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Comparison of Sums of Independent and Disjoint Functions in Symmetric Spaces

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Abstract—The sums of independent functions (random variables) in a symmetric space X on $[0,1]$ are studied. We use the operator approach closely connected with the methods developed, primarily, by Braverman. Our main results concern the Orlicz exponential spaces $\exp(L_p)$, $1 \le p \le \infty$, and Lorentz spaces Λ_ψ. As a corollary, we obtain results that supplement the well-known Johnson–Schechtman theorem stating that the condition $L_p \subset X$, $p < \infty$, implies the equivalence of the norms of sums of independent functions and their disjoint "copies." In addition, a statement converse, in a certain sense, to this theorem is proved.

Key words: *sums of independent random variables, disjoint random variables, symmetric spaces, Orlicz spaces, Lorentz spaces, Johnson–Schechtman theorem.*

1. INTRODUCTION

This paper is devoted to the study of sums of independent functions in symmetric spaces. It is well known that these sums often behave like sums of disjoint functions with the same distribution. In 1970, Rosenthal [1] proved a remarkable inequality, from which it follows that for an arbitrary sequence $\{f_n\}_{n=1}^\infty$ of independent functions from $L_p[0,1]$, $p \ge 2$, such that $\int_0^1 f_k(t)\,dt = 0$, the map $f_k \to \bar f_k$, where

$$\bar f_k(t) := f_k(t-k+1)\chi_{[k-1,k)}(t), \qquad t \in \mathbb{R},$$

can be extended to an isomorphism between the closed linear span of $[f_k]_{k=1}^\infty$ (taken in $L_p[0,1]$) and the closed linear span of $[\bar f_k]_{k=1}^\infty$ (taken in $L_p[0,\infty) \cap L_2[0,\infty)$). In [2], the Rosenthal inequality was extended to the case of the Lorentz spaces $L_{p,q}$, $2 < p < \infty$, $0 < q < \infty$. Later, significant progress in this direction was made by Johnson and Schechtman [3, Theorem 1], who introduced the spaces Y_X and Z_X on $[0,\infty)$ constructed from a given symmetric space X on $[0,1]$, and showed that, under the condition that X contains $L_p[0,1]$ for a certain $p < \infty$, any finite sequence $\{f_k\}_{k=1}^n$ of independent functions with zero (respectively, positive) mean from X is equivalent (uniformly in n) to the sequence of their disjoint translates from Y_X (respectively, Z_X). In particular, it follows directly from their result that if $\{f_k\}_{k=1}^\infty$ is a sequence of independent functions in a symmetric space X such that for all $n \in \mathbb{N}$ we have

$$\sum_{k=1}^n \lambda(\{f_k \ne 0\}) \le 1 \tag{1}$$

(where λ is the Lebesgue measure), then, under the condition that X contains $L_p[0,1]$ for a certain $p < \infty$, the map $f_k \to \bar f_k$, $k \ge 1$, from the sequence $\{f_k\}_{k=1}^\infty$ to the sequence $\{\bar f_k\}_{k=1}^\infty$ of disjoint "copies" of its terms can be extended to an isomorphism between their closed linear spans $[f_k]_{k=1}^\infty$ and $[\bar f_k]_{k=1}^\infty$ in X.

In this paper, we study a more general question: *For what symmetric spaces X and Y on $[0,1]$ does there exist a constant $C=C(X,Y)>0$ such that for each sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions satisfying condition* (1), *we have*

$$\Big\|\sum_{k=1}^n f_k\Big\|_Y \le C\Big\|\sum_{k=1}^n \bar f_k\Big\|_X ? \tag{2}$$

Our approach is based on the study of a certain linear operator and is close to the approach developed by Braverman [4] in his investigation of the Rosenthal inequality and its modifications in symmetric spaces by using, in turn, some earlier ideas and probabilistic constructions of Kruglov [5]. We will give more details in Sec. 3, after introducing all the necessary definitions in Sec. 2. The main results of the paper concerning Orlicz exponential spaces $\exp(L_p)$ and Lorentz spaces Λ_ψ are formulated in Secs. 4 and 5. In the case $X=Y$, one of their corollaries is a statement converse, in a sense, to a result of Johnson and Schechtman [3, Theorem 1].

In Sec. 6, the last part of the paper, we extend our results to arbitrary sequences of independent functions (which, generally speaking, do not satisfy condition (1)); this amplifies results of [3].

2. DEFINITIONS AND NOTATION

We denote by $S(\Omega)$ $(=S(\Omega,\mathcal{P}))$ the linear space of all measurable functions finite almost everywhere on a measure space $(\Omega,\mathcal{P})$ equipped with the topology of measure convergence on sets of finite measure.

A Banach space $(E,\|\cdot\|_E)$ of real-valued Lebesgue measurable functions on the interval $[0,\alpha)$, $0<\alpha\le\infty$, is said to be *symmetric* (or *rearrangement invariant*) if

(i) for any $y\in E$ and $|x|\le|y|$, we have $x\in E$ and $\|x\|_E\le\|y\|_E$;
(ii) for any $y\in E$ and $x^*=y^*$, we have $x\in E$ and $\|x\|_E=\|y\|_E$.

Hereafter, λ is the Lebesgue measure, and x^* is a nondecreasing right continuous *rearrangement* of $|x|$, i.e.,

$$x^*(t)=\inf\big\{s\ge 0\colon \lambda(\{|x|>s\})\le t\big\},\qquad t>0.$$

If E is a symmetric space on the interval $[0,\alpha)$, then its *dual* (or *associated*) space $E^\times$ consists of all measurable functions y such that

$$\|y\|_{E^\times}:=\sup\bigg\{\int_0^\alpha |x(t)y(t)|\,dt\colon\ x\in E,\ \|x\|_E\le 1\bigg\}<\infty.$$

If E^* is the space conjugate to E, then $E^\times\subset E^*$ and $E^\times=E^*$ if and only if E is separable. The natural embedding of E into its second dual space $E^{\times\times}$ is an isometric surjection if and only if E is *maximal* (or has the *Fatou property*), i.e., the conditions

$$\{f_n\}_{n\ge1}\subseteq E,\qquad f\in S[0,\alpha),\qquad f_n\to f\quad\text{a.e. on } [0,\alpha),\qquad \sup_n\|f_n\|_E<\infty,$$

imply

$$f\in E\qquad\text{and}\qquad \|f\|_E\le\liminf_{n\to\infty}\|f_n\|_E.$$

The norm $\|\cdot\|_E$ in a symmetric space E is said to be *order semicontinuous* if the unit ball E is closed in E with respect to convergence almost everywhere. This is equivalent to the fact that the natural embedding of E into its second dual space is an isometry.

For any symmetric space E on $[0,\alpha)$, the following continuous embeddings hold:

$$L_1[0,\alpha)\cap L_\infty[0,\alpha)\subseteq E\subseteq L_1[0,\alpha)+L_\infty[0,\alpha).$$

In what follows, we denote by E^0 the closure of $L_1[0,\alpha)\cap L_\infty[0,\alpha)$ in E. If $E\neq L_\infty$, then E^0 is separable. Finally, if E is a symmetric space, then the function $\phi_E(t) := \|\chi_A(\,\cdot\,)\|_E$, where a measure set A satisfies the condition $\lambda(A)=t$, and χ_A is the characteristic function of this set, is called the *fundamental function* of E. For more details about symmetric spaces see the monographs [6–8].

3. KRUGLOV PROPERTY AND THE OPERATOR $\mathcal{K}$

Let f be a measurable function (random variable) on $[0,1]$, and let $\mathcal{F}_f$ be its distribution function. We denote by $\pi(f)$ any random variable on $[0,1]$ whose characteristic function is given by the relation

$$\theta_{\pi(f)}(t)=\exp\biggl(\int_{-\infty}^{\infty}(e^{itx}-1)\,d\mathcal{F}_f(x)\biggr).$$

The following property was intensely studied and used by Braverman [4].

Definition. A symmetric space X satisfies the *Kruglov property* ($X\in\mathbb{K}$) if from $f\in X$, it follows that $\pi(f)\in X$.

Let us define a positive linear operator closely connected with the Kruglov property. Let $\{E_n\}$ be a sequence of pairwise disjoint subsets of $[0,1]$, and let

$$m(E_n)=\frac{1}{e\cdot n!},\qquad n\in\mathbb{N}.$$

For $f\in S([0,1],\lambda)$, we set

$$\mathcal{K}f(\omega_0,\omega_1,\omega_2,\dots):=\sum_{n=1}^{\infty}\sum_{k=1}^{n}f(\omega_k)\chi_{E_n}(\omega_0). \tag{3}$$

Then

$$\mathcal{K}\colon S([0,1],\lambda)\to S(\Omega,\mathcal{P}),\qquad \text{where}\quad (\Omega,\mathcal{P}):=\prod_{k=0}^{\infty}([0,1],\lambda_k)$$

(λ_k is the Lebesgue measure on $[0,1]$ for each $k\ge 0$).

The operator $\mathcal{K}$ (more exactly, an operator close to it) can be defined in a slightly different way. Let $f\in S([0,1],\lambda)$, and let $\{f_{n,k}\}_{k=1}^{n}$ be a sequence of measurable functions on $[0,1]$ such that

(i) the functions $f_{n,1}, f_{n,2},\dots,f_{n,n}$, χ_{E_n} are independent for each $n\in\mathbb{N}$,

(ii) $\mathcal{F}_{f_{n,k}}=\mathcal{F}_f$ for all $n\in\mathbb{N}$, $k=1,2,\dots,n$.

Set

$$\mathcal{K}'f(x):=\sum_{n=1}^{\infty}\sum_{k=1}^{n}f_{n,k}(x)\chi_{E_n}(x),\qquad x\in[0,1]. \tag{3$'$}$$

Clearly, the distribution functions $\mathcal{K}f$ and $\mathcal{K}'f$ are the same for any $f\in S([0,1],\lambda)$. In what follows, we deal with symmetric spaces; therefore, we can identify $\mathcal{K}'f$ and $\mathcal{K}f$.

Direct calculation using (3) (or ($3'$)) shows that $\theta_{\mathcal{K}f}(t)=\theta_{\pi(f)}(t)$, $t\in\mathbb{R}$. Thus, we obtain a simple, but important statement that enables us to apply the interpolation technique to our investigation.

Proposition 1. *If X is a symmetric space on $[0,1]$, then the operator $\mathcal{K}$ acts in X boundedly if and only if $X\in\mathbb{K}$.*

The following proposition is of paramount importance for the entire paper.

Theorem 2. *Let $X \subseteq Y$, where X and Y are symmetric spaces on $[0,1]$. Consider the following conditions*:

(i) *there exists a $C>0$ such that* (2) *holds for an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions satisfying* (1);

(ii) *there exists a $C>0$ such that* (2) *holds for an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of independent identically distributed functions satisfying* (1);

(iii) *the operator $\mathcal{K}$ acts boundedly from X to $Y^{\times\times}$*;

(iii$'$) *the operator $\mathcal{K}$ acts boundedly from X to Y.*

The following implications hold:

$$(\mathrm{iii}') \implies (\mathrm{i}) \iff (\mathrm{ii}).$$

If the norm in the space Y is order semicontinuous, then

$$(\mathrm{i}) \iff (\mathrm{ii}) \iff (\mathrm{iii}).$$

4. OPERATOR $\mathcal{K}$ IN ORLICZ EXPONENTIAL SPACES

Let Φ be an *Orlicz function* on $[0,\infty)$, i.e., a continuous convex increasing function on $[0,\infty)$ such that $\Phi(0)=0$ and $\Phi(\infty)=\infty$. The *Orlicz space* $L_\Phi = L_\Phi[0,1]$ consists of all functions f measurable on $[0,1]$ whose norm

$$\|f\|_{L_\Phi} = \inf\left\{\rho>0\colon \int_0^1 \Phi\left(\frac{|f(t)|}{\rho}\right)dt \le 1\right\}$$

is finite. The space L_Φ is always maximal, and it is separable if and only if Φ satisfies the Δ_2-condition at ∞ (see [7] or [8]).

The *Orlicz exponential space* $\exp(L_p)$ is generated by the function $N_p(t) := e^{t^p}-1$, which is convex for all $t \ge 0$ if $p \ge 1$ and for sufficiently large t if $p\in(0,1)$. Also, we set $\exp(L_\infty) := L_\infty$.

It follows from the results of [5] and [4] that the operator $\mathcal{K}$ acts boundedly in $\exp(L_p)$ if and only if $p\in(0,1]$. To characterize the behavior of this operator in the spaces $\exp(L_p)$, $p\in(1,\infty]$, let us introduce the set $\mathcal{Y}_p$ consisting of all symmetric spaces Y such that the operator $\mathcal{K}$ acts boundedly from $\exp(L_p)$ to Y. Let us also define the Orlicz functions

$$M_p(t) := \exp\{|t|\ln^{1/p}(e+|t|)\}-1, \qquad t\in\mathbb{R}, \quad p>0.$$

The following statement (in the case $p=\infty$) demonstrates the important role played by the space L_{M_1} in the study of systems of uniformly bounded independent functions (cf. [9, Corollary 3.5.2]).

Theorem 3. *The unique minimal element of the set $\mathcal{Y}_p$, $1<p\le\infty$, ordered by inclusion is the Orlicz space L_{M_q}, where $1/p+1/q=1$.*

Corollary 4. *Suppose that $p\in(1,\infty]$ and $q=p/(p-1)$. There exists a $C_p>0$ such that for any finite sequence of independent functions $\{f_k\}_{k=1}^n \subset \exp(L_p)$ satisfying* (1), *we have*

$$\left\|\sum_{k=1}^n f_k\right\|_{L^0_{M_q}} \le C_p \left\|\sum_{k=1}^n \bar f_k\right\|_{\exp(L_p)}.$$

Furthermore, if a certain symmetric space Y with an order semicontinuous norm is such that

$$\left\|\sum_{k=1}^n f_k\right\|_Y \le C \left\|\sum_{k=1}^n \bar f_k\right\|_{\exp(L_p)}$$

for all the sequences $\{f_k\}_{k=1}^n \subset \exp(L_p)$ described above, then $L^0_{M_q} \subseteq Y$.

5. OPERATOR $\mathcal{K}$ IN LORENTZ SPACES

Denote by Ψ the set of all increasing concave functions ψ on $[0,1]$ such that $\psi(0)=\psi(+0)=0$. If $\psi\in\Psi$, then the *Lorentz space* $\Lambda_\psi=\Lambda_\psi[0,1]$ consists of all functions $x(t)$ measurable on $[0,1]$ for which

$$\|x\|_{\Lambda_\psi}:=\int_0^1 x^*(t)\,d\psi(t)<\infty.$$

Theorem 5. *If $\psi\in\Psi$, then the operator $\mathcal{K}$ is bounded in the Lorentz space Λ_ψ (i.e., $\Lambda_\psi\in\mathbb{K}$) if and only if there exists a $C>0$ such that*

$$\sum_{k=1}^{\infty}\psi\biggl(\frac{u^k}{k!}\biggr)\le C\psi(u),\qquad u\in(0,1]. \tag{4}$$

Notice that condition (4) for functions $\psi\in\Psi$ also appeared in [10] in the study of random rearrangements in symmetric spaces.

The following statement is, in a sense, converse to the assertion of the Johnson and Schechtman theorem [3] (see the Introduction).

Corollary 6. *Suppose that a symmetric space E has the following property: for each maximal symmetric space $X\supset E$ there exists a $C>0$ such that the inequality*

$$\biggl\|\sum_{k=1}^{n}f_k\biggr\|_X\le C\biggl\|\sum_{k=1}^{n}\bar f_k\biggr\|_X$$

holds for any sequence $\{f_k\}_{k=1}^n\subset X$ of independent functions satisfying (1). *Then E contains $L_p[0,1]$ for a certain $p\in[1,\infty)$.*

The last statement of this section shows that it is impossible to give a criterion for the boundedness of the operator $\mathcal{K}$ in terms of embeddings.

Corollary 7. *If $\psi\in\Psi$ satisfies $\sup_{t\in(0,1]}t^{-\alpha}\psi(t)=\infty$ for each $\alpha\in(0,1]$, then there exists a $\phi\in\Psi$ such that $\phi\le C\psi$ for a certain $C>0$ and the operator $\mathcal{K}$ is unbounded in any symmetric space X with the fundamental function $\phi_X=\phi$.*

6. GENERAL CASE

Here we consider the main question (see the Introduction) in the case in which inequality (1), generally speaking, does not hold. Following [3], for an arbitrary symmetric space X on $[0,1]$ we define a functional space Z_X on $[0,\infty)$:

$$Z_X:=\{f\in L_1[0,\infty)+L_\infty[0,\infty)\colon \|f\|'_{Z_X}:=\|f^*\chi_{[0,1]}\|_X+\|f^*\chi_{[1,\infty)}\|_1<\infty\}.$$

Since the quasinorm $\|\cdot\|'_{Z_X}$ is equivalent to the norm $\|f\|_{Z_X}:=\|f^*\chi_{[0,1]}\|_X+\|f\|_{L_1(0,\infty)}$, $f\in Z_X$, the space Z_X is a symmetric space on $[0,\infty)$.

The main result here supplements Johnson and Schechtman's theorem [3, inequality (4)]: it shows that the (modified) inequality (2) under the condition that $\mathcal{K}$ is bounded remains true in the general situation as well.

Theorem 8. *Suppose that X and Y are symmetric spaces on $[0,1]$, $X\subseteq Y$. If*

(i) *the operator $\mathcal{K}$ acts boundedly from X to $Y^{\times\times}$ and the norm in Y is order semicontinuous, or*

(ii) *the operator $\mathcal{K}$ acts boundedly from X to Y,* *then there exists a $C > 0$ such that the inequality*

$$\Big\|\sum_{i=1}^{n} g_i\Big\|_Y \le C\Big\|\sum_{i=1}^{n} \bar{g}_i\Big\|_{Z_X}$$

holds for each sequence $\{g_i\}_{i=1}^n \subset X$, $n \in \mathbb{N}$, of independent functions.

Corollary 9. *Let X be interpolation space with respect to the Banach pair $(L_1(0,1), L_\infty(0,1))$. If the operator $\mathcal{K}$ is bounded in X, then there exists a constant $C > 0$ such that for an arbitrary sequence of independent functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, and each sequence*

$$\{g_k\}_{k=1}^n, \qquad g_k \ge 0, \quad g_k^* = f_k^*, \quad 1 \le k \le n,$$

we have

$$\Big\|\sum_{i=1}^{n} f_i\Big\|_X \le C\Big\|\sum_{i=1}^{n} g_i\Big\|_X.$$

In the case of the symmetric space X such that $X \supseteq L_p[0,1]$, $p < \infty$, the last statement was proved in [3] (see inequality (10)).

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REFERENCES

1. H. P. Rosenthal, "On the subspaces of L_p $(p > 2)$ spanned by sequences of independent random variables," *Israel J. Math.*, **8** (1970), 273–303.
2. N. L. Carothers and S. J. Dilworth, "Inequalities for sums of independent random variables," *Proc. Amer. Math. Soc.*, **194** (1988), 221–226.
3. W. B. Johnson and G. Schechtman, "Sums of independent random variables in rearrangement invariant function spaces," *Ann. Probab.*, **17** (1989), 789–808.
4. M. Sh. Braverman, *Independent Random Variables and Rearrangement Invariant Spaces*, Cambridge University Press, Cambridge, 1994.
5. V. M. Kruglov, "Note on infinitely divisible distributions," *Teor. Veroyatnost. i Primenen.* [*Theory Probab. Appl.*], **15** (1970), no. 2, 331–336.
6. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators* [in Russian], Nauka, Moscow, 1978.
7. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II. Function spaces*, Springer-Verlag, Berlin–Heidelberg–New York, 1979.
8. C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York, 1988.
9. S. Kwapień and W. A. Woyczyński, *Random Series and Stochastic Integrals: Single and Multiple*, Birkhäuser, 1992.
10. S. Montgomery-Smith and E. M. Semenov, "Random rearrangements and operators," *Amer. Math. Soc. Trans. (2)*, **184** (1998), 157–183.

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