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On the Multiplier Space Generated by the Rademacher System

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Abstract—In this paper, we consider the space of multipliers of symmetric spaces with respect to the Rademacher systems. We obtain sufficient conditions under which the space in question coincides with the space L_∞ (with equivalence of norms). These conditions are stated in terms of operator interpolation theory and are essentially weaker than the conditions for the solution of this problem recently obtained by other authors.

KEY WORDS: *Rademacher function, rearrangement invariant space, Orlicz space, Marcinkiewicz space, interpolation of operators, K-functional.*

1. INTRODUCTION

Suppose that X is a symmetric space of measurable functions on $[0,1]$. Following [1], we denote by $\Lambda(R,X)$ the set of all measurable functions $f\colon [0,1]\to\mathbb{R}$ such that $fg\in X$ for all functions of the form

$$g(t)=\sum_{k=1}^{\infty}c_kr_k(t),$$

where the $r_k(t)$, $k=1,2,\dots$, are Rademacher functions, i.e.,

$$r_k(t)=\operatorname{sign}\sin(2^k\pi t)\qquad (t\in[0,1]).$$

Then $\Lambda(R,X)$ is a Banach functional lattice with respect to the norm

$$\|f\|_\Lambda=\sup\biggl\{\|fg\|_X : g=\sum_{k=1}^{\infty}c_kr_k\in X,\quad \|g\|_X\le 1\biggr\}.$$

In other words, it is the space of multipliers in X acting boundedly from the closed linear hull $[r_k]_X$ spanned by the Rademacher system to the whole space.

In [1], it was shown that the space $\Lambda(R,X)$ is not symmetric for a wide class of symmetric spaces X such that $[r_k]_X$ is isomorphic to l_2 (necessary and sufficient conditions for the latter property were obtained in [2]). To be more specific, the fact that the function $f(t)$ belongs to $\Lambda(R,X)$, does not, generally, imply that the nonincreasing rearrangement of its modulus, $f^*(t)$, belongs to $\Lambda(R,X)$ as well. At the same time, it turns out that, under certain conditions, $\Lambda(R,X)=L_\infty$ with equivalence of norms [3, 4]. To state these conditions, we need the following quasilinear operator also introduced earlier in the study of the Rademacher system in [5].

Let us define the following operator on the set of all measurable functions $f\colon[0,1]\to\mathbb{R}$:

$$Qf(t)=\biggl\{\frac{1}{\theta(t)}\int_0^t (f^*(s))^2\,d\theta(s)\biggr\}^{1/2},\qquad \text{where}\quad \theta(t)=\ln^{-1}\frac{e}{t}. \tag{1}$$

We now state the main assertion from [3, 4].

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Theorem (Curbera and Rodin). *If the operator Q acts boundedly on a symmetric space X, then*

$$\Lambda(R, X) = L_\infty \tag{2}$$

with equivalence of norms.

Here relation (2) will be proved for spaces that are interpolation spaces with respect to the Banach couple (L_∞, L_N), where L_N is the Orlicz space constructed by the function $N(u) = e^{u^2} - 1$ (Theorem 1). In the proof, we use the description of subspaces of the form $[r_k]_X$ for such spaces obtained earlier in the author's papers [6, 7]. The condition for a space to be an interpolation space with respect to the Banach couple (L_∞, L_N) is less restrictive than that of the boundedness of the operator Q (see Theorem 2 and Remark 1). Hence Theorem 1 strengthens the Curbera–Rodin theorem given above.

2. DEFINITIONS AND NOTATION

For a detailed treatment of the theory of symmetric function spaces and of operator interpolation theory, see, for example, the monographs [8–10].

If $x = x(t)$ is a measurable function on $[0, 1]$, then, by definition,

$$n_{|x|}(\tau) = \mu\{t \in [0,1] \colon |x(t)| > \tau\} \qquad \text{for} \quad \tau > 0$$

(μ is the Lebesgue measure). The functions $x(t)$ and $y(t)$ are said to be *equimeasurable* if we have $n_{|x|}(\tau) = n_{|y|}(\tau)$, $\tau > 0$. In particular, any function $x(t)$ is equimeasurable with its nonincreasing rearrangement

$$x^*(t) = \inf\{\tau \ge 0 \colon n_{|x|}(\tau) < t\}.$$

A Banach space X of measurable functions defined on $[0, 1]$ is called a *symmetric space* if the following conditions are satisfied:

(a) the relations $y = y(t) \in X$ and $|x(t)| \le |y(t)|$ yield $x = x(t) \in X$ and $\|x\| \le \|y\|$;
(b) if $y = y(t) \in X$ and the functions $x = x(t)$ and $y = y(t)$ are equimeasurable, then $x \in X$ and $\|x\| = \|y\|$.

The *fundamental function* of a symmetric space X is defined by the relation $\varphi_X(t) = \|\chi_{(0,t)}\|_X$, where, as is customary, $\chi_G(s) = 1$ if $s \in G$, and $\chi_G(s) = 0$ if $s \notin G$. Recall that for any symmetric space the function $\varphi_X(t)$ is quasiconcave on $(0, 1]$ [8, p. 137], i.e., is nonnegative, and increasing, while the function $\varphi_X(t)/t$ is decreasing on this closed interval. As is well known [8, p. 70], a quasiconcave function $f(t)$ is equivalent to its least concave majorant $\tilde{f}(t)$ or, to be exact,

$$f(t) \le \tilde{f}(t) \le 2f(t), \qquad t \in [0,1]. \tag{3}$$

In any symmetric space X, the dilation operator $\sigma_\tau x(t) = x(t/\tau)\chi_{[0,1]}(t/\tau)$, $\tau > 0$ is continuous [8, pp. 131–133]. Moreover, for all $\tau > 0$, we have

$$\|\sigma_\tau\|_{X \to X} \le \max(1, \tau). \tag{4}$$

An important example of a symmetric space is the Orlicz space. Suppose that $M(t)$ is an increasing convex function on $[0, \infty)$, $M(0) = 0$. The Orlicz space L_M consists of all measurable functions $x = x(t)$ on $[0, 1]$ such that the norm

$$\|x\|_{L_M} = \inf\left\{u > 0 \colon \int_0^1 M\left(\frac{|x(t)|}{u}\right) dt \le 1\right\}$$

is finite. In the special case $M(t) = t^p$, $1 \le p < \infty$, we obtain ordinary L_p-spaces. A direct calculation shows that $\varphi_{L_M}(t) = 1/M^{-1}(1/t)$, where $M^{-1}(u)$ is the inverse of $M(u)$, is the fundamental function.

Suppose that $\psi(t)$ is a quasiconcave function on $[0,1]$. The *Marcinkiewicz space* $M(\psi)$ consists of all measurable functions $x = x(s)$ on $[0,1]$ for which

$$\|x\|_{M(\psi)} = \sup_{0<t\le 1} \frac{\int_0^t x^*(s)\,ds}{\psi(t)} < \infty.$$

Note that this space is dual to the Lorentz space $\Lambda(\tilde\psi)$ (see [8 p. 152–154]) and

$$\|x\|_{\Lambda(\tilde\psi)} = \int_0^1 x^*(s)\,d\tilde\psi(s).$$

The fundamental function of the space $M(\psi)$ is equal to $t/\psi(t)$.

Suppose that (X_0, X_1) is a *Banach couple*, i.e., a pair of Banach spaces linearly and continuously embedded in some linear topological Hausdorff space, and $x \in X_0 + X_1$, $t > 0$. We define the $\mathcal{K}$-*Peetre functional*

$$\mathcal{K}(t, x\,;\, X_0, X_1) = \inf\bigl\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} :\ x = x_0 + x_1, \quad x_0 \in X_0, \quad x_1 \in X_1\bigr\}.$$

This functional plays an important role in operator interpolation theory and some other areas of calculus. For a fixed $x \in X_0 + X_1$, the $\mathcal{K}$-functional is an increasing concave function with respect to the variable t [9, Chap. 3, Lemma 3.1.1]. But if we choose $t > 0$, then $\mathcal{K}(t, x\,;\, X_0, X_1)$ is a norm on the sum of the spaces $X_0 + tX_1$.

A Banach space X is called an *interpolation space with respect to the Banach couple* (X_0, X_1) if the embedding $X_0 \cap X_1 \subset X \subset X_0 + X_1$ holds and the boundedness of a linear operator in X yields its boundedness in the spaces X_0 and X_1.

One of the most important methods of derivating interpolation spaces is the so-called $\mathcal{K}$-method based on the use of the Peetre $\mathcal{K}$-functional.

Suppose that F is the Banach lattice of two-sided numerical sequences

$$a = (a_k)_{k=-\infty}^{\infty}$$

and $F \supset l_\infty \cap l_\infty(2^{-k})$ (if $1 \le p \le \infty$ and $w_k > 0$, $k = 0, \pm 1, \dots$, then the "weight" space $l_p(w_k)$ consists of all sequences $a = (a_k)_{k=-\infty}^{\infty}$ such that $(a_k w_k)_k \in l_p$ and $\|a\|_{l_p(w_k)} = \|(a_k w_k)\|_{l_p}$). If (X_0, X_1) is an arbitrary Banach couple, then the *space of the* $\mathcal{K}$-*method* $(X_0, X_1)_F^{\mathcal{K}}$ consists of all $x \in X_0 + X_1$ for which the sequence $(\mathcal{K}(2^k, x\,;\, X_0, X_1))_k$ belongs to F and the norm of the element in F is given by the relation

$$\|x\| = \|(\mathcal{K}(2^k, x\,;\, X_0, X_1))_k\|_F.$$

It is readily verified that the space $(X_0, X_1)_F^{\mathcal{K}}$ is an interpolation space with respect to the couple (X_0, X_1).

In particular, for $0 < \theta < 1$, $1 \le p \le \infty$ we obtain the classical spaces $(X_0, X_1)_{\theta,p} = (X_0, X_1)^{\mathcal{K}}_{l_p(2^{-k\theta})}$. Their properties are considered in detail in the monograph [9].

In conclusion, let us dwell briefly on the notation. In what follows, an expression of the form $F_1 \asymp F_2$ means that for some $C_1 > 0$ and $C_2 > 0$ the estimates $C_1 F_1 \le F_2 \le C_2 F_1$ are satisfied; here, as a rule, the constants C_1 and C_2, are independent of all or part of the arguments of F_1 and F_2.

3. AUXILIARY RESULTS

First, we obtain a suitable expression for the $\mathcal{K}$-functional $\mathcal{K}(t, f\,; L_\infty, L_N)$, where L_N is the Orlicz space determined by the function $N(u) = e^{u^2} - 1$.

Suppose that $\psi(t)$ and $\varphi(t)$ are two increasing concave nonnegative functions on $[0, 1]$. In view of [8, pp. 165–166], the norm of the intersection $\Lambda(\psi) \cap t^{-1}\Lambda(\varphi)$, equal, by definition, to $\max\bigl(\|f\|_{\Lambda(\psi)}, t^{-1}\|f\|_{\Lambda(\varphi)}\bigr)$, is equivalent to the norm of the Lorentz space $\Lambda\bigl(\max(\psi, t^{-1}\varphi)\bigr)$. Hence, passing to the dual spaces [8, p. 27] and taking inequality (3) into account, we obtain

$$\mathcal{K}(t, f\,; M(\psi), M(\varphi)) \asymp \sup_{0<u\le 1} \frac{\int_0^u f^*(s)\,ds}{\max(\psi(u), t^{-1}\varphi(u))} \tag{5}$$

with constants independent of $t > 0$ and $f \in M(\psi) + M(\varphi)$.

Lemma 1. *The following relation holds with constants independent of $t > 0$ and $f \in L_N$:*

$$\mathcal{K}(t, f\,; L_\infty, L_N) \asymp t \sup_{0<u\le \min(1, 2^{1-t^2})} \biggl(f^*(u) \log_2^{-1/2} \frac{2}{u}\biggr).$$

Proof. First, we note that the space L_N is simultaneously (with equivalence of norms) the Marcinkiewicz space $M(\psi_1)$ for $\psi_1(u) = u \log_2^{1/2}(2/u)$ [11]. In addition, it is obvious that $L_\infty = M(\psi_0)$, $\psi_0(u) = u$. Hence it follows from relation (5) that

$$\mathcal{K}(t, f\,; L_\infty, L_N) \asymp \|f\|_{M(\psi_t)}, \qquad \text{where} \quad \psi_t(u) = u \max\biggl\{1, t^{-1} \log_2^{1/2} \frac{2}{u}\biggr\}. \tag{6}$$

Suppose that $\mathcal{M}_{\psi_t}(s)$ is the dilate of the function ψ_t, i.e.,

$$\mathcal{M}_{\psi_t}(s) = \sup_{0<u\le \min(1, 1/s)} \frac{\psi_t(su)}{\psi_t(u)}.$$

Let

$$\mathcal{M}_{\psi_t}\biggl(\frac{1}{2}\biggr) = \frac{1}{2} \sup_{0<u\le 1} F_t(u), \qquad \text{where} \quad F_t(u) = \frac{\max(1, t^{-1} \log_2^{1/2}(4/u))}{\max(1, t^{-1} \log_2^{1/2}(2/u))}.$$

Since

$$\log_2^{1/2}(4/u) \le \sqrt{2} \log_2^{1/2}(2/u),$$

we have $F_t(u) \le \sqrt{2}$. Therefore, for any $t > 0$ we have $\mathcal{M}_{\psi_t}(1/2) \le 1/\sqrt{2}$, and hence

$$\|f\|_{M(\psi_t)} \asymp \sup_{0<u\le 1} \biggl\{f^*(u) \min\biggl(1, t \log_2^{-1/2} \frac{2}{u}\biggr)\biggr\} = t \sup_{0<u\le \min(1, 2^{1-t^2})} \biggl(f^*(u) \log_2^{-1/2} \frac{2}{u}\biggr)$$

with constants independent of $t > 0$ and f [8, p. 156]. Therefore, in view of (6), we obtain the proof of the lemma. $\square$

Further, we need the following relation for the $\mathcal{K}$-functional $\mathcal{K}_{1,2}(t, a) = \mathcal{K}(t, a\,; l_1, l_2)$, constructed from the Banach couple (l_1, l_2), with some constant $c > 0$ independent of the element $a = (a_k)_{k=1}^\infty \in l_2$ and of $t > 0$ [11]:

$$c\Biggl(\sum_{i=1}^{[t^2]} a_i^* + t\Biggl(\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\Biggr)^{1/2}\Biggr) \le \mathcal{K}_{1,2}(t, a) \le \sum_{i=1}^{[t^2]} a_i^* + t\Biggl(\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\Biggr)^{1/2}, \tag{7}$$

where $(a_i^*)_{i=1}^\infty$ is the decreasing rearrangement of the sequence $(|a_k|)_{k=1}^\infty$, and $[z]$ is the integral part of the number z.

Suppose that $n \in \mathbb{N}$ and $b_n = (1/n)\sum_{k=1}^n e_k$, where the e_k are the standard unit vectors in the space of numerical sequences.

Lemma 2. *The following relation holds with constants independent of* $t > 0$ *and* $n \in \mathbb{N}$:

$$\mathcal{K}_{1,2}(t, b_n) \asymp \min\left(1, \frac{t}{\sqrt{n}}\right). \tag{8}$$

Proof. First, suppose that $t \in \mathbb{N}$. Then, if $t^2 \geq n$, then relation (8) is a direct consequence of relation (7). Now, suppose that $t^2 < n$. Then, by (7), we have

$$c(v^2 + v(1-v^2)^{1/2}) \leq \mathcal{K}_{1,2}(t, b_n) \leq v^2 + v(1-v^2)^{1/2},$$

where $v = t/\sqrt{n} < 1$. In the case $v \leq 1/\sqrt{2}$, it follows that

$$\frac{c}{\sqrt{2}} v \leq \mathcal{K}_{1,2}(t, b_n) \leq v^2 + v \leq \left(1 + \frac{1}{\sqrt{2}}\right) v.$$

But if $1/\sqrt{2} < v < 1$, then again

$$\frac{c}{\sqrt{2}} v \leq cv^2 \leq \mathcal{K}_{1,2}(t, b_n) \leq 2v.$$

Thus, relation (8) is proved for natural, and hence for all, values of $t \geq 1$, since the $\mathcal{K}$-functional is quasiconcave in t. It remains to note that for $0 < t \leq 1$

$$\mathcal{K}_{1,2}(t, b_n) = t\|b_n\|_{l_2} = v$$

by the definition of the $\mathcal{K}$-functional. The lemma is proved. □

4. MAIN RESULTS

Theorem 1. *If* X *is an interpolation space with respect to the Banach couple* (L_∞, L_N)*, then* $\Lambda(R, X) = L_\infty$ *(with equivalence of norms).*

Proof. Suppose that $h_n(t) = \chi_{[0,2^{-n}]}(t)$, $n \in \mathbb{N}$. By Lemma 1, we have

$$\mathcal{K}(2^k, h_n; L_\infty, L_N) \asymp 2^k \min\left(\frac{1}{\sqrt{n}}, \frac{1}{2^k}\right)$$

with constants independent of $n \in \mathbb{N}$ and $k = 0, \pm 1, \pm 2, \dots$. Therefore, if, as before, $b_n = (1/n)\sum_{i=1}^n e_i$, then, in view of Lemma 2, we can write

$$\mathcal{K}(2^k, h_n; L_\infty, L_N) \asymp \mathcal{K}_{1,2}(2^k, b_n) \tag{9}$$

with constants independent of $n \in \mathbb{N}$ and $k = 0, \pm 1, \pm 2, \dots$.

By assumption, the space X is an interpolation space with respect to the Banach couple (L_∞, L_N), and hence

$$X = (L_\infty, L_N)_F^{\mathcal{K}}$$

for some Banach lattice F of two-sided numerical sequences [6, 7]. Hence, if $\varphi_X(t)$ is a fundamental function X, then it follows from (9) that

$$\varphi_X(2^{-n}) \asymp \|(\mathcal{K}(2^k, h_n; L_\infty, L_N))_k\|_F \asymp \|(\mathcal{K}_{1,2}(2^k, b_n))_k\|_F.$$

On the other hand [6, 7],

$$\left\| \frac{1}{n} \sum_{i=1}^{n} r_i \right\|_X \asymp \|(\mathcal{K}_{1,2}(2^k, b_n))_k\|_F.$$

Thus,

$$\left\| \sum_{i=1}^{n} r_i \right\|_X \asymp n\varphi_X(2^{-n}). \tag{10}$$

Now, suppose that $G \subset [0,1]$ is an arbitrary measurable set, with $\mu(G) > 0$. Since almost all the points of a measurable set are points of density with respect to the Lebesgue measure, we can find a binary interval I such that

$$\mu(G \cap I) \geq \frac{1}{2}\mu(I). \tag{11}$$

Suppose that the rank I is equal to n i.e., $I = [(i-1)2^{-n}, i2^{-n})$ for some $i = 1, 2, \dots, 2^n$. We choose $\varepsilon_i = \pm 1$, $i = 1, 2, \dots, n$ so that for all $t \in I$ the equality

$$\sum_{i=1}^{n} \varepsilon_i r_i(t) = n$$

is satisfied. Then, if $K = \| \sum_{i=1}^{n} \varepsilon_i r_i \|_X$, it follows from the definition of $\Lambda(R, X)$, the symmetry of X, the inequality $\|\sigma_\tau\|^{-1} \leq \|\sigma_{\tau^{-1}}\|$, and also from relations (4) and (11) that

$$\|\chi_G\|_\Lambda \geq \|\chi_{G\cap I}\|_\Lambda \geq \frac{1}{K} \left\| \chi_{G\cap I} \sum_{i=1}^{n} \varepsilon_i r_i \right\|_X \geq \frac{1}{K} \|\sigma_2\|_{X\to X}^{-1} \left\| \chi_I \sum_{i=1}^{n} \varepsilon_i r_i \right\|_X \geq \frac{n}{2K} \varphi_X(2^{-n}).$$

By the definition of the Rademacher functions and the symmetry of the space X, we have

$$K = \left\| \sum_{i=1}^{n} r_i \right\|_X.$$

Hence the last inequality and (10) yield

$$\|\chi_G\|_{\Lambda(R,X)} \geq c_0, \tag{12}$$

where the constant $c_0 > 0$ is independent of the set G.

Since $L_\infty \subset \Lambda(R, X)$, it follows that, to prove the theorem, it suffices to show that the multiplier space $\Lambda(R, X)$ does not contain a single unbounded function. Suppose that f belongs to $\Lambda(R, X)$ and is unbounded. Then, for each $n \in \mathbb{N}$, the measure of the set

$$G_n = \{t \in [0,1] \colon |f(t)| \geq n\}$$

is positive. Hence, since $\Lambda(R, X)$ is a Banach lattice, it follows from (12) that

$$\|f\|_\Lambda \geq n \|\chi_{G_n}\|_\Lambda \geq nc_0 \qquad \text{for any} \quad n \in \mathbb{N}.$$

This contradicts the fact that f belongs to $\Lambda(R, X)$. The theorem is proved. □

In studying problems concerning the Rademacher system in symmetric spaces (see [3–5]), we can effectively use the quasilinear operator Q defined by relation (1).

In [4], it was shown that if the operator Q acts boundedly on a symmetric space X, then $L_\infty \subset X \subset L_N$. Here we shall prove a stronger assertion.

Theorem 2. *If the operator Q acts boundedly on a symmetric space X, then the space X is an interpolation space with respect to the Banach couple (L_∞, L_N).*

Proof. Let us introduce the Lorentz space $\Lambda_2(\theta)$ with norm

$$\|f\|_{\Lambda_2(\theta)} = \left\{ \int_0^1 (f^*(s))^2 \, d\theta(s) \right\}^{1/2}, \qquad \theta(t) = \ln^{-1} \frac{e}{t},$$

and also the class Σ of all bounded linear operators from L_∞ to L_∞ and from $\Lambda_2(\theta)$ to L_N. Let us show that a linear operator T belongs to Σ if and only if there exists a constant $C > 0$ such that for all $f \in \Lambda_2(\theta)$ and $0 < t \le 1$

$$(Tf)^*(t) \le C\mathcal{K}\left(\log_2^{1/2} \frac{2}{t}, f\,;\, L_\infty, \Lambda_2(\theta)\right). \tag{13}$$

As was already stated in the proof of Lemma 1, the space L_N coincides with the Marcinkiewicz space $M(\psi_1)$, $\psi_1(u) = u \log_2^{1/2}(2/u)$. Moreover, we have (see [8, p. 156] or Lemma 1)

$$\|f\|_{L_N} \asymp \sup_{0<s\le 1} \left(f^*(s) \log_2^{-1/2} \frac{2}{s} \right). \tag{14}$$

Suppose that $T \in \Sigma$. We express $f \in \Lambda_2(\theta)$ as

$$f = f_0 + f_1, \qquad \text{where} \quad f_0 \in L_\infty, \quad f_1 \in \Lambda_2(\theta). \tag{15}$$

Taking relations (4) and (14) into account, for each $t > 0$ we obtain

$$\begin{aligned} (Tf)^*(t) &\le (|Tf_0| + |Tf_1|)^*(t) \le (Tf_0)^*\left(\frac{t}{2}\right) + (Tf_1)^*\left(\frac{t}{2}\right) \\ &\le C_1 \max\bigl(\|T\|_{L_\infty \to L_\infty}, \|T\|_{\Lambda_2(\theta)\to L_N}\bigr) \left(\|f_0\|_{L_\infty} + \left(\log_2^{1/2} \frac{2}{t}\right) \|f_1\|_{\Lambda_2(\theta)} \right). \end{aligned}$$

Passing to the infimum over all representations (15), we obtain (13) with a suitable constant. The converse assertion follows immediately from the fact that

$$\mathcal{K}\left(\log_2^{1/2} \frac{2}{t}, f\,;\, L_\infty, \Lambda_2(\theta)\right) \le \|f\|_{L_\infty}, \qquad f \in L_\infty,$$
$$\mathcal{K}\left(\log_2^{1/2} \frac{2}{t}, f\,;\, L_\infty, \Lambda_2(\theta)\right) \le \left(\log_2^{1/2} \frac{2}{t}\right) \|f\|_{L_N}, \qquad f \in L_N.$$

Standard arguments (see, for example, [9, Chap. 5, Theorem 5.2.1]) show that

$$\mathcal{K}(t, f\,;\, L_\infty, \Lambda_2(\theta)) \asymp t \left\{ \int_0^{2^{1-t^2}} (f^*(s))^2 \, d\theta(s) \right\}^{1/2}.$$

Hence, in view of (1), inequality (13) is equivalent to the following inequality:

$$(Tf)^*(t) \le CQf(t). \tag{16}$$

Note that we have the following relation for the fundamental function:

$$\varphi_{\Lambda_2(\theta)}(t) = \theta^{1/2}(t) \asymp t/\psi_1(t).$$

Therefore, (see [8, pp. 160–162]), the following continuous embeddings take place:

$$\Lambda\left(\frac{t}{\psi_1(t)}\right) \subset \Lambda_2(\theta) \subset M(\psi_1) = L_N. \tag{17}$$

Suppose now that the operator Q is bounded on the symmetric space X. If T is a linear operator bounded on L_∞ and L_N, then the embeddings (17) yield $T \in \Sigma$, and hence it follows from inequality (16) that T is bounded in X. Thus, the space X is an interpolation space with respect to the couple (L_∞, L_N). $\square$

Remark 1. The converse statement is false. For example, the space $X = L_N$ is, obviously, an interpolation space with respect to the Banach couple (L_∞, L_N), but, as is easy to verify [4], the operator Q is not bounded on it.

Remark 2. Theorem 2 shows that Theorem 1 proved in this paper is a strengthening of the Curbera–Rodin theorem [3, 4] stated in the introduction.

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