(x^m) \cap \text{Ker } \varphi)_{\theta, q} = (L_{p_0}(x^l), L_{p_1}(x^m))_{\theta, q} \cap \text{Ker } \varphi$$

if and only if $\theta \neq \frac{p_1(l - p_0 + 1)}{p_1(l + 1) - p_0(m + 1)}$.

Remark. For $p_0 = p_1 = q$, the last statement was proved in [1, Corollary 1] by a different method.

Now, we apply the results of §3 to the evaluation of the $\mathcal{K}$ -functional on the couple of intersections $(L_p(w_0) \cap \text{Ker } \varphi_g, L_p(w_1) \cap \text{Ker } \varphi_g)$ for weighted L_p -spaces, where $\varphi_g(f) = \int_0^\infty f(x)g(x) dx$. Consider two cases.

a) $p_0 = p_1 = p > 1$. As before, the condition $\varphi_g \in (L_p(w_0) \cap L_p(w_1))^*$ is equivalent to the condition

$$g \in L_{p'}(w_0^{-1/(p-1)}) + L_{p'}(w_1^{-1/(p-1)}) = L_{p'}(\max(w_0, w_1)^{-1/(p-1)}).$$

Then, as was mentioned already in [2],

$$\begin{aligned} & \mathcal{K}(t^{-1}, \varphi_g; L_p(w_0)^*, L_p(w_1)^*) \\ &= \mathcal{K}(t^{-1}, g; L_{p'}(w_0^{-1/(p-1)}), L_{p'}(w_1^{-1/(p-1)})) \\ &\asymp \left(\int_0^\infty |g(x)|^{p'} \min(w_0^{-1/(p-1)}(x), t^{-p'} w_1^{-1/(p-1)}(x)) dx \right)^{1/p'}, \end{aligned}$$

and the latter function is finite for all $t > 0$. In a similar way, we obtain

$$\mathcal{K}(t, f; L_p(w_0), L_p(w_1)) \asymp \left(\int_0^\infty |f(x)|^p \min(w_0(x), t^p w_1(x)) dx \right)^{1/p}.$$

Therefore, under the assumption

$$(4.12) \quad \text{the function } w(x) := \frac{w_0(x)}{w_1(x)} \text{ is monotonic and } \lim_{t \rightarrow 0} w(x) = \lim_{t \rightarrow \infty} \frac{1}{w(x)} = 0,$$

we have

$$\begin{aligned}
& \mathcal{K}(t, f; L_p(w_0), L_p(w_1)) \\
& \asymp \left(\int_0^{w^{-1}(t^p)} |f(x)|^p w_0(x) dx + t^p \int_{w^{-1}(t^p)}^\infty |f(x)|^p w_1(x) dx \right)^{1/p} \\
& = (\|f_0^t\|_{L_p(w_0)}^p + t^p \|f_1^t\|_{L_p(w_1)}^p)^{1/p} \\
& \asymp \|f_0^t\|_{L_p(w_0)} + t \|f_1^t\|_{L_p(w_1)},
\end{aligned}$$

where $f_0^t(x) = f(x)\chi_{(0, w^{-1}(t^p))}(x)$ and $f_1^t(x) = f(x)\chi_{(w^{-1}(t^p), \infty)}(x)$. The last display shows that the decomposition $f = f_0^t + f_1^t$ is optimal (see (3.2)). Moreover, if $f \in L_p(w_0) \cap \text{Ker } \varphi_g + L_p(w_1) \cap \text{Ker } \varphi_g$, then f_0^t belongs to the domain D_g of φ_g for every $t > 0$. Thus, Theorem 3 implies the following result.

Theorem 7. *Suppose $p \in (1, \infty)$, the function $w(x) = \frac{w_0(x)}{w_1(x)}$ satisfies (4.12), and $g \in L_{p'}(\max(w_0, w_1)^{-1/(p-1)})$. Then*

$$\begin{aligned}
& \mathcal{K}(t, f; L_p(w_0) \cap \text{Ker } \varphi_g, L_p(w_1) \cap \text{Ker } \varphi_g) \\
& \asymp \left(\int_0^\infty |f(x)|^p \min(w_0(x), t^p w_1(x)) dx \right)^{1/p} \\
& \quad + \frac{|\int_0^{w^{-1}(t^p)} g(x)f(x) dx|}{G_p(t)},
\end{aligned}$$

where

$$G_p(t) = \left(\int_0^\infty |g(x)|^{p'} \min(w_0^{-1/(p-1)}(x), t^{-p'} w_1^{-1/(p-1)}(x)) dx \right)^{1/p'}.$$

If $g(x) \equiv 1$, the formula for the $\mathcal{K}$ -functional in Theorem 7 simplifies under certain conditions on $p \in (1, \infty)$ and the weight functions $w_0(x)$ and $w_1(x)$. Specifically, suppose that

(***) $w_0(x)$ is monotone increasing and

$$\int_x^\infty w_0(s)^{\frac{1}{1-p}} ds \leq C_1 x w_0(x)^{\frac{1}{1-p}} \quad (x > 0);$$

next, either $w_1(x)$ is monotone decreasing and $w_1(x) \leq C_2 w_1(2x)$ ($x > 0$), or both $w_1(x)$ and $w(x) = \frac{w_0(x)}{w_1(x)}$ are monotone increasing, and

$$\int_0^x w_1(s)^{\frac{1}{1-p}} ds \leq C_3 x w_1(x)^{\frac{1}{1-p}} \quad (x > 0).$$

Arguing as we did before Corollary 6, we easily obtain the following result, which was established in [1, Theorem 3] by a different method, namely, on the basis of direct estimates.

Corollary 9. *Suppose $p \in (1, \infty)$, $w_0(x)$, and $w_1(x)$ satisfy (***) . Then*

$$\begin{aligned}
& \mathcal{K}(t, f; L_p(w_0) \cap \text{Ker } \varphi, L_p(w_1) \cap \text{Ker } \varphi) \\
& \asymp \left(\int_0^\infty |f(x)|^p \min(w_0(x), t^p w_1(x)) dx \right)^{1/p} \\
& \quad + (w^{-1}(t^p))^{1/p-1} w_0^{1/p}(w^{-1}(t^p)) \left| \int_0^{w^{-1}(t^p)} f(x) dx \right|.
\end{aligned}$$

b) $p_0 = p_1 = 1$. In this case the condition $\varphi_g \in (L_1(w_0) \cap L_1(w_1))^*$ emerges, which is equivalent to

$$g \in L_\infty(1/w_0) + L_\infty(1/w_1) = L_\infty(1/\max(w_0, w_1)).$$

Now, for all $t > 0$ the following function is finite:

$$\begin{aligned} \mathcal{K}(t, \varphi_g; L_1(w_0)^*, L_1(w_1)^*) \\ = \mathcal{K}(t, g; L_\infty(1/w_0), L_\infty(1/w_1)) \\ = \sup_{x>0} \left(|g(x)| \min \left(\frac{1}{w_0(x)}, \frac{t}{w_1(x)} \right) \right). \end{aligned}$$

As in the case of $p > 1$, if (4.12) is fulfilled, then

$$\mathcal{K}(t, f; L_1(w_0), L_1(w_1)) = \|f_0^t\|_{L_1(w_0)} + t\|f_1^t\|_{L_1(w_1)},$$

where $f_0^t(x) = f(x)\chi_{(0, w^{-1}(t))}(x)$ and $f_1^t(x) = f(x)\chi_{(w^{-1}(t), \infty)}(x)$. Thus, the decomposition $f = f_0^t + f_1^t$ is optimal. Moreover, if $f \in L_1(w_0) \cap \text{Ker } \varphi_g + L_1(w_1) \cap \text{Ker } \varphi_g$, then $f_0^t \in D_g$ for every $t > 0$. Therefore, Theorem 3 yields yet another theorem.

Theorem 8. *If (4.12) is fulfilled and $g \in L_\infty(1/\max(w_0, w_1))$, then*

$$\begin{aligned} \mathcal{K}(t, f; L_1(w_0) \cap \text{Ker } \varphi_g, L_1(w_1) \cap \text{Ker } \varphi_g) \\ \asymp \left(\int_0^\infty |f(x)| \min(w_0(x), tw_1(x)) dx \right) \\ + \frac{|\int_0^{w^{-1}(t)} g(x)f(x) dx|}{G_1(t)}, \end{aligned}$$

where

$$G_1(t) = \sup_{x>0} \left(|g(x)| \min \left(\frac{1}{w_0(x)}, \frac{1}{tw_1(x)} \right) \right).$$

If $g(x) \equiv 1$ and the weight functions are sufficiently regular, the formula for the $\mathcal{K}$ -functional on the couple of intersections simplifies somewhat.

Corollary 10. *If $w_0(x)$ and $w_1(x)$ satisfy (4.10), then*

$$\begin{aligned} \mathcal{K}(t, f; L_1(w_0) \cap \text{Ker } \varphi, L_1(w_1) \cap \text{Ker } \varphi) \\ \asymp \left(\int_0^\infty |f(x)| \min(w_0(x), tw_1(x)) dx \right) \\ + w_0(w^{-1}(t)) \left| \int_0^{w^{-1}(t)} f(x) dx \right|. \end{aligned}$$

Remark. Under the additional assumption $w_1(x/2) \leq Aw_1(x)$ ($x > 0$), Corollary 10 was proved by a different method in [1] (see Theorem 3).

The author is grateful to Prof. L. Maligranda for a useful discussion of certain issues related to the proof of Theorem 2.

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Received 30/SEP/2003

Translated by S. V. KISLYAKOV