Siberian Mathematical Journal, Vol. 46, *No.* 2, *pp.* 205–225, 2005
Original Russian Text Copyright © 2005 *Astashkin S. V.*

EXTRAPOLATION FUNCTORS ON A FAMILY OF SCALES GENERATED BY THE REAL INTERPOLATION METHOD

S. V. Astashkin UDC 517.982.27

Abstract: A new class of extrapolation functors is defined on a family of scales generated by the real interpolation method. We prove extrapolation relations for the $\mathscr{K}$- and $\mathscr{J}$-functionals corresponding to some natural pairs of limit spaces which make it possible to describe the values of these functors. We can consider these relations as new assertions similar to the classical Yano theorem on estimates for the norms of operators in interpolation scales of spaces.

Keywords: operator extrapolation, extrapolation functor, rearrangement invariant space, operator interpolation, real interpolation method

The starting point for extrapolation theory is the classical Yano theorem (see [1] or [2, Chapter 12, Theorem 4.41]) about the Zygmund spaces $L(\log L)^\alpha$ and $\operatorname{Exp} L^\beta$ with the norms

$$\|f\|_{L(\log L)^\alpha} = \int_0^1 \log_2^\alpha\left(\frac{2}{t}\right) f^*(t)\,dt \quad \text{and} \quad \|f\|_{\operatorname{Exp} L^\beta} = \sup_{0<t\le 1} \log_2^{-1/\beta}\left(\frac{2}{t}\right) f^*(t)$$

which are important in applications. Here $\alpha, \beta > 0$ and $f^*(t)$ is the nonincreasing rearrangement of a function $|f(t)|$ on the interval $[0,1]$. Suppose that T is a bounded linear operator in the $L_p = L_p[0,1]$ spaces for all p in some right half-neighborhood of 1 and $\|T\|_{L_p\to L_p} = \mathscr{O}((p-1)^{-\alpha})$ $(p \to 1+)$ for some $\alpha > 0$. Then we can define T on the Zygmund space $L(\log L)^\alpha$ so that it becomes a bounded operator from this space to L_1. The dual assertion about the dual space $\operatorname{Exp} L^{1/\alpha}$ to $L(\log L)^\alpha$ is also valid. If a linear operator T is bounded in L_p for all sufficiently large p and $\|T\|_{L_p\to L_p} = \mathscr{O}(p^\alpha)$ $(p \to \infty)$ for some $\alpha > 0$ then $T : L_\infty \to \operatorname{Exp} L^{1/\alpha}$.

In the 1990s some general approaches of extrapolation theory began to be developed which are mainly associated with Jawerth and Milman [3–5]. The primary goal of this theory consists in studying the natural limit spaces associated with the interpolation scales of spaces and obtaining estimates for the norms of operators in these spaces. Jawerth and Milman showed that a "source" for the Yano-type theorems is existence of extrapolation constructions (functors) whose values are the limit spaces of these theorems. In particular, using the functors of intersection Δ and sum Σ, they obtained the following extrapolation description for the Zygmund spaces of the Yano theorem (for example, see [5, pp. 22–23]):

$$\Delta_{1<p<\infty}\big(p^{-\alpha}L_p\big) = \operatorname{Exp} L^{1/\alpha} \quad \text{and} \quad \Sigma_{1<p<\infty}\big((p-1)^{-\alpha}L_p\big) = L(\log L)^\alpha \ (\alpha > 0).$$

Jawerth and Milman considered as extrapolation functors only the extreme functors, the sum and intersection of families of Banach spaces (for example, see [4, § 2] or [5, Chapter 2]). This choice leads only to the generalized Lorentz and Marcinkiewicz spaces (see the definitions below) as the resultant extrapolation spaces. A broader class of functors (defined however only on the scale of L_p-spaces) was introduced in [6]. It is closely connected with the real interpolation method and enables the limit space to be almost every rearrangement invariant (symmetric) space "close" to L_∞ and L_1. This article is devoted to extending and developing these ideas.

First of all, we extend the definition of functors of [6] to the following family of discrete scales of Banach spaces. Let Φ be an Orlicz function on $[0,\infty)$. Denote by $\mathscr{A}$ the family of sequences (discrete scales) of the form

$$\vec{A}_n^{\mathscr{K}} = (A_0, A_1)^{\mathscr{K}}_{1-1/\Phi(2^n),\Phi(2^n)}, \quad n = 1, 2, \dots;$$

Samara. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 46, No. 2, pp. 264–289, March–April, 2005.
Original article submitted August 27, 2004.

and by $\mathscr{B}$, the family of the form

$$\vec{A}_n^{\mathscr{J}} = (A_0, A_1)^{\mathscr{J}}_{1/\Phi(2^n),\Phi(2^n)/(\Phi(2^n)-1)}, \quad n = 1, 2, \dots,$$

where $\vec{A} = (A_0, A_1)$ is an arbitrary Banach pair such that $A_1 \overset{1}{\subset} A_0$ and $(A_0, A_1)^{\mathscr{K}}_{\theta,p}$ and $(A_0, A_1)^{\mathscr{J}}_{\theta,p}$ $(0 < \theta < 1,\ 1 \le p \le \infty)$ are the spaces of the real $\mathscr{K}$- and $\mathscr{J}$-methods.

The main results of this article consist in description for the values of some extrapolation functors (defined below) on the scales of these families. We prove that in the case of $\mathscr{A}$ (of $\mathscr{B}$) we precisely obtain the interpolation spaces with respect to the pair $(A_1, M_\varphi(\vec{A}))$ (the pair $(A_0, \Lambda_\varphi(\vec{A}))$, where $M_\varphi(\vec{A})$ and $\Lambda_\varphi(\vec{A})$ are the generalized Marcinkiewicz and Lorentz spaces constructed for the function $\varphi(u) = u\Phi^{-1}(\log(1 + (e-1)/u))$ $(0 < u \le 1)$. The key point here is the extrapolation relations we obtain below for $\mathscr{K}$- and $\mathscr{J}$-functionals of the pairs $(A_1, M_\varphi(\vec{A}))$ and $(A_0, \Lambda_\varphi(\vec{A}))$. In the last part of this article we show that the above-introduced functors can also be defined on some broader families of scales without changing the values of these functors. This is one of the consequences of the important stability property of extrapolation constructions.

The results of this article in the particular case of the scale of L_p-spaces (i.e., for the pair $\vec{A} = (L_1, L_\infty)$) were partially announced in [7].

§ 1. Definitions, Notations, and Preliminaries

A detailed exposition of the theory of operator interpolation and the theory of rearrangement invariant spaces can be found in [8–11].

Henceforth we assume that each embedding of one Banach space into the other is assumed continuous; i.e., $X_1 \subset X_0$ means that $x \in X_1$ implies that $x \in X_0$ and $\|x\|_{X_0} \le C\|x\|_{X_1}$ for some $C > 0$. To specify the constant of the embedding, we sometimes write $X_1 \overset{C}{\subset} X_0$.

Let X_0 and X_1 be Banach spaces such that $X_1 \subset X_0$. Then we define the *completion of* X_1 *with respect to* X_0 (or *Gagliardo completion*) as the set $\widetilde{X}_1$ of all $x \in X_0$ such that there is a sequence $\{x_n\} \subset X_0$ with the properties: $\|x_n\|_{X_1} \le C$ $(n = 1, 2, \dots)$ for some $C > 0$ and $x_n \to x$ in X_0. The space X_1 is said to be *complete with respect to* X_0 if $\widetilde{X}_1 = X_1$.

If (X_0, X_1) is a Banach pair (i.e., the Banach spaces X_0 and X_1 are linearly and continuously embedded in some Hausdorff topological vector space) then we can naturally define the intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ with the respective norms

$$\|x\|_{X_0 \cap X_1} = \max_{i=0,1} \|x\|_{X_i},$$

$$\|x\|_{X_0 + X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1,\ x_i \in X_i,\ i = 0, 1\}.$$

Let (X_0, X_1) and (Y_0, Y_1) be Banach pairs. A triple (X_0, X_1, X) of spaces, $X_0 \cap X_1 \subset X \subset X_0 + X_1$, is called an *interpolation* (*exact interpolation*) triple *with respect to* (Y_0, Y_1, Y), $Y_0 \cap Y_1 \subset Y \subset Y_0 + Y_1$, if every linear operator T on $X_0 + X_1$ bounded from X_0 to Y_0 and from X_1 to Y_1 is also bounded from X to Y (and moreover $\|T\|_{X \to Y} \le \max_{i=0,1} \|T\|_{X_i \to Y_i}$). If $X_i = Y_i$ $(i = 0, 1)$ and $X = Y$ then we say that X is an *interpolation space with respect to* (X_0, X_1).

An (*exact*) *interpolation functor* is an arbitrary mapping F of the set of Banach pairs to the set of Banach spaces such that for all Banach pairs $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ the triple $(X_0, X_1, F(\vec{X}))$ is an (exact) interpolation triple with respect to $(Y_0, Y_1, F(\vec{Y}))$. The characteristic function $\rho(t)$ of the interpolation functor F is defined by the relation $F(\mathbb{R}, (1/t)\mathbb{R}) = (1/\rho(t))\mathbb{R}$, $t > 0$. If F is an exact functor then $\rho(t)$ is a quasiconcave function on $(0, \infty)$; i.e., $\rho(t)$ increases and $\rho(t)/t$ decreases for $t > 0$.

An important way for obtaining interpolation spaces is the real interpolation method based on application of the Peetre $\mathscr{K}$- and $\mathscr{J}$-functionals:

$$\mathscr{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\},$$

$$\mathscr{J}(t, x; X_0, X_1) = \max\{\|x\|_{X_0}, t\|x\|_{X_1}\}.$$

For a fixed $t > 0$, the first of them is the norm on the sum X_0+tX_1 and the second is the norm on the intersection $X_0\cap tX_1$ (if X is a Banach space and $\alpha > 0$ then the space αX consists of the same elements as X but has the norm $\|x\|_{\alpha X} = \alpha\|x\|_X$). If we take $x \in X_0 + X_1$ ($x \in X_0 \cap X_1$) then $\mathscr{K}(t, x; X_0, X_1)$ is an increasing concave function of t ($\mathscr{J}(t, x; X_0, X_1)$ is an increasing convex function).

Let E be a Banach lattice of two-sided real sequences $\alpha = (\alpha_j)_{j=-\infty}^{\infty}$. If (X_0, X_1) is an arbitrary Banach pair then the space $(X_0, X_1)^{\mathscr{K}}_E$ of the $\mathscr{K}$-method consists of all $x \in X_0 + X_1$ satisfying $(\mathscr{K}(2^j, x; X_0, X_1))_j \in E$ and

$$\|x\| = \|(\mathscr{K}(2^j, x; X_0, X_1))_j\|_E < \infty.$$

The space $(X_0, X_1)^{\mathscr{J}}_E$ of the $\mathscr{J}$-method consists of all $x \in X_0 + X_1$ representable as

$$x = \sum_{j=-\infty}^{\infty} u_j \quad \text{(w.r.t. the convergence of } X_0 + X_1\text{)}, \quad \text{where} \quad u_j \in X_0 \cap X_1. \tag{1}$$

The norm in $(X_0, X_1)^{\mathscr{J}}_E$ is defined as $\inf_{\{u_j\}} \|(\mathscr{J}(2^j, u_j; X_0, X_1))_j\|_E$, where the greatest lower bound is calculated over all sequences $\{u_j\}_{j=-\infty}^{\infty}$ such that (1) is valid.

If E is a Banach lattice of two-sided sequences and $\alpha = (\alpha_k)_{k=-\infty}^{\infty}$ is a sequence of nonnegative numbers then the space $E(\alpha_k)$ consists of all $a = (a_k)_{k=-\infty}^{\infty}$ such that $(a_k\alpha_k)_{k=-\infty}^{\infty} \in E$ and $\|a\|_{E(\alpha_k)} = \|(a_k\alpha_k)\|_E$. Suppose that $E \supset \Delta(\vec{l}_\infty) := l_\infty \cap l_\infty(2^{-k})$ ($\{0\} \neq E \subset \Sigma(\vec{l_1}) := l_1 + l_1(2^{-k})$). Then the mapping $(X_0, X_1) \mapsto (X_0, X_1)^{\mathscr{K}}_E$ ($(X_0, X_1) \mapsto (X_0, X_1)^{\mathscr{J}}_E$) defines an exact interpolation functor. The collection of all such functors is called the *real $\mathscr{K}$-method* (*$\mathscr{J}$-method*). It is important to notice that in the case of the $\mathscr{K}$-method ($\mathscr{J}$-method) the parameter E can always be considered as an interpolation space with respect to the pair $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$ ($\vec{l_1} = (l_1, l_1(2^{-k}))$) [9, Corollaries 3.3.10 and 3.4.6]. Moreover, if $\rho(t)$ is a quasiconcave function then the functors $(\cdot,\cdot)^{\mathscr{J}}_{l_1(1/\rho(2^k))}$ and $(\cdot,\cdot)^{\mathscr{K}}_{l_\infty(1/\rho(2^k))}$ are extreme in the sense that, for every exact interpolation functor F with the characteristic function $\rho(t)$, we have

$$(X_0, X_1)^{\mathscr{J}}_{l_1(1/\rho(2^k))} \overset{1}{\subset} F(X_0, X_1) \overset{1}{\subset} (X_0, X_1)^{\mathscr{K}}_{l_\infty(1/\rho(2^k))}$$

for every Banach pair (X_0, X_1).

In particular, for $0 < \theta < 1$ and $1 \le p \le \infty$ we obtain the classical interpolation spaces

$$(X_0, X_1)^{\mathscr{K}}_{\theta,p} = (X_0, X_1)^{\mathscr{K}}_{l_p(2^{-k\theta})} \quad \text{and} \quad (X_0, X_1)^{\mathscr{J}}_{\theta,p} = (X_0, X_1)^{\mathscr{J}}_{l_p(2^{-k\theta})}$$

whose properties are studied in detail in [8]. It is well known [8, Theorem 3.3.1] that $(X_0, X_1)^{\mathscr{K}}_{\theta,p} = (X_0, X_1)^{\mathscr{J}}_{\theta,p}$ for arbitrary $0 < \theta < 1$ and $1 \le p \le \infty$. Moreover, it is important to observe that the constant of the equivalence of their norms depends on θ and p and may tend to ∞.

Henceforth we speak in particular of the pairs of rearrangement invariant (symmetric) function spaces on the interval $[0, 1]$. Recall that a Banach space X of measurable functions on $[0, 1]$ is called rearrangement invariant provided that

(a) if $y = y(t) \in X$ and $|x(t)| \le |y(t)|$ then $x = x(t) \in X$ and $\|x\| \le \|y\|$;

(b) if $y = y(t) \in X$ and $x^*(t) = y^*(t)$ then $x \in X$ and $\|x\| = \|y\|$.

The simplest and important examples of the rearrangement invariant spaces are the L_p-spaces ($1 \le p \le \infty$) with the usual norms:

$$\|x\|_p = \Big(\int_0^1 |x(t)|^p\, dt\Big)^{1/p} \ (1 \le p < \infty) \quad \text{and} \quad \|x\|_\infty = \operatorname*{ess\,sup}_{0\le t\le 1} |x(t)|.$$

Their natural generalizations are Orlicz spaces. Henceforth by an Orlicz function we mean an increasing continuous convex function $N(u) \ge 0$ on $[0, \infty)$ such that $N(0) = 0$ and $N(1) = 1$ (thus N always has the inverse N^{-1} which is concave). The corresponding Orlicz space L_N consists of all measurable functions $x = x(t)$ on $[0,1]$ such that the norm

$$\|x\|_{L_N} = \inf\left\{u > 0 : \int_0^1 N\left(\frac{|x(t)|}{u}\right) dt \le 1\right\}$$

is finite. In particular, if $N(t) = t^p$ $(1 \le p < \infty)$ then we obtain the L_p-space.

Other examples of rearrangement invariant spaces are Lorentz and Marcinkiewicz spaces. If $\psi(t)$ is a nonnegative increasing concave function on $[0,1]$ then the Marcinkiewicz space $M(\psi)$ consists of all measurable functions $x = x(s)$ on $[0,1]$ satisfying

$$\|x\|_{M(\psi)} = \sup_{0<t\le 1} \frac{\int_0^t x^*(s)\,ds}{\psi(t)} < \infty,$$

and the Lorentz space $\Lambda(\psi)$ consists of all $x = x(s)$ satisfying

$$\|x\|_{\Lambda(\psi)} = \int_0^1 x^*(s)\,d\psi(s) < \infty.$$

Henceforth we are interested in the exponential Orlicz spaces $\operatorname{Exp} L^\Phi = L_{N_\Phi}$, where $N_\Phi(u) = (e^{\Phi(u)} - 1)/(e-1)$, and $\Phi(u)$ is an Orlicz function. It is well known [12, 13] that the norm in $\operatorname{Exp} L^\Phi$ is equivalent to that in the Marcinkiewicz space $M(\varphi)$ constructed with the concave function $\varphi(u) = u\Phi^{-1}(\log(1 + (e-1)/u))$. Hence, it follows in particular that $\operatorname{Exp} L^\Phi = \Lambda(\varphi)^*$ [10, pp. 152–154]. For $\Phi(u) = u^\alpha$ $(\alpha \ge 1)$ we obtain the well-known Zygmund spaces $\operatorname{Exp} L^\alpha$ and $L(\log L)^{1/\alpha}$ [14] mentioned already in the introduction in connection with the Yano extrapolation theorem.

In line with [5, p. 8], we recall the definitions of extrapolation space and extrapolation functor (in the case of discrete scales). A sequence (or discrete scale) $\{A_n\}_{n=1}^\infty$ of Banach spaces is *strictly compatible* if there are Banach spaces U_A and V_A such that the continuous embeddings $V_A \subset A_n \subset U_A$ $(n = 1, 2, \dots)$ hold. Suppose that $\{B_n\}_{n=1}^\infty$ is another strictly compatible sequence of Banach spaces such that $V_B \subset B_n \subset U_B$ $(n = 1, 2, \dots)$. Then A and B are called *extrapolation spaces* (with respect to the scales $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$) if $V_A \subset A \subset U_A$ and $V_B \subset B \subset U_B$ and every linear operator T, $T : U_A \to U_B$, sending A_n to B_n with $\|T\|_{A_n \to B_n} \le 1$ for all $n = 1, 2, \dots$, is bounded from A to B. An *extrapolation functor* on the family $\mathscr{A}$ of strictly compatible scales is a mapping $\{A_n\} \overset{\mathscr{F}}{\mapsto} \mathscr{F}(\{A_n\})$, $\{A_n\} \in \mathscr{A}$, such that $\mathscr{F}(\{A_n\})$ and $\mathscr{F}(\{B_n\})$ are extrapolation spaces whenever $\{A_n\}$ and $\{B_n\}$ belong to the family $\mathscr{A}$.

The simplest extrapolation functors are the sum and intersection functors. Let X_k $(k = 1, 2, \dots)$ be Banach spaces embedded linearly and continuously in a Hausdorff topological vector space $\mathscr{T}$. Then their *intersection* is the Banach space $\Delta_{k=1}^\infty X_k$ constituted by all $x \in \bigcap_{k=1}^\infty X_k$ satisfying

$$\|x\| = \sup_{k=1,2,\dots} \|x\|_{X_k} < \infty.$$

Suppose additionally that there is a Banach space X_0 embedded into $\mathscr{T}$ and such that $X_k \overset{1}{\subset} X_0$ $(k = 1, 2, \dots)$. Then the sum $\sum_{k=1}^\infty X_k$ is defined as the set of all $x \in X_0$ representable as $x = \sum_{k=1}^\infty x_k$ $(x_k \in X_k)$ where $\sum_{k=1}^\infty \|x_k\|_{X_k} < \infty$. This space becomes a Banach space with the norm $\|x\| = \inf \sum_{k=1}^\infty \|x_k\|_{X_k}$, where the greatest lower bound is calculated over all possible representations of x.

Denote by e^k $(k = 0, \pm 1, \pm 2, \dots)$ the standard basis vectors in the space of two-sided real sequences; i.e., $e^k = (e_j^k)$, $e_k^k = 1$ and $e_j^k = 0$ $(j \ne k)$. If $1 \le p \le \infty$ then p' is the conjugate of p; i.e., $1/p' + 1/p = 1$. The dilation function of a positive function $\psi(s)$, $s \in (0, \infty)$, is defined by the relation $\mathscr{M}_\psi(t) = \sup_{s>0} \psi(st)/\psi(s)$. Finally, the relation $F_1 \asymp F_2$ means henceforth that $cF_1 \le F_2 \le CF_1$ for some $c > 0$ and $C > 0$; moreover, as a rule, the constants c and C are independent of all or a part of the arguments of F_1 and F_2.

§ 2. Auxiliary Results

In extrapolation theory, it is customary to consider, alongside the usual spaces of the real method, the modified spaces [5, p. 13]:

$$\big\langle (X_0, X_1)^{\mathscr K}_{\theta,p} \big\rangle = c_{\theta.p}(X_0, X_1)^{\mathscr K}_{\theta,p}, \quad \text{where } c_{\theta.p} = (\theta(1-\theta)p)^{1/p}(1 \le p < \infty) \text{ and } c_{\theta,\infty} = 1, \tag{2}$$

and

$$\big\langle (X_0, X_1)^{\mathscr J}_{\theta,p} \big\rangle = c'_{\theta.p}(X_0, X_1)^{\mathscr J}_{\theta,p}, \quad \text{where } c'_{\theta.p} = (\theta(1-\theta)p')^{-1/p'}(1 < p \le \infty) \text{ and } c'_{\theta,1} = 1.$$

The characteristic function of the functors $\big\langle (\cdot,\cdot)^{\mathscr K}_{\theta,p} \big\rangle$ and $\big\langle (\cdot,\cdot)^{\mathscr J}_{\theta,p} \big\rangle$ is exactly t^θ [5]; moreover [15, Example 7],

$$\big\langle (L_1, L_\infty)^{\mathscr K}_{1/q',q} \big\rangle = L_q \quad (1 \le q \le \infty) \tag{3}$$

and the norms of the spaces in (3) are equivalent with a constant independent of q. Note that for $\theta = 1/q'$ the constant $c_{\theta.q}$ is equal to $(q')^{-1/q}$, whence $1/\sqrt{2} \le c_{\theta.q} < 1$ if $q \ge 2$. Thereby, by (3),

$$(L_1, L_\infty)^{\mathscr K}_{1/q',q} = L_q \tag{4}$$

with a constant of the equivalence of norms independent of $q \ge 2$.

Moreover, we need some auxiliary assertions. The first contains a formula for calculation of the $\mathscr K$-functional of the pair $(L_\infty, M(\psi))$ ($M(\psi)$ is the Marcinkiewicz space on $[0,1]$). In the particular case of $\psi(t) = t\log_2^{1/2}(2/t)$, this was proved in [16] and applied therein to studying the space of multipliers generated by the Rademacher system.

Lemma 1. *Let ψ be an increasing concave function on $[0,1]$ such that its dilation function satisfies the condition $\mathscr M_\psi(1/2) < 1$. Then the relation*

$$\mathscr K(t, f; L_\infty, M(\psi)) \asymp t \sup_{u:\psi(u)\ge tu} \frac{f^*(u)u}{\psi(u)}$$

is valid with constants independent of $f \in M(\psi)$ and $t > 0$.

Proof. First of all, since $L_\infty = M(\psi_0)$ where $\psi_0(u) = u$, we have

$$\mathscr K(t, f; L_\infty, M(\psi)) \asymp \|f\|_{M(\psi_t)}, \quad \text{where} \quad \psi_t(u) = \max(u, \psi(u)/t), \tag{5}$$

with constants independent of $f \in M(\psi)$ and $t > 0$ [17].

Verify that the dilation function $\mathscr M_{\psi_t}(u)$ of ψ_t satisfies

$$\mathscr M_{\psi_t}(1/2) \le \mathscr M_\psi(1/2) < 1 \quad (t > 0). \tag{6}$$

To this end, represent

$$\mathscr M_{\psi_t}(1/2) = \frac{1}{2} \sup_{0<u\le 1} F_t(u), \quad \text{where} \quad F_t(u) = \frac{\max\big(1, \frac{2\psi(u/2)}{tu}\big)}{\max\big(1, \frac{\psi(u)}{tu}\big)}. \tag{7}$$

Since ψ is concave, the following three cases are possible:

(a) $\psi(u) \ge tu$;

(b) $\psi(u) < tu \le 2\psi(u/2)$;

(c) $2\psi(u/2) < tu$.

In the first two cases we have:

$$F_t(u) = \frac{2\psi(u/2)}{\psi(u)} \le 2\mathscr M_\psi(1/2)$$

and

$$F_t(u) = \frac{2\psi(u/2)}{tu} \le 2\mathscr M_\psi(1/2)\frac{\psi(u)}{tu} < 2\mathscr M_\psi(1/2),$$

respectively. In the last case $F_t(u) = 1 \le 2\mathscr M_\psi(1/2)$. Thereby, (6) is a consequence of (7).

Using (6) and arguing as in the proof of Lemma 1.4 of [10], we obtain

$$\int_0^s \frac{\psi_t(u)}{u}\,du \le C\psi_t(s),$$

where $C > 0$ is independent of $s \in [0,1]$ and $t > 0$. Hence, by the definition of the norm of the Marcinkiewicz space,

$$\|f\|_{M(\psi_t)} = \sup_{0<s\le 1} \frac{1}{\psi_t(s)} \int_0^s f^*(u)\,du$$

$$\le \sup_{0<s\le 1} \frac{1}{\psi_t(s)} \int_0^s \frac{\psi_t(u)}{u}\,du \sup_{0<u\le 1} \frac{uf^*(u)}{\psi_t(u)} \le C \sup_{0<u\le 1} \frac{uf^*(u)}{\psi_t(u)}.$$

Since the reverse inequality $\|f\|_{M(\psi_t)} \ge \sup_{0<u\le 1} \frac{uf^*(u)}{\psi_t(u)}$ is obvious, it follows from (5) that

$$\mathscr{K}(t, f; L_\infty, M(\psi)) \asymp \sup_{0<u\le 1} \frac{uf^*(u)}{\psi_t(u)} = \sup_{0<u\le 1} \frac{f^*(u)}{\max(1, \psi(u)/(ut))} = t \sup_{u:\psi(u)\ge ut} \frac{uf^*(u)}{\psi(u)}. \qquad \square$$

Lemma 2. *Let $\operatorname{Exp} L^\Phi$ be the exponential Orlicz space corresponding to an Orlicz function Φ. Then the inequality*

$$\|f\|_{\operatorname{Exp} L^\Phi} \le C \sup_{q\ge 1} \frac{\|f\|_q}{\Phi^{-1}(q)}$$

is valid with some $C > 0$ independent of f.

Proof. It is easy to verify that $\log(1 + 2t) \le c_0 \log(1 + t)$ $(t \ge 1)$, where $c_0 := \log 3/\log 2 < 2$. Therefore, by concavity of Φ^{-1}, the inequality $\mathscr{M}_\varphi(1/2) \le c_0/2 < 1$ is valid for the function $\varphi(u) = u\Phi^{-1}(\log(1 + (e - 1)/u))$. Consequently, by [13] and [10, Theorem 2.5.3],

$$\|f\|_{\operatorname{Exp} L^\Phi} \asymp \|f\|_{M(\varphi)} \asymp \sup_{0<u\le 1} \frac{f^*(u)}{\Phi^{-1}(\log(1 + (e - 1)/u))}, \tag{8}$$

whence

$$\|f\|_{\operatorname{Exp} L^\Phi} \le C_1 \sup_{q\ge 1} \frac{f^*((e - 1)/(e^q - 1))}{\Phi^{-1}(q)} \le C_1 \sup_{q\ge 1} \frac{f^*(e^{-q})}{\Phi^{-1}(q)}.$$

The assertion of the lemma now follows from the fact that

$$\|f\|_q \ge \left(\int_0^{e^{-q}} f^*(t)^q\,dt\right)^{1/q} \ge \frac{f^*(e^{-q})}{e}. \qquad \square$$

In line with [18], we now consider an approach in which the spaces of the real $\mathscr{K}$-method are parametrized by rearrangement invariant spaces on the interval $[0, 1]$. Namely, with each rearrangement invariant space X on $[0, 1]$ and an arbitrary Banach pair $\vec{A} = (A_0, A_1)$ we associate the space

$$X(\vec{A}) := \{a \in A_0 + A_1 : \lim_{t\to+0} \mathscr{K}(t, a; \vec{A}) = 0 \quad \text{and} \quad \mathscr{K}'(t, a; \vec{A})\chi_{[0,1]}(t) \in X\}$$

with the norm $\|a\|_{X(\vec{A})} := \|\mathscr{K}'(\cdot, a; \vec{A})\chi_{[0,1]}\|_X$.

As usual, we denote by c_0 the space of all sequences $(\alpha_k)_{k=-\infty}^{\infty}$ such that $\lim_{k\to\pm\infty} \alpha_k = 0$.

Lemma 3. *Let F be a Banach lattice of two-sided real sequences such that $F \subset c_0 + l_\infty(2^{-k})$ and*

$$\Big\|\sum_{k=1}^{\infty} e^k\Big\|_F \le K\|e^0\|_F. \tag{9}$$

Then for every Banach pair $\vec{A} = (A_0, A_1)$ such that $A_1 \subset A_0$ we have

$$(A_0, A_1)_F^{\mathscr{K}} = X(\vec{A}), \quad \text{where } X := (L_1, L_\infty)_F^{\mathscr{K}};$$

moreover, the norms of these spaces satisfy the inequality

$$\frac{1}{K+1}\|a\|_{X(\vec{A})} \le \|a\|_{(A_0,A_1)_F^{\mathscr{K}}} \le (K+1)\|a\|_{X(\vec{A})}. \tag{10}$$

Proof. If $a \in (A_0, A_1)_F^{\mathscr{K}}$ then, by hypothesis,

$$\lim_{t\to+0} \mathscr{K}(t, a; \vec{A}) = 0, \quad \|a\|_{(A_0,A_1)_F^{\mathscr{K}}} := \|(\mathscr{K}(2^k, a; \vec{A}))_k\|_F < \infty.$$

Since $A_1 \overset{1}{\subset} A_0$, we have $\mathscr{K}(t, a; \vec{A}) = \|a\|_{A_0}$ $(t \ge 1)$ and, by (9),

$$\Big\|\sum_{k=1}^{\infty} \mathscr{K}(2^k, a; \vec{A})e^k\Big\|_F = \|a\|_{A_0}\Big\|\sum_{k=1}^{\infty} e^k\Big\|_F \le K\|a\|_{A_0}\|e^0\|_F \le K\Big\|\sum_{k=0}^{-\infty} \mathscr{K}(2^k, a; \vec{A})e^k\Big\|_F,$$

whence

$$\Big\|\sum_{k=0}^{-\infty} \mathscr{K}(2^k, a; \vec{A})e^k\Big\|_F \le \|a\|_{(A_0,A_1)_F^{\mathscr{K}}} \le (K+1)\Big\|\sum_{k=0}^{-\infty} \mathscr{K}(2^k, a; \vec{A})e^k\Big\|_F. \tag{11}$$

In particular, using the well-known equality $\mathscr{K}(t, f; L_1, L_\infty) = \int_0^t f^*(s)\,ds$ [8, Theorem 5.2.1], we obtain

$$\Big\|\sum_{k=0}^{-\infty} \int_0^{2^k} f^*(s)\,ds\, e^k\Big\|_F \le \|f\|_X \le (K+1)\Big\|\sum_{k=0}^{-\infty} \int_0^{2^k} f^*(s)\,ds\, e^k\Big\|_F. \tag{12}$$

Since the function $\mathscr{K}(t, a; \vec{A})$ is absolutely continuous in t [10, p. 67], we now have $\mathscr{K}(t, a; \vec{A}) = \int_0^t \mathscr{K}'(s, a; \vec{A})\,ds$. Using the fact that the derivative $\mathscr{K}'(s, a; \vec{A})$ of a concave function decreases, from (11) and (12) we therefore obtain

$$\|a\|_{X(\vec{A})} = \|\mathscr{K}'(\cdot, a; \vec{A})\chi_{[0,1]}\|_X \le (K+1)\Big\|\sum_{k=0}^{-\infty} \int_0^{2^k} \mathscr{K}'(s, a; \vec{A})\,ds\, e^k\Big\|_F \le (K+1)\|a\|_{(A_0,A_1)_F^{\mathscr{K}}}.$$

Thus, the embedding $(A_0, A_1)_F^{\mathscr{K}} \subset X(\vec{A})$ and the left inequality in (10) are proved. The reverse embedding and the right inequality in (10) can be verified similarly by means of (11) and (12). □

§ 3. Extrapolation Functors on a Family of Scales Generated by the Real $\mathscr{K}$-Method

Suppose that $\Phi(u)$ is an Orlicz function on $[0, \infty)$ and $\vec{A} = (A_0, A_1)$ is an arbitrary Banach pair such that $A_1 \overset{1}{\subset} A_0$. Introduce the discrete scale of spaces constructed for the pair $\vec{A}$ by the real $\mathscr{K}$-method:

$$\vec{A}_n^{\mathscr{K}} = (A_0, A_1)_{1-1/\Phi(2^n),\Phi(2^n)}^{\mathscr{K}}, \quad n = 1, 2, \dots.$$

Denote by $\mathscr{A}$ the family of all such scales.

Definition 1. Let F be an intermediate Banach lattice with respect to the Banach pair $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$; i.e., $\Delta(\vec{l}_\infty) \subset F \subset \Sigma(\vec{l}_\infty)$. Define $\mathscr{L}^{\mathscr{K}}_{\Phi,F}(\{\vec{A}^{\mathscr{K}}_n\})$ as the set of all $a \in A_0$ such that the sequence $u_a = \sum_{n=1}^{\infty} \|a\|_{\vec{A}^{\mathscr{K}}_n} e^n$ belongs to F.

It is easy to verify that $\mathscr{L}^{\mathscr{K}}_{\Phi,F}\big(\{\vec{A}^{\mathscr{K}}_n\}\big)$ is a Banach space with the norm $\|a\|_{\mathscr{L}^{\mathscr{K}}_{\Phi,F}(\{\vec{A}^{\mathscr{K}}_n\})} := \|u_a\|_F$ and the mapping $\{\vec{A}^{\mathscr{K}}_n\} \mapsto \mathscr{L}^{\mathscr{K}}_{\Phi,F}\big(\{\vec{A}^{\mathscr{K}}_n\}\big)$ is an extrapolation functor on the family $\mathscr{A}$ in the sense of the definition of [5, p. 8] (see § 1).

Our main goal is to describe the corresponding limit (or extrapolation) spaces of these scales. We will show that, under some conditions, all of them are precisely interpolation spaces with respect to the pair $(A_1, M_\varphi(\vec{A}))$ where $M_\varphi(\vec{A})$ is the generalized Marcinkiewicz space constructed with the increasing concave function

$$\varphi(u) = u\Phi^{-1}(\log(1 + (e-1)/u)) \quad (0 < u \le 1).$$

Recall [19, p. 422] that

$$M_\varphi(\vec{A}) = (A_0, A_1)^{\mathscr{K}}_{l_\infty(1/\varphi(2^k))}.$$

Note that, by (11) and the fact that $A_1 \overset{1}{\subset} A_0$, the last space is well-defined in spite of the fact that the function $\varphi(u)$ is defined only on the interval $[0,1]$.

First of all, we find an extrapolation description of the $\mathscr{K}$-functional for the pair $(A_1, M_\varphi(\vec{A}))$, whence, in particular, it will follow that $M_\varphi(\vec{A})$ is a limit space for the scale $\{\vec{A}^{\mathscr{K}}_n\}$ as $n \to \infty$. Recall that $\widetilde{A}_1$ is the completion of A_1 with respect to A_0 (see § 1).

Theorem 1. *Let $\Phi(u)$ be an Orlicz function on $[0,\infty)$. For a Banach pair $\vec{A} = (A_0, A_1)$ such that $A_1 \overset{1}{\subset} A_0$ and $\widetilde{A}_1 = A_1$, the relation*

$$\mathscr{K}(2^k, a; A_1, M_\varphi(\vec{A})) \asymp \sup_{n \ge k} 2^{k-n} \|a\|_{\vec{A}^{\mathscr{K}}_n}$$

is valid with constants independent of $\vec{A}$, $a \in M_\varphi(\vec{A})$, and $k = 1, 2, \dots$.

First, consider the particular case in which $\vec{A}$ is the pair (L_1, L_∞) of spaces of functions on $[0,1]$. Then $M_\varphi(\vec{A})$ coincides with the usual Marcinkiewicz space $M(\varphi)$ or the exponential Orlicz space $\operatorname{Exp} L^\Phi$ (with equivalence of norms) [12, 13].

Proposition 1. *If Φ is an arbitrary Orlicz function then the following holds:*

$$\mathscr{K}(p, f; L_\infty, \operatorname{Exp} L^\Phi) \asymp p \sup_{q \ge p} \frac{\|f\|_{\Phi(q)}}{q}$$

with constants independent of $f \in \operatorname{Exp} L^\Phi$ and $p \ge 1$.

Proof. By Lemma 1,

$$\mathscr{K}(\Phi^{-1}(t), f; L_\infty, \operatorname{Exp} L^\Phi) \asymp \Phi^{-1}(t) \sup_{0 < u \le (e-1)/(e^t - 1)} \frac{f^*(u)}{\Phi^{-1}(\log(1 + (e-1)/u))} \quad (t > 0). \tag{13}$$

Given $q \ge 1$, we estimate

$$\|f\|_q \le \bigg(e^{2q} \int_0^{e^{-2q}} (f^*(s))^q \, ds\bigg)^{1/q}$$

$$\le e^2 \bigg(\int_0^{e^{-2q}} (\Phi^{-1}(\log(1 + (e-1)/u)))^q \, du\bigg)^{1/q} \sup_{0 < u \le e^{-q}} \frac{f^*(u)}{\Phi^{-1}(\log(1 + (e-1)/u))}. \tag{14}$$

By change of variables, we obtain

$$\int_0^{e^{-2q}} (\Phi^{-1}(\log(1+(e-1)/u)))^q\,du = (e-1)\int_{\log(1+e^{2q})}^{\infty} (\Phi^{-1}(v))^q \frac{e^v}{(e^v-1)^2}\,dv \le 8I(\Phi,q), \tag{15}$$

where $I(\Phi,q) := \int_{2q}^{\infty}(\Phi^{-1}(v))^q e^{-v}\,dv$.

Since $\Phi^{-1}(v)$ is concave and $\Phi^{-1}(v)\Phi'(\Phi^{-1}(v))$ is increasing; therefore, integrating by parts, we infer the estimate

$$I(\Phi,q) = e^{-2q}(\Phi^{-1}(2q))^q + q\int_{2q}^{\infty} e^{-v}\frac{(\Phi^{-1}(v))^{q-1}}{\Phi'(\Phi^{-1}(v))}\,dv \le e^{-2q}2^q(\Phi^{-1}(q))^q + \frac{q}{\Phi^{-1}(2q)\Phi'(\Phi^{-1}(2q))}I(\Phi,q).$$

Moreover, since $v\Phi'(v) \ge \Phi(v)$, we have $\Phi^{-1}(2q)\Phi'(\Phi^{-1}(2q)) \ge 2q$ and

$$I(\Phi,q) \le e^{-2q}2^q(\Phi^{-1}(q))^q + \frac{1}{2}I(\Phi,q),$$

whence

$$I(\Phi,q) \le e^{-2q}2^{1+q}(\Phi^{-1}(q))^q.$$

The last inequality together with (13)–(15) yields

$$\|f\|_q \le C_1\mathscr{K}(\Phi^{-1}(q), f; L_\infty, \operatorname{Exp} L^\Phi) \quad (q\ge 1).$$

Since $\mathscr{K}(t,x;X_0,X_1)$ is a concave function of t, we have

$$\frac{\|f\|_q}{\Phi^{-1}(q)} \le C_1\frac{\mathscr{K}(\Phi^{-1}(q), f; L_\infty, \operatorname{Exp} L^\Phi)}{\Phi^{-1}(q)} \le C_1\frac{\mathscr{K}(\Phi^{-1}(p), f; L_\infty, \operatorname{Exp} L^\Phi)}{\Phi^{-1}(p)}$$

for all $q\ge p\ge 1$. Changing variables and using the properties of Φ, we arrive eventually at the estimate

$$p\sup_{q\ge p}\frac{\|f\|_{\Phi(q)}}{q} \le C_1\mathscr{K}(p,f;L_\infty,\operatorname{Exp} L^\Phi).$$

We have to prove the reverse inequality. To this end, by (13), it is sufficient to show that the inequality

$$f^*(u) \le C_2\sup_{q\ge p}\frac{\|f\|_q}{\Phi^{-1}(q)}\Phi^{-1}(\log(1+(e-1)/u)) \tag{16}$$

is valid for all $0<u\le (e-1)/(e^p-1)$.

From Lemma 2 and (8) we obtain

$$f^*(u) \le C_3\sup_{q\ge 1}\frac{\|f\|_q}{\Phi^{-1}(q)}\Phi^{-1}(\log(1+(e-1)/u)) \quad (0<u\le 1). \tag{17}$$

In the proof of (16) we can assume that $f=f^*$ and $\{s\in[0,1] : f(s)\ne 0\}\subset[0,2^{1-p}]$ (since $(e-1)/(e^p-1) \le 2^{1-p}$). If $1\le q\le p$ then, by Hölder's inequality,

$$\|f\|_q \le 2^{\frac{(1-p)(p-q)}{pq}}\|f\|_p \le 2^{2-\frac{p}{q}}\|f\|_p.$$

Moreover, $\Phi^{-1}(p)\le \frac{p}{q}\Phi^{-1}(q)\le 2^{p/q}\Phi^{-1}(q)$ by concavity of Φ^{-1}. Hence, for $1\le q\le p$

$$\frac{\|f\|_q}{\Phi^{-1}(q)} \le \frac{4\|f\|_p}{\Phi^{-1}(p)},$$

whence

$$\sup_{1\le q\le p}\frac{\|f\|_q}{\Phi^{-1}(q)} \le \frac{4\|f\|_p}{\Phi^{-1}(p)}.$$

Thereby (16) follows from (17) and the proposition is proved. $\square$

As a consequence, we obtain an extrapolation description for the exponential Orlicz spaces.

Corollary 1. *For every Orlicz function Φ we have*

$$\|f\|_{\operatorname{Exp} L^\Phi} \asymp \sup_{p\ge 1} \frac{\|f\|_{\Phi(p)}}{p}.$$

Proof. Since $L_\infty \overset{1}{\subset} \operatorname{Exp} L^\Phi$, by concavity of the $\mathscr{K}$-functional, we have

$$\|f\|_{\operatorname{Exp} L^\Phi} = \mathscr{K}(1, f; L_\infty, \operatorname{Exp} L^\Phi) \le \mathscr{K}(2, f; L_\infty, \operatorname{Exp} L^\Phi) \le 2\|f\|_{\operatorname{Exp} L^\Phi},$$

and the claim follows from Proposition 1. $\square$

Proof of Theorem 1. Using the definition of generalized Marcinkiewicz space, [19, p. 384], and the condition $\widetilde{A}_1 = A_1$, we find that

$$M_\varphi(\vec{A}) = (A_0, A_1)^{\mathscr{K}}_{l_\infty(1/\varphi(2^k))} \quad \text{and} \quad A_1 = (A_0, A_1)^{\mathscr{K}}_{l_\infty(2^{-k})} \tag{18}$$

isometrically. In particular,

$$\operatorname{Exp} L^\Phi = (L_1, L_\infty)^{\mathscr{K}}_{l_\infty(1/\varphi(2^k))} \quad \text{and} \quad L_\infty = (L_1, L_\infty)^{\mathscr{K}}_{l_\infty(2^{-k})}. \tag{19}$$

Since φ is a concave function and $\lim_{s\to+0}\varphi(s) = 0$, the conditions of Lemma 3 are satisfied for the spaces $l_\infty(2^{-k})$ and $l_\infty(1/\varphi(2^k))$; hence,

$$M_\varphi(\vec{A}) = \operatorname{Exp} L^\Phi(\vec{A}), \quad A_1 = L_\infty(\vec{A}), \tag{20}$$

and the norms of these spaces are equivalent with a constant independent of $\vec{A}$. Moreover, $l_\infty(2^{-k})$ and $l_\infty(1/\varphi(2^k))$ are interpolation spaces with respect to $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$ (for example, see [19, p. 422]). Therefore, putting $f_a(s) := \mathscr{K}(s, a; \vec{A})$ ($a \in A_0$) and applying the reiteration arguments [19, Corollary 7.1.1], from (18), (19), and Proposition 1 we obtain

$$\begin{aligned}\mathscr{K}(t, a; A_1, M_\varphi(\vec{A})) &= \mathscr{K}\big(t, a; (A_0, A_1)^{\mathscr{K}}_{l_\infty(2^{-k})}, (A_0, A_1)^{\mathscr{K}}_{l_\infty(1/\varphi(2^k))}\big) \\ &\asymp \mathscr{K}(t, (f_a(2^k))_k; l_\infty(2^{-k}), l_\infty(1/\varphi(2^k)))\end{aligned}$$

$$= \mathscr{K}\left(t, \left(\int_0^{2^k} f_a'(s)\,ds\right)_k; l_\infty(2^{-k}), l_\infty(1/\varphi(2^k))\right) \asymp \mathscr{K}(t, f_a'; L_\infty, \operatorname{Exp} L^\Phi) \asymp t \sup_{q\ge t} \frac{\|f_a'\|_{\Phi(q)}}{q}.$$

Rewrite the last relation in discrete form:

$$\mathscr{K}(2^k, a; A_1, M_\varphi(\vec{A})) \asymp \sup_{n\ge k} 2^{k-n}\|f_a'\|_{\Phi(2^n)} \quad (k = 1, 2, \dots). \tag{21}$$

It is easy to verify that the parameter $l_q(2^{(1/q-1)k})$ satisfies (9) with the constant $K = 3$ for every $q \ge 2$. Therefore, by Lemma 3,

$$(A_0, A_1)^{\mathscr{K}}_{1/q',q} = (L_1, L_\infty)^{\mathscr{K}}_{1/q',q}(\vec{A})$$

and the norms of these spaces are equivalent with the same constant for all $q \ge 2$. Using this together with (4) and (21), we obtain the claim of the theorem. $\square$

In particular, the generalized Marcinkiewicz space $M_\varphi(\vec{A})$ is an extrapolation space with respect to the scale $\{\vec{A}^{\mathscr{K}}_n\}$.

Corollary 2. *The following equivalence holds for every Orlicz function* Φ *and an arbitrary Banach pair* $\vec{A} = (A_0, A_1)$ *such that* $A_1 \overset{1}{\subset} A_0$ *and* $\widetilde{A}_1 = A_1$*:*

$$\|a\|_{M_\varphi(\vec{A})} \asymp \sup_{n\geq 1} 2^{-n}\|a\|_{\vec{A}_n^{\mathscr{K}}}.$$

Proof. It is easy to verify that $A_1 \overset{1}{\subset} M_\varphi(\vec{A})$. Therefore, the arguments are exactly the same as in the proof of Corollary 1. $\square$

Observe that A_1 (if $\widetilde{A}_1 = A_1$) and the Marcinkiewicz space $M_\varphi(\vec{A})$ are the least and greatest extrapolation spaces of the scale $\{\vec{A}_n^{\mathscr{K}}\}$ which correspond to the family $\mathscr{L}_{\Phi,F}^{\mathscr{K}}$ of extrapolation functors (with a fixed function Φ). Namely, we have

$$\mathscr{L}^{\mathscr{K}}{}_{\Phi,l_\infty}\big(\{\vec{A}_n^{\mathscr{K}}\}\big) = A_1 \quad \text{and} \quad \mathscr{L}^{\mathscr{K}}{}_{\Phi,l_\infty(2^{-k})}\big(\{\vec{A}_n^{\mathscr{K}}\}\big) = M_\varphi(\vec{A}). \tag{22}$$

Indeed, by (4), Lemma 3, and Corollary 1,

$$\|u_a\|_{l_\infty} = \sup_{n=1,2,\dots} \|a\|_{\vec{A}_n^{\mathscr{K}}} \asymp \sup_{k=1,2,\dots} \|\mathscr{K}'(\cdot, a; \vec{A})\chi_{[0,1]}\|_{\Phi(2^k)} = \|\mathscr{K}'(\cdot, a; \vec{A})\chi_{[0,1]}\|_{L_\infty}$$

and

$$\|u_a\|_{l_\infty(2^{-k})} = \sup_{n=1,2,\dots} 2^{-n}\|a\|_{\vec{A}_n^{\mathscr{K}}} \asymp \sup_{n=1,2,\dots} 2^{-n}\|\mathscr{K}'(\cdot, a; \vec{A})\chi_{[0,1]}\|_{\Phi(2^n)} = \|\mathscr{K}'(\cdot, a; \vec{A})\chi_{[0,1]}\|_{\operatorname{Exp} L^\Phi}.$$

We are done on applying relations (20).

From (22) we infer that the following embeddings hold for every Banach pair $\vec{A} = (A_0, A_1)$ such that $A_1 \overset{1}{\subset} A_0$ and every Banach lattice F intermediate between l_∞ and $l_\infty(2^{-k})$:

$$A_1 \subset \mathscr{L}_{\Phi,F}^{\mathscr{K}}\big(\{\vec{A}_n^{\mathscr{K}}\}\big) \subset M_\varphi(\vec{A}).$$

Moreover, Theorem 2, the main result of this section, implies that $\mathscr{L}_{\Phi,F}^{\mathscr{K}}\big(\vec{A}_n^{\mathscr{K}}\big)$ is an interpolation space with respect to $(A_1, M_\varphi(\vec{A}))$, provided that F is an interpolation lattice with respect to $\vec{l}_\infty$.

Theorem 2. *Suppose that* Φ *is an Orlicz function and* $\vec{A} = (A_0, A_1)$ *is a Banach pair such that* $A_1 \overset{1}{\subset} A_0$ *and* A_1 *is complete with respect to* A_0*. Then for every interpolation Banach lattice* F *with respect to* $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$*, we have*

$$\mathscr{L}_{\Phi,F}^{\mathscr{K}}\big(\{\vec{A}_n^{\mathscr{K}}\}\big) = (A_1, M_\varphi(\vec{A}))_F^{\mathscr{K}} \quad (\textit{with equivalence of norms}).$$

Proof. It is sufficient to show that the relation

$$\|u_a\|_F \asymp \|v_a\|_F \tag{23}$$

holds with some constants independent of $a \in A_0$, where

$$u_a = \sum_{n=1}^{\infty} \|a\|_{\vec{A}_n^{\mathscr{K}}} e^n \quad \text{and} \quad v_a = \sum_{n=-\infty}^{\infty} \mathscr{K}(2^n, a; A_1, M_\varphi(\vec{A}))e^n.$$

Since $A_1 \overset{1}{\subset} M_\varphi(\vec{A})$, we have $(v_a)_n = 2^n(v_a)_0 = 2^n\|a\|_{M_\varphi(\vec{A})}$ for $n = 0, -1, -2, \dots$. Therefore, by Lemma 2 of [6], we obtain

$$\|v_a\|_F \asymp \|P_+ v_a\|_F \tag{24}$$

with a constant independent of a, where $P_+ w = \sum_{j=1}^{\infty} w_j e^j$ and $w = (w_j)_{j=-\infty}^{\infty}$.

It is well known [9, Remark 3.3.8] that

$$\|w\|_F \asymp \|\bar{w}\|_F, \quad \text{where } \overline{w} = (\overline{w}_k)_{k=-\infty}^{\infty}, \bar{w}_k = \sup_{n=0,\pm 1,\dots} \{\min[1, 2^{k-n}]|w_n|\},$$

provided that F is an interpolation space with respect to $\vec{l}_\infty$. Consequently, $\|u_a\|_F \asymp \|\bar{u_a}\|_F$, whence, by the inequalities $\bar{u_a} \ge P_+\bar{u_a} \ge u_a$, we obtain

$$\|u_a\|_F \asymp \|P_+\bar{u_a}\|_F. \tag{25}$$

Since the norm of a function in $L_q[0,1]$ increases in q, application of Lemma 3 to the parameter $l_q(2^{(1/q-1)k})$ $(k \ge 2)$ using (4) demonstrates that the sequence u_a increases in the generalized sense: $\|a\|_{\vec{A}_n^{\mathscr{K}}} \le C\|a\|_{\vec{A}_m^{\mathscr{K}}}$ for some $C > 0$ and arbitrary $1 \le n \le m$. Therefore,

$$P_+\bar{u_a} \asymp \sum_{k=1}^{\infty} 2^k \sup_{n \ge k} \frac{\|a\|_{\vec{A}_n^{\mathscr{K}}}}{2^n} e^k.$$

Hence, by Theorem 1, $\|P_+\bar{u_a}\|_F \asymp \|P_+v_a\|_F$ with universal constants. Thereby relation (23) follows from (24) and (25). $\square$

Consider the important particular case: $\vec{A} = \vec{L} := (L_1, L_\infty)$. By (4), $\vec{L}_n^{\mathscr{K}} = L_{\Phi(2^n)}$ with a constant independent of $n = 1, 2, \dots$. Consequently, $\mathscr{L}_{\Phi,F}^{\mathscr{K}}(\{\vec{L}_n^{\mathscr{K}}\})$ is a rearrangement invariant space on $[0,1]$ consisting of all measurable functions f on $[0,1]$ such that the sequence $u_f = \sum_{k=1}^{\infty} \|f\|_{\Phi(2^k)} e^k$ belongs to F and $\|f\|_{\mathscr{L}_{\Phi,F}^{\mathscr{K}}(\{\vec{L}_n^{\mathscr{K}}\})} = \|u_f\|_F$.

Corollary 3. *The following is valid for an arbitrary Orlicz function Φ and every interpolation Banach lattice F with respect to $(l_\infty, l_\infty(2^{-k}))$:*

$$\mathscr{L}_{\Phi,F}^{\mathscr{K}}(\{L_{\Phi(2^n)}\}) = (L_\infty, \operatorname{Exp} L^{\Phi})_F^{\mathscr{K}} \quad (\textit{with equivalence of norms}).$$

Interpolation for the pair $(L_\infty, \operatorname{Exp} L^{\Phi})$ is described by the real $\mathscr{K}$-method [17] (also see [20, Proposition 1]). This means that every interpolation space X with respect to this pair is representable in the form $X = (L_\infty, \operatorname{Exp} L^{\Phi})_F^{\mathscr{K}}$, where F is a Banach lattice of two-sided real sequences. Moreover, as observed in § 1, we can assume that F is an interpolation lattice with respect to $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$. Therefore, from Theorem 2 we obtain the following extrapolation description for interpolation spaces with respect to $(L_\infty, \operatorname{Exp} L^{\Phi})$.

Corollary 4. *Let Φ be an arbitrary Orlicz function. For every interpolation space X with respect to $(L_\infty, \operatorname{Exp} L^{\Phi})$, there is a Banach lattice F such that $X = \mathscr{L}_{\Phi,F}^{\mathscr{K}}(\{L_{\Phi(2^n)}\})$ (with equivalence of the norms).*

§ 4. Extrapolation Functors on a Family of Scales Generated by the Real $\mathscr{J}$-Method

Suppose that Φ is an Orlicz function and $\vec{A} = (A_0, A_1)$ is a Banach pair, $A_1 \overset{1}{\subset} A_0$. Introduce the scale of spaces constructed for the pair $\vec{A}$ by the real $\mathscr{J}$-method:

$$\vec{A}_n^{\mathscr{J}} = (A_0, A_1)_{1/\Phi(2^n), \Phi(2^n)/(\Phi(2^n)-1)}^{\mathscr{J}}, \quad n = 1, 2, \dots.$$

Denote the family of all such scales by $\mathscr{B}$.

Definition 2. Let a Banach lattice G be intermediate with respect to the Banach pair $\vec{l_1} = (l_1, l_1(2^{-k}))$; i.e., $\Delta(\vec{l_1}) \subset G \subset \Sigma(\vec{l_1})$. Define $\mathscr{L}_{\Phi,G}^{\mathscr{J}}(\{\vec{A}_n^{\mathscr{J}}\})$ as the set of all $b \in A_0$ for which there is a representation

$$b = \sum_{n=1}^{\infty} b_n \quad (\text{w.r.t. the convergence of } A_0), \quad b_n \in \vec{A}_n^{\mathscr{J}} \quad \text{and} \quad \sum_{n=1}^{\infty} \|b_n\|_{\vec{A}_n^{\mathscr{J}}} e^{-n} \in G. \tag{26}$$

It is easy to verify that $\mathscr{L}^{\mathscr{J}}_{\Phi,G}\big(\{\vec{A}^{\mathscr{J}}_n\}\big)$ is a Banach space with the norm

$$\|b\|_{\mathscr{L}^{\mathscr{J}}_{\Phi,G}(\{\vec{A}^{\mathscr{J}}_n\})} = \inf \Big\| \sum_{n=1}^{\infty} \|b_n\|_{\vec{A}^{\mathscr{J}}_n} e^{-n} \Big\|_G,$$

where the greatest lower bound is taken over all representations (26) and the mapping $\{\vec{A}^{\mathscr{J}}_n\} \mapsto \mathscr{L}^{\mathscr{J}}_{\Phi,G}\big(\{\vec{A}^{\mathscr{J}}_n\}\big)$ is the extrapolation functor on $\mathscr{B}$.

Show that the corresponding extrapolation spaces of the scale $\{\vec{A}^{\mathscr{J}}_n\}$ are interpolation spaces with respect to $(A_0, \Lambda_\varphi(\vec{A}))$, where $\Lambda_\varphi(\vec{A})$ is the generalized Lorentz space [19, p. 430],

$$\Lambda_\varphi(\vec{A}) = (A_0, A_1)^{\mathscr{J}}_{l_1(2^{-k}\varphi(2^k))}, \quad \text{where} \quad \varphi(u) = u\Phi^{-1}(\log(1 + (e-1)/u)) \ (0 < u \le 1).$$

Since $A_1 \overset{1}{\subset} A_0$, in the definition of this space we can confine ourselves to representations of the form $x = \sum_{k=0}^{-\infty} u_k$ ($u_k \in A_1$) (see the proof of Theorem 4 below).

We start with a theorem on description of the $\mathscr{J}$-functional for the pair $(A_0, \Lambda_\varphi(\vec{A}))$. In its proof we need the following well-known assertion on duality of the $\mathscr{J}$- and $\mathscr{K}$-methods [8, Theorem 3.7.1]. Suppose that $\vec{A} = (A_0, A_1)$ is a Banach pair such that $A_0 \cap A_1$ is everywhere dense in A_0 and A_1. Then the dual spaces A_0^* and A_1^* also constitute a Banach pair and the following holds:

$$\big((A_0, A_1)^{\mathscr{J}}_{\theta,r}\big)^* = \big(A_0^*, A_1^*\big)^{\mathscr{K}}_{\theta,r'}, \quad \frac{1}{r} + \frac{1}{r'} = 1 \quad (1 \le r < \infty). \tag{27}$$

It is important to observe that (27) holds isometrically for all $0 < \theta < 1$ and $1 \le r < \infty$. Indeed, the embedding $\big((A_0, A_1)^{\mathscr{J}}_{\theta,r}\big)^* \overset{1}{\subset} (A_0^*, A_1^*)^{\mathscr{K}}_{\theta,r'}$ is proved in [8, Theorem 3.7.1, relation (3)]. The reverse embedding (also with constant 1) can be verified in exactly the same way as (4) in [8].

Theorem 3. *Let $\Phi(u)$ be an Orlicz function on $[0, \infty)$. Given a Banach pair $\vec{A} = (A_0, A_1)$ such that $A_1 \overset{1}{\subset} A_0$ and A_1 is everywhere dense in A_0, we have*

$$\mathscr{J}(2^{-k}, b; A_0, \Lambda_\varphi(\vec{A})) \asymp \|b\|_{U_k},$$

where $U_k = \sum_{n \ge k} 2^{n-k} \vec{A}^{\mathscr{J}}_n$, with constants independent of $\vec{A}$, $b \in A_0$, and $k = 1, 2, \dots$.

Proof. For the duality reasons, it is sufficient to verify the equivalence of the norms of the corresponding dual spaces. In other words [8, p. 74], it is necessary to show that

$$\mathscr{K}(2^k, a; A_0^*, (\Lambda_\varphi(\vec{A}))^*) \asymp \|a\|_{(U_k)^*} \quad (k = 1, 2, \dots). \tag{28}$$

Since, by hypothesis, A_1 is everywhere dense in A_0 and $l_1(2^{-k}\varphi(2^k))$ is an interpolation space with respect to $\vec{l_1}$ [19, p. 430], by the duality theorem for the real method [19, Theorem 7.5.1],

$$(\Lambda_\varphi(\vec{A}))^* = (A_0^*, A_1^*)^{\mathscr{K}}_F,$$

where F is the dual Banach lattice to $l_1(2^{-k}\varphi(2^k))$ with respect to the bilinear form $\langle \alpha, \beta \rangle = \sum_{k=-\infty}^{\infty} \alpha_{-k}\beta_k$. It is easy to verify that $F = l_\infty(2^{-k}/\varphi(2^{-k}))$; i.e.,

$$(\Lambda_\varphi(\vec{A}))^* = M_\psi(A_0^*, A_1^*),$$

where $\psi(u) = u\varphi(1/u)$. Using the definition of generalized Marcinkiewicz space and the equality

$$\mathscr{K}(t, b; A_0, A_1) = t\mathscr{K}(1/t, b; A_1, A_0) \quad (t > 0), \tag{29}$$

we rewrite the last relation as

$$(\Lambda_\varphi(\vec{A}))^* = M_\varphi(\vec{A}^*), \quad \text{where} \quad \vec{A}^* := (A_1^*, A_0^*). \tag{30}$$

Since A_1 is everywhere dense in $\vec{A}_n^{\mathscr{J}}$ $(n = 1, 2, \dots)$ [8, Theorem 3.4.2]; therefore, applying the duality theorem for infinite sums and intersections of Banach spaces [3] and relation (27), we obtain

$$(U_k)^* = \varDelta_{n\geq k} 2^{k-n} \big(\vec{A}_n^{\mathscr{J}}\big)^* = \varDelta_{n\geq k} 2^{k-n} (A_0^*, A_1^*)^{\mathscr{K}}_{1/\Phi(2^n),\Phi(2^n)}$$

or, using (29) again,

$$(U_k)^* = \varDelta_{n\geq k} 2^{k-n} (A_1^*, A_0^*)^{\mathscr{K}}_{1-1/\Phi(2^n),\Phi(2^n)} = \varDelta_{n\geq k} 2^{k-n} \vec{A}_n^{*\mathscr{K}}.$$

By (30), we see that (28) becomes the equivalence

$$\mathscr{K}(2^k, a; A_0^*, M_\varphi(\vec{A}^*)) \asymp \sup_{n\geq k} 2^{k-n} \|a\|_{\vec{A}_n^{*\mathscr{K}}}.$$

The space A_0^* is complete with respect to A_1^* [10, Theorem 1.2.2] and so applying Theorem 1 to $\vec{A}^*$, we conclude that the last relation is valid with universal constants. Consequently, Theorem 3 is proved. ☐

Corollary 5. *Suppose that $\vec{A} = (A_0, A_1)$ is a Banach pair, $A_1 \overset{1}{\subset} A_0$, and A_1 is everywhere dense in A_0. If Φ is an Orlicz function and $\varphi(u) = u\Phi^{-1}(\log(1 + (e-1)/u))$ then*

$$\Lambda_\varphi(\vec{A}) = \sum_{n=1}^{\infty} 2^n \vec{A}_n^{\mathscr{J}} \quad (\textit{with equivalence of norms}).$$

Proof. Since $\Lambda_\varphi(\vec{A}) \overset{1}{\subset} A_0$, we have $\mathscr{J}(1, b; A_0, \Lambda_\varphi(\vec{A})) = \|b\|_{\Lambda_\varphi(\vec{A})}$. By the definition of $\mathscr{J}$-functional,

$$\frac{1}{2}\|b\|_{\Lambda_\varphi(\vec{A})} \leq \mathscr{J}(1/2, b; , A_0, \Lambda_\varphi(\vec{A})) \leq \|b\|_{\Lambda_\varphi(\vec{A})},$$

and the claim follows from Theorem 3. ☐

In the particular case $\vec{A} = (L_1, L_\infty)$ the space $\Lambda_\varphi(\vec{A})$ coincides with the rearrangement invariant Lorentz space $\Lambda(\varphi)$. From (4) and (27) we find that

$$(L_1, L_\infty)^{\mathscr{J}}_{1/q,q'} = L_{q'}$$

with a constant independent of $q \geq 2$. Therefore, Theorem 3 yields

Corollary 6. *If Φ is an arbitrary Orlicz function then the relations*

$$\mathscr{J}(2^{-k}, g; L_1, \Lambda(\varphi)) \asymp \|g\|_{U_k},$$

where $U_k = \sum_{j\geq k} \big(2^{j-k} L_{r_j}\big)$ and $r_j = \Phi(2^j)/(\Phi(2^j) - 1)$ $(j = 1, 2, \dots)$, are valid with a constant independent of $g \in \Lambda(\varphi)$ and $k = 1, 2, \dots$.

We now prove the main result of this section by using Theorem 3.

Theorem 4. *Suppose that G is an interpolation Banach lattice with respect to $\vec{l_1} = (l_1, l_1(2^{-k}))$ and $\vec{A} = (A_0, A_1)$ is a Banach pair such that $A_1 \overset{1}{\subset} A_0$ and A_1 is everywhere dense in A_0. Then $\mathscr{L}^{\mathscr{J}}_{\Phi,G}\big(\{\vec{A}_n^{\mathscr{J}}\}\big) = (A_0, \Lambda_\varphi(\vec{A}))^{\mathscr{J}}_G$ (with equivalence of norms).*

Proof. Denote $Y = (A_0, \Lambda_\varphi(\vec{A}))^{\mathscr{J}}_G$ and put $\mathscr{J}(t, b) = \mathscr{J}(t, b; A_0, \Lambda_\varphi(\vec{A}))$ for arbitrary $b \in \Lambda_\varphi(\vec{A})$ and $t > 0$. By the definition of spaces of the $\mathscr{J}$-method,

$$\|b\|_Y = \inf \|(\mathscr{J}(2^k, b_k))_k\|_G,$$

where the greatest lower bound is taken over all representations

$$b = \sum_{k=-\infty}^{\infty} b_k \quad (\text{w.r.t. the convergence of } A_0), \quad b_k \in \Lambda_\varphi(\vec{A}). \tag{31}$$

If $b \in \mathscr{L}^{\mathscr{J}}_{\Phi,G}\big(\{\vec{A}^{\mathscr{J}}_n\}\big)$ then, by definition,

$$b = \sum_{n=1}^{\infty} c_n,\ c_n \in \vec{A}^{\mathscr{J}}_n \quad \text{and} \quad \Big\|\sum_{n=1}^{\infty} \|c_n\|_{\vec{A}^{\mathscr{J}}_n} e^{-n}\Big\|_G < \infty.$$

Then $c_n \in U_n$ and, by Theorem 3, $\mathscr{J}(2^{-n}, c_n) \le C_1 \|c_n\|_{\vec{A}^{\mathscr{J}}_n}$ for some $C_1 > 0$. Thereby (31) is valid for the sequence $\{b_n\}$ such that $b_n = c_{-n}$, $n = -1, -2, \dots$, and $b_n = 0$, $n = 0, 1, 2, \dots$; moreover, $b_n \in \Lambda_\varphi(\vec{A})$ and

$$\|(\mathscr{J}(2^n, b_n))_n\|_G \le C_1 \Big\|\sum_{n=1}^{\infty} \|c_n\|_{\vec{A}^{\mathscr{J}}_n} e^{-n}\Big\|_G.$$

Hence, $\mathscr{L}^{\mathscr{J}}_{\Phi,G}\big(\{\vec{A}^{\mathscr{J}}_n\}\big) \subset Y$ and $\|b\|_Y \le C_1 \|b\|_{\mathscr{L}^{\mathscr{J}}_{\Phi,G}(\{\vec{A}^{\mathscr{J}}_n\})}$.

To prove the reverse embedding, we show first that the norm of Y can be calculated (to within equivalence) using only representations (31) with $b_k = 0$ for $k = 0, 1, 2, \dots$.

Since $\Lambda_\varphi(\vec{A}) \overset{1}{\subset} A_0$, we obtain $\mathscr{J}(2^k, b) = 2^k \|b\|_{\Lambda_\varphi(\vec{A})}$ $(k = 0, 1, 2, \dots)$. Suppose that $\{b_k\} \subset \Lambda_\varphi(\vec{A})$ satisfies (31). If C_2 is the constant of the embedding $G \subset \Sigma(\vec{l_1})$ then

$$\begin{aligned}\Big\|\sum_{k=0}^{\infty} \mathscr{J}(2^k, b_k) e^k\Big\|_G &= \Big\|\sum_{k=0}^{\infty} 2^k \|b_k\|_{\Lambda_\varphi(\vec{A})} e^k\Big\|_G \\ \ge C_2^{-1} \Big\|\sum_{k=0}^{\infty} 2^k \|b_k\|_{\Lambda_\varphi(\vec{A})} e^k\Big\|_{l_1(2^{-k})} &= C_2^{-1} \sum_{k=0}^{\infty} \|b_k\|_{\Lambda_\varphi(\vec{A})}.\end{aligned} \tag{32}$$

We now consider a new representation:

$$b = \sum_{k=-\infty}^{\infty} b'_k, \quad b'_k = b_k \quad \text{for} \quad k = -2, -3, \dots,$$

$$b'_{-1} = \sum_{i=-1}^{\infty} b_i \quad \text{and} \quad b'_k = 0 \quad \text{for} \quad k = 0, 1, \dots.$$

Then from (32) and the fact that G is a Banach lattice we obtain

$$\begin{aligned}&\|(\mathscr{J}(2^k, b'_k))_k\|_G \le \|(\mathscr{J}(2^k, b_k))_k\|_G \\ &+ \sum_{k=0}^{\infty} \|b_k\|_{\Lambda_\varphi(\vec{A})} \|e^{-1}\|_G \le (1 + 2C_2C_3) \|(\mathscr{J}(2^k, b_k))_k\|_G,\end{aligned}$$

where C_3 is the constant of the embedding $\varDelta(\vec{l_1}) \subset G$.

Thus, if $b \in Y$ then there is some representation of the form (31) such that $b_k = 0$ for $k = 0, 1, \dots$ and

$$\|b\|_Y \ge C_4^{-1} \Big\|\sum_{k=1}^{\infty} \mathscr{J}(2^{-k}, b_{-k}) e^{-k}\Big\|_G. \tag{33}$$

By Theorem 3, for each $k = 1, 2, \dots$, we can find a representation

$$b_{-k} = \sum_{n=k}^{\infty} b_{k,n}, \quad \text{where} \quad b_{k,n} \in \vec{A}^{\mathscr{J}}_n, \tag{34}$$

such that

$$\mathscr{J}(2^{-k}, b_{-k}) \ge C_5^{-1} \sum_{n=k}^{\infty} 2^{n-k} \|b_{k,n}\|_{\vec{A}^{\mathscr{J}}_n}. \tag{35}$$

Since $G \subset \Sigma(\vec{l_1})$, from (33) and (35) we derive

$$\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\|b_{k,n}\|_{\vec{A}_n^{\mathscr{J}}} \le \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}2^{n-k}\|b_{k,n}\|_{\vec{A}_n^{\mathscr{J}}} \le C_5\sum_{k=1}^{\infty}\mathscr{J}(2^{-k}, b_{-k})$$
$$\le C_5C_2\Big\|\sum_{k=1}^{\infty}\mathscr{J}(2^{-k}, b_{-k})e^{-k}\Big\|_G \le C_5C_4C_2\|b\|_Y < \infty. \tag{36}$$

In particular, this demonstrates that the sum of the series $\sum_{k=1}^{\infty}\sum\limits_{n=k}^{\infty} b_{k,n}$ is independent of the summation order. Therefore,

$$b = \sum_{n=1}^{\infty} c_n \quad \text{(w.r.t. the convergence of } A_0), \quad \text{where } c_n = \sum_{k=1}^{n} b_{k,n}.$$

Moreover, by (34), we have $c_n \in \vec{A}_n^{\mathscr{J}}$ and

$$\|c_n\|_{\vec{A}_n^{\mathscr{J}}} \le \sum_{k=1}^{n}\|b_{k,n}\|_{\vec{A}_n^{\mathscr{J}}}. \tag{37}$$

Now, consider the sequence

$$\alpha_b = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\|b_{k,n}\|_{\vec{A}_n^{\mathscr{J}}}e^{-n}. \tag{38}$$

By (36),

$$\alpha_b = \sum_{k=1}^{\infty}\alpha^k \quad \text{(w.r.t. the convergence of } \Sigma(\vec{l_1})), \quad \text{where } \alpha^k = \sum_{n=k}^{\infty}\|b_{k,n}\|_{\vec{A}_n^{\mathscr{J}}}e^{-n}.$$

It follows from (35) that $\alpha^k \in \Delta(\vec{l_1})$, $k = 1, 2, \ldots$. Since G is an interpolation space with respect to the pair $\vec{l_1}$, applying Lemma 4 of [6], we obtain

$$\|\alpha_b\|_G \le C_6\Big\|\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\max(1, 2^{n-k})\|b_{k,n}\|_{\vec{A}_n^{\mathscr{J}}}e^{-k}\Big\|_G = C_6\Big\|\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}2^{n-k}\|b_{k,n}\|_{\vec{A}_n^{\mathscr{J}}}e^{-k}\Big\|_G.$$

Therefore, (37), (38), (35), and (33) imply eventually that

$$\Big\|\sum_{n=1}^{\infty}\|c_n\|_{\vec{A}_n^{\mathscr{J}}}e^{-n}\Big\|_G \le \|\alpha_b\|_G \le C_4C_5C_6\|b\|_Y,$$

whence $b \in \mathscr{L}_{\Phi,G}^{\mathscr{J}}\big(\{\vec{A}_n^{\mathscr{J}}\}\big)$ and $\|b\|_{\mathscr{L}_{\Phi,G}^{\mathscr{J}}(\{\vec{A}_n^{\mathscr{J}}\})} \le C_4C_5C_6\|b\|_Y$. $\square$

In the particular case $\vec{A} = \vec{L} := (L_1, L_\infty)$, relations (4) and (27) imply the equality $\vec{L}_n^{\mathscr{J}} = L_{r_n}$, where $r_n = \Phi(2^n)/(\Phi(2^n) - 1)$, with constants independent of $n = 1, 2, \ldots$. Therefore, the value of the above-introduced functor at this pair is the rearrangement invariant space $\mathscr{L}_{\Phi,G}^{\mathscr{J}}\big(\{\vec{L}_n^{\mathscr{J}}\}\big)$ consisting of all measurable functions g on $[0, 1]$ with a representation $g = \sum_{k=1}^{\infty} g_k$ (w.r.t. the convergence of L_1), $g_k \in L_{r_k}$ and $\sum_{k=1}^{\infty}\|g_k\|_{r_k}e^{-k} \in G$. The norm in $\mathscr{L}_{\Phi,G}^{\mathscr{J}}\big(\{\vec{L}_n^{\mathscr{J}}\}\big)$ is given by the relation

$$\|g\|_{\mathscr{L}_{\Phi,G}^{\mathscr{J}}(\{\vec{L}_n^{\mathscr{J}}\})} = \inf\Big\|\sum_{k=1}^{\infty}\|g_k\|_{r_k}e^{-k}\Big\|_G,$$

where the greatest lower bound is taken over all possible representations. Eventually we obtain

Corollary 7. *The equality* $\mathscr{L}^{\mathscr{J}}_{\Phi,G}(\{L_{r_n}\}) = (L_1, \Lambda(\varphi))^{\mathscr{J}}_G$ (*with equivalence of norms*) *is valid for an arbitrary Orlicz function* Φ *and every interpolation Banach lattice* G *with respect to* $(l_1, l_1(2^{-k}))$.

Since the space $\Lambda(\varphi)$ is simultaneously an Orlicz space [12], the interpolation in the pair $(L_1, \Lambda(\varphi))$ is described by the real $\mathscr{K}$-method [21, Corollary 5.10]. By the theorems on connection between the $\mathscr{K}$- and $\mathscr{J}$-functors [9], it is also described by the $\mathscr{J}$-method. In other words, if Y is an interpolation space with respect to the pair $(L_1, \Lambda(\varphi))$ then $Y = (L_1, \Lambda(\varphi))^{\mathscr{J}}_G$ for some Banach lattice G interpolation with respect to the pair $\vec{l_1} = (l_1, l_1(2^{-k}))$. Therefore, Theorem 4 gives an extrapolation description for such spaces.

Corollary 8. *Let* Φ *be an arbitrary Orlicz function. For every space* Y *that is an interpolation space with respect to* $(L_1, \Lambda(\varphi))$, *there is a Banach lattice* G *satisfying* $Y = \mathscr{L}^{\mathscr{J}}_{\Phi,G}(\{L_{r_n}\})$ (*with equivalence of norms*).

§ 5. Further Generalizations and Stability of Extrapolation Functors

Here we show that the results of two previous sections are valid for a more general family of scales generated by the real method of interpolation. The key role in the proof is played by a certain stability of extrapolation intersections of families of Banach spaces. Namely, if the latter are the values of interpolation functors at the same Banach pair then, under some conditions, the intersection depends only on the characteristic functions of these functors.

From a technical point of view, in this section it is more convenient to use functors of the real interpolation method with functional parameters. For example, in this case the space $\vec{A}^{\mathscr{K}}_{\theta,q} := (A_0, A_1)^{\mathscr{K}}_{\theta,q}$ of the $\mathscr{K}$-method consists of all $a \in A_0 + A_1$ such that

$$\|a\|^{\mathscr{K}}_{\theta,q} = \left\{ \int_0^\infty (t^{-\theta} \mathscr{K}(t, a; \vec{A}))^q \frac{dt}{t} \right\}^{1/q} < \infty \quad (0 < \theta < 1,\ 1 \le q < \infty)$$

and

$$\|a\|^{\mathscr{K}}_{\theta,\infty} = \sup_{t>0} (t^{-\theta} \mathscr{K}(t, a; \vec{A})) < \infty \quad (0 < \theta < 1).$$

Note that this norm is equivalent to the norm of the corresponding space with the discrete parameter $l_q(2^{-\theta k})$ (see § 1). Moreover, the equivalence constant can be chosen to be the same for all $0 < \theta < 1$ and $1 \le q \le \infty$ [8, Lemmas 3.1.3 and 3.2.3]. Thereby we can identify these spaces from both interpolation and extrapolation points of view.

In the proof of the following theorem we will use some arguments of the proof of Theorem 21 of [5]:

Theorem 5. *Suppose that* Φ *is an Orlicz function on* $[0, \infty)$, $\theta_n = 1 - 1/\Phi(2^n)$, *and* F_n *is an exact interpolation functor with the characteristic function* t^{θ_n} $(n = 1, 2, \dots)$. *Then for a Banach pair* $\vec{A} = (A_0, A_1)$ *such that* $A_1 \overset{1}{\subset} A_0$ *and* $\widetilde{A}_1 = A_1$,

$$\varDelta_{n \ge k} 2^{-n} F_n(\vec{A}) \asymp \varDelta_{n \ge k} 2^{-n} \vec{A}^{\mathscr{K}}_{\theta_n, 1/(1-\theta_n)}$$

with constants independent of $\vec{A}$ *and* $k = 1, 2, \dots$.

Proof. It suffices to show (see § 1) that

$$\varDelta_{n \ge k} 2^{-n} \vec{A}^{\mathscr{K}}_{\theta_n, \infty} \subset \varDelta_{n \ge k} 2^{-n} \vec{A}^{\mathscr{J}}_{\theta_n, 1} \tag{39}$$

with a constant independent of $\vec{A}$ and $k = 1, 2, \dots$. First, prove that

$$\langle \vec{A}^{\mathscr{K}}_{\theta,1} \rangle \subset \vec{A}^{\mathscr{J}}_{\theta,1} \tag{40}$$

with a universal constant, where $\langle \vec{A}^{\mathscr{K}}_{\theta,p} \rangle$ stands, as above, for the modified spaces of the $\mathscr{K}$-method (see (2)).

Let $a \in \vec{A}^{\mathscr{K}}_{\theta,1}$. Then by a strengthened version of the fundamental lemma of interpolation theory [5, Lemma 1], there is a representation $a = \int_0^\infty u(s)\,ds/s$ such that

$$\int_0^\infty \min(1, t/s)\,\mathscr{J}(s, u(s); \vec{A})\,\frac{ds}{s} \le \gamma \mathscr{K}(t, a; \vec{A})$$

with a universal constant γ. Therefore, using (2), we obtain

$$\|a\|^{\mathscr{J}}_{\theta,1} \le \int_0^\infty s^{-\theta} \mathscr{J}(s, u(s); \vec{A}) \frac{ds}{s} = \theta(1-\theta) \int_0^\infty \mathscr{J}(s, u(s); \vec{A}) \int_0^\infty \min(1, t/s) t^{-\theta} \frac{dt}{t}\frac{ds}{s}$$

$$= \theta(1-\theta) \int_0^\infty \int_0^\infty \mathscr{J}(s, u(s); \vec{A}) \min(1, t/s)\,\frac{ds}{s} t^{-\theta}\,\frac{dt}{t}$$

$$\le \gamma\theta(1-\theta) \int_0^\infty \mathscr{K}(t, a; \vec{A}) t^{-\theta}\,\frac{dt}{t} = \gamma \|a\|_{\langle \vec{A}^{\mathscr{K}}_{\theta,1}\rangle}.$$

Thereby (40) is proved. Therefore, to prove (39) it is sufficient to verify that

$$\varDelta_{n\ge k} 2^{-n} \vec{A}^{\mathscr{K}}_{\theta_n,\infty} \subset \varDelta_{n\ge k} 2^{-n} \langle \vec{A}^{\mathscr{K}}_{\theta_n,1}\rangle \tag{41}$$

with a constant independent of $\vec{A}$ and $k = 1, 2, \ldots$.

Show that in the case when $A_1 \overset{1}{\subset} A_0$ the inequality

$$\|a\|^{\mathscr{K}}_{\theta,q} \le 9 \bigg\{ \int_0^1 (t^{-\theta} \mathscr{K}(t, a; \vec{A}))^q \frac{dt}{t} \bigg\}^{1/q} \tag{42}$$

holds for all $\theta \in [1/2, 1)$ and $1 \le q \le \infty$ (we used similar arguments in the proof of Lemma 3).

Indeed, since $\mathscr{K}(t, a; \vec{A}) = \|a\|_{A_0}$ $(t \ge 1)$ and $\theta \in [1/2, 1)$, we obtain

$$\bigg\{ \int_0^1 (t^{-\theta} \mathscr{K}(t, a; \vec{A}))^q \frac{dt}{t} \bigg\}^{1/q} \ge \bigg\{ \int_{1/4}^1 (t^{-\theta} \mathscr{K}(t, a; \vec{A}))^q \frac{dt}{t} \bigg\}^{1/q}$$

$$\ge \mathscr{K}(1/4, a; \vec{A}) \bigg\{ \int_{1/4}^1 (t^{-\theta})^q \frac{dt}{t} \bigg\}^{1/q} \ge \frac{1}{4}\|a\|_{A_0}.$$

Therefore,

$$\bigg\{ \int_1^\infty (t^{-\theta} \mathscr{K}(t, a; \vec{A}))^q \frac{dt}{t} \bigg\}^{1/q} \le \|a\|_{A_0} \bigg\{ \int_1^\infty t^{-q/2}\,\frac{dt}{t} \bigg\}^{1/q} \le 2\|a\|_{A_0} \le 8 \bigg\{ \int_0^1 (t^{-\theta} \mathscr{K}(t, a; \vec{A}))^q \frac{dt}{t} \bigg\}^{1/q}$$

and (42) follows from the definition of $\|a\|^{\mathscr{K}}_{\theta,q}$.

For each $k = 1, 2, \ldots$, we define the functions $\tau_k(t) = \inf_{n\ge k} 2^n t^{\theta_n}$ and $\tilde{\tau}_k(t) = \inf_{\theta_k \le u < 1} M(u) t^u$ on the interval $(0, 1]$, where $M(u) = \Phi^{-1}(1/(1-u))$.

First of all,

$$\tilde{\tau}_k(t) \le \tau_k(t) \le 2\tilde{\tau}_k(t). \tag{43}$$

Since $M(\theta_n) = 2^n$, the left inequality is obvious. To prove the right, we take $u \in [\theta_k, 1)$. Then $\theta_n \le u < \theta_{n+1}$ for some $n \ge k$ and, since the function $M(u)$ is increasing,

$$M(u)t^u \ge M(\theta_n) t^{\theta_{n+1}} = 2^n t^{\theta_{n+1}} = \frac{1}{2} M(\theta_{n+1}) t^{\theta_{n+1}}.$$

Consequently, (43) is proved.

Show now that

$$\tilde{\tau}_k(t^2) \le 2t\tilde{\tau}_k(t), \quad 0 \le t \le 1, \ k = 1, 2, \dots. \tag{44}$$

Indeed, since $\theta_k \ge 1/2$, we obtain

$$\tilde{\tau}_k(t^2) = \inf_{\theta_k \le u < 1} M(u) t^{2u} = t \inf_{\theta_k \le u < 1} M(u) t^{2u-1} = t \inf_{2\theta_k - 1 \le u < 1} M\left(\frac{u+1}{2}\right) t^u \le 2t\tilde{\tau}_k(t),$$

where in the last inequality we used the fact that $2\theta_k - 1 < \theta_k$ and

$$M((u+1)/2) = \Phi^{-1}(2/(1-u)) \le 2M(u).$$

From (43) and (44) we find that

$$\tau_k(t^2) \le 4t\tau_k(t), \quad 0 \le t \le 1, \ k = 1, 2, 3, \dots.$$

This and the definition of the function $\tau_k(t)$ imply that

$$\int_0^1 \tau_k(t) t^{-\theta_k} \frac{dt}{t} \le 4 \int_0^1 t^{1/2} \tau_k(t^{1/2}) t^{-\theta_k} \frac{dt}{t} \le 4 \sup_{0 < t \le 1} (t^{-\theta_k/2} \tau_k(t^{1/2})) \int_0^1 t^{(1-\theta_k)/2} \frac{dt}{t}$$
$$= \frac{8}{1-\theta_k} \sup_{0<t\le 1} (t^{-\theta_k} \tau_k(t)) \le \frac{8}{1-\theta_k} 2^k.$$

Using the last inequality and (42), we finally obtain

$$\sup_{n \ge k} 2^{-n} \|a\|_{\langle \vec{A}^{\mathscr{K}}_{\theta_n,1} \rangle} \le 9 \sup_{n \ge k} 2^{-n} (1-\theta_n)\theta_n \int_0^1 \mathscr{K}(t, a; \vec{A}) t^{-\theta_n} \frac{dt}{t}$$
$$\le 9 \sup_{n \ge k} 2^{-n} (1-\theta_n)\theta_n \int_0^1 \tau_n(t) t^{-\theta_n} \frac{dt}{t} \sup_{0<s\le 1} \frac{\mathscr{K}(s, a; \vec{A})}{\tau_n(s)}$$
$$\le 72 \sup_{0<s\le 1} \frac{\mathscr{K}(s, a; \vec{A})}{\tau_n(s)} = 72 \sup_{0<s\le 1} \sup_{n\ge k} \mathscr{K}(s, a; \vec{A}) 2^{-n} s^{-\theta_n}$$
$$= 72 \sup_{n\ge k} 2^{-n} \sup_{0<s\le 1} \mathscr{K}(s, a; \vec{A}) s^{-\theta_n} = 72 \sup_{n \ge k} 2^{-n} \|a\|_{\vec{A}^{\mathscr{K}}_{\theta_n,\infty}},$$

which proves (41) and the theorem. □

For an arbitrary sequence $\vec{q} = \{q_n\}_{n=1}^{\infty}$, $1 \le q_n \le \infty$, and every Banach pair $\vec{A} = (A_0, A_1)$, $A_1 \overset{1}{\subset} A_0$, we introduce the discrete scale of spaces

$$\vec{A}^{\mathscr{K}}_n(\vec{q}) = (A_0, A_1)^{\mathscr{K}}_{1-1/\Phi(2^n), q_n}, \quad n = 1, 2, \dots.$$

In particular, if $q_n = \Phi(2^n)$ then we obtain the scale $\vec{A}^{\mathscr{K}}_n$ of § 3.

Theorems 1 and 4 yield

Corollary 9. *For an arbitrary Orlicz function Φ and a Banach pair $\vec{A} = (A_0, A_1)$ such that $A_1 \overset{1}{\subset} A_0$ and $\widetilde{A}_1 = A_1$, the following relation is valid:*

$$\mathscr{K}(2^k, a; A_1, M_\varphi(\vec{A})) \asymp \sup_{n \ge k} 2^{k-n} \|a\|_{\langle \vec{A}^{\mathscr{K}}_n(\vec{q}) \rangle}.$$

with a constant independent of $\vec{A}$, $a \in M_\varphi(\vec{A})$, a sequence $\vec{q}$, and $k = 1, 2, \dots$.

Under some conditions on $\vec{q}$, a similar assertion is valid for the ordinary (nonmodified) $\mathscr{K}$-method.

Corollary 10. *If $\Phi(2^n) \le q_n \le \infty$ $(n = 1, 2, \dots)$ then for every Banach pair $\vec{A} = (A_0, A_1)$ such that $A_1 \overset{1}{\subset} A_0$ and $\widetilde{A}_1 = A_1$, we have*

$$\mathscr{K}(2^k, a; A_1, M_\varphi(\vec{A})) \asymp \sup_{n \ge k} 2^{k-n} \|a\|_{\vec{A}_n^{\mathscr{K}}(\vec{q})}$$

with universal constants.

Proof. Since $q_n \ge 2$ and $\theta_n := 1 - 1/\Phi(2^n) \ge 1/2$, it is easy to verify that $1/\sqrt{2} \le c_{\theta_n, q_n} \le e^{1/e}$ (see (2)). Thereby we can replace the modified spaces of the $\mathscr{K}$-method in the previous corollary with the ordinary spaces. $\square$

In particular, under the same conditions on the Banach pair $\vec{A}$, we obtain

$$\mathscr{K}(2^k, a; A_1, M_\varphi(\vec{A})) \asymp \sup_{n \ge k} 2^{k-n} \|a\|_{(A_0, A_1)^{\mathscr{K}}_{1-1/\Phi(2^n),\infty}}.$$

Theorem 6. *Suppose that Φ is an Orlicz function and $\vec{A} = (A_0, A_1)$ is a Banach pair such that $A_1 \overset{1}{\subset} A_0$ and $\widetilde{A}_1 = A_1$. If $\Phi(2^n) \le q_n \le \infty$ $(n = 1, 2, \dots)$ then, for every interpolation Banach lattice F with respect to $\vec{l_\infty} = (l_\infty, l_\infty(2^{-k}))$, the equality $\mathscr{L}^{\mathscr{K}}_{\Phi,F}(\{\vec{A}_n^{\mathscr{K}}(\vec{q})\}) = (A_1, M_\varphi(\vec{A}))^{\mathscr{K}}_F$ is valid (with equivalence of norms).*

Proof. Corollary 10 and inspection of the proof of Theorem 2 demonstrate that it suffices to validate the inequality

$$\|a\|_{\vec{A}_n^{\mathscr{K}}(\vec{q})} \le C \|a\|_{\vec{A}_m^{\mathscr{K}}(\vec{q})} \quad (1 \le n \le m).$$

To this end, by hypothesis and Theorem 3.4.1 of [8], it suffices to show that

$$\|a\|^{\mathscr{K}}_{\theta_n, q_n^0} \le 18 \|a\|^{\mathscr{K}}_{\theta_m, \infty} \quad (1 \le n < m),$$

where, as above, $\theta_n = 1 - 1/\Phi(2^n)$ and $q_n^0 = \Phi(2^n)$. The last inequality is obtained from the following estimates in which we use (42) and the inequality $\Phi(2^m) \ge 2 \cdot \Phi(2^n)$:

$$\|a\|^{\mathscr{K}}_{\theta_n, q_n^0} \le 9 \left\{ \int_0^1 (t^{-\theta_n} \mathscr{K}(t, a; \vec{A}))^{q_n^0} \frac{dt}{t} \right\}^{1/q_n^0}$$

$$\le 9 \sup_{0 < t \le 1} (t^{-\theta_m} \mathscr{K}(t, a; \vec{A})) \left\{ \int_0^1 t^{1 - \Phi(2^n)/\Phi(2^m)} \frac{dt}{t} \right\}^{1/\Phi(2^n)} \le 18 \|a\|^{\mathscr{K}}_{\theta_m, \infty}. \quad \square$$

In conclusion, we define the dual scale of spaces of the $\mathscr{J}$-method: if $\vec{s} = \{s_n\}_{n=1}^\infty$, $1 \le s_n \le \infty$, and $\vec{A} = (A_0, A_1)$ is a Banach pair such that $A_1 \overset{1}{\subset} A_0$ then

$$\vec{A}_n^{\mathscr{J}}(\vec{s}) = (A_0, A_1)^{\mathscr{J}}_{1/\Phi(2^n), s_n}, \quad n = 1, 2, \dots.$$

By analogy with Theorems 3 and 4, we can prove the following

Theorem 7. *Suppose that Φ is an Orlicz function and $1 \le s_n \le \Phi(2^n)/(\Phi(2^n) - 1)$, $n = 1, 2, \dots$. Given a Banach pair $\vec{A} = (A_0, A_1)$ such that $A_1 \overset{1}{\subset} A_0$ and A_1 is everywhere dense in A_0, we then have*

$$\mathscr{J}(2^{-k}, b; A_0, \Lambda_\varphi(\vec{A})) \asymp \|b\|_{U_k},$$

where $U_k = \sum_{n \ge k} 2^{n-k} \vec{A}_n^{\mathscr{J}}(\vec{s})$, with constants independent of $\vec{A}$, $b \in A_0$, a sequence $\vec{s}$, and $k = 1, 2, \dots$.

Theorem 8. *Suppose that Φ is an Orlicz function and $1 \le s_n \le \Phi(2^n)/(\Phi(2^n) - 1)$, $n = 1, 2, \dots$. Moreover, suppose that G is an interpolation Banach lattice with respect to the pair $\vec{l_1} = (l_1, l_1(2^{-k}))$, and $\vec{A} = (A_0, A_1)$ is a Banach pair such that $A_1 \overset{1}{\subset} A_0$ and A_1 is everywhere dense in A_0. Then*

$$\mathscr{L}^{\mathscr{J}}_{\Phi,G}(\{\vec{A}_n^{\mathscr{J}}(\vec{s})\}) = (A_0, \Lambda_\varphi(\vec{A}))^{\mathscr{J}}_G$$

(with equivalence of norms).

References

1. *Yano S.*, "An extrapolation theorem," J. Math. Soc. Japan, **3**, 296–305 (1951).
2. *Zygmund A.*, Trigonometric Series [Russian translation], Mir, Moscow (1965).
3. *Jawerth B. and Milman M.,* Extrapolation Spaces with Applications, Amer. Math. Soc., Providence RI (1991). (Mem. Amer. Math. Soc.; 440.)
4. *Jawerth B. and Milman M.*, "New results and applications of extrapolation theory," Israel Math. Conf. Proc., **5**, 81–105 (1992).
5. *Milman M.,* Extrapolation and Optimal Decompositions with Applications to Analysis, Springer-Verlag, Berlin (1994). (Lecture Notes in Math.; 1580.)
6. *Astashkin S. V.*, "On extrapolation properties of the scale of L_p-spaces," Mat. Sb., **194**, No. 6, 23–42 (2003).
7. *Astashkin S. V.*, "Some new extrapolation estimates for the scale of L_p-spaces," Funktsional. Anal. i Prilozhen., **37**, No. 3, 73–77 (2003).
8. *Bergh J. and Löfström J.*, Interpolation Spaces. An Introduction [Russian translation], Mir, Moscow (1980).
9. *Brudnyi Yu. A. and Krugliak N. Ya.*, Interpolation Functors and Interpolation Spaces, North-Holland, Amsterdam (1991).
10. *Kreĭn S. G., Petunin Yu. I., and Semenov E. M.*, Interpolation of Linear Operators [in Russian], Nauka, Moscow (1978).
11. *Lindenstrauss J. and Tzafriri L.*, Classical Banach Spaces. 2. Function Spaces, Springer-Verlag, Berlin (1979).
12. *Lorentz G. G.*, "Relations between function spaces," Proc. Amer. Math. Soc., **12**, 127–132 (1961).
13. *Rutitskiĭ Ya. B.*, "On some classes of measurable functions," Uspekhi Mat. Nauk, **20**, No. 4, 205–208 (1965).
14. *Bennett C. and Rudnick K.,* "On Lorentz–Zygmund spaces," Dissertationes Math., **175**, 5–67 (1980).
15. *Milman M.*, "A note on extrapolation theory," J. Math. Anal. Appl., **282**, 26–47 (2003).
16. *Astashkin S. V.*, "A space of multipliers generated by Rademacher system," Mat. Zametki, **75**, No. 2, 173–181 (2004).
17. *Cwikel M. and Nilsson P.*, "Interpolation of Marcinkiewicz spaces," Math. Scand., **56**, 29–42 (1985).
18. *Bennett C.*, "Banach function spaces and interpolation methods," J. Funct. Anal., **17**, 409–440 (1974).
19. *Ovchinnikov V. I.*, "The method of orbits in interpolation theory," Math. Rep. Ser. 2, **1**, 349–515 (1984).
20. *Astashkin S. V.*, "On interpolation of subspaces of rearrangement invariant spaces generated by Rademacher system," Izv. Ross. Akad. Estestv. Nauk Ser. MMMIU, **1**, No. 1, 18–35 (1997).
21. *Kalton N. J.*, "Calderon couples of rearrangement invariant spaces," Studia Math., **106**, No. 3, 233–277 (1993).