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2003 Sb. Math. 194 813

(http://iopscience.iop.org/1064-5616/194/6/A02)

Sbornik: *Mathematics* **194**:6 813–832

Matematicheskiĭ Sbornik **194**:6 23–42
DOI 10.1070/SM2003v194n06ABEH000740

Extrapolation properties of the scale of L_p-spaces

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Abstract. A new class of extrapolation functors on the scale of L_p-spaces ($1<p<\infty$) is introduced, allowing one to take for its 'limiting spaces' two symmetric spaces 'close' to L_∞ and L_1. Crucial here are the extrapolation relations for the Peetre $\mathcal{K}$- and $\mathcal{J}$-functionals for the Banach couples $(L_\infty, \operatorname{Exp} L^\beta)$ and $(L_1, L(\log L)^{1/\beta})$, respectively ($\operatorname{Exp} L^\beta$ and $L(\log L)^{1/\beta}$, $\beta > 0$, are Zygmund spaces).

The real method of operator interpolation is used.

Bibliography: 22 titles.

Introduction

We recall Yano's classical extrapolation theorem (see [1] or [2], Chapter 12, Theorem 4.41), which is in a certain sense a conversion of Marcinkiewicz's theorem on interpolation of linear operators ([2], Chapter 12, Theorem 4.6). Let T be a linear operator bounded in the spaces $L_p = L_p[0,1]$ for all p in some right half-neighbourhood of 1, and let $\|T\|_{L_p\to L_p} = O((p-1)^{-\alpha})$ (as $p \to 1+$) for some $\alpha > 0$. Then T can be defined in the Zygmund space $L(\log L)^\alpha$ (see the definitions below) as a bounded operator from that space into L_1. The dual result for the space $\operatorname{Exp} L^{1/\alpha}$ dual to $L(\log L)^\alpha$ also holds. If T is a linear operator bounded in L_p for all sufficiently large p and $\|T\|_{L_p\to L_p} = O(p^\alpha)$ (as $p \to \infty$) for some $\alpha > 0$, then $T\colon L_\infty \to \operatorname{Exp} L^{1/\alpha}$.

General approaches of extrapolation theory have been developed since the 1990s. This is primarily due to Jawerth and Milman [3]–[5], who initiated a systematic investigation of the natural limiting spaces associated with scales of interpolation spaces, which can serve as a basis for estimates of the norms of the operators in these spaces. For a bibliography of numerous applications of extrapolation theory to analysis one can also consult [3]–[5].

Note that the tools (the extrapolation functors) used by Jawerth and Milman and bringing about new limiting spaces are the sum and the intersection of families of Banach spaces (see, for instance, [4], § 2 or [5], Chapter 2). This allows one to obtain as extrapolation spaces only spaces of the form $(X_0, X_1)^{\mathcal{J}}_{l_1(1/\rho)}$ and $(X_0, X_1)^{\mathcal{K}}_{l_\infty(1/\rho)}$, where (X_0, X_1) is an arbitrary Banach couple and ρ is a quasiconcave function. Here we overcome certain limitations of these methods and introduce a much broader class of extrapolation functors (at any rate on the scale of L_p-spaces) closely connected with the real interpolation method. As a result we obtain as limiting spaces

AMS 2000 *Mathematics Subject Classification.* Primary 46M35, 46E30.

'almost all' symmetric (or rearrangement invariant) spaces 'close' to L_∞ and L_1. Crucial here are the extrapolation relations for the $\mathcal{K}$- and $\mathcal{J}$-functionals for the couples $(L_\infty, \operatorname{Exp} L^\beta)$ and $(L_1, L(\log L)^{1/\beta})$, respectively, established in this paper.

En route, for the norms of spaces in the above-mentioned class we find fairly simple equivalent relations involving only the L_p-norms of the function. As a consequence one obtains a 'series' of extrapolation results similar to Yano's theorem.

In what follows we consider spaces of functions on $[0,1]$. For generalizations and refinements of Yano's theorem in the case of functions on a space with σ-finite measure we refer the reader to [6], [7].

We shall now discuss in this introduction facts and observations prompting the idea of the existence of an equivalent expression for the $\mathcal{K}$-functional for the Banach couple $(L_\infty, \operatorname{Exp} L^2)$ depending only on the L_p-norms of the corresponding functions.

In 1923, Khinchine [8] proved the following inequalities, which later became classical:

$$A_p\biggl(\sum_{k=1}^\infty |a_k|^2\biggr)^{1/2} \leqslant \biggl\|\sum_{k=1}^\infty a_k r_k\biggr\|_{L_p[0,1]} \leqslant B_p\biggl(\sum_{k=1}^\infty |a_k|^2\biggr)^{1/2}, \tag{1}$$

where the r_k $(k=1,2,\dots)$ are the Rademacher functions on $[0,1]$, that is,

$$r_k(t) = \operatorname{sign}\sin(2^{k-1}\pi t) \qquad (t\in[0,1]),$$

$a_k\in\mathbb{R}$, and A_p and B_p are constants depending only on $p\in[1,\infty)$. These inequalities were the starting point for many researchers, who repeatedly generalized and improved them. It is important for us that the constant B_p on the right-hand side of (1) approaches infinity as $p\to\infty$. More precisely, $B_p \asymp \sqrt{p}$ as $p\to\infty$, that is, for all $p\geqslant 1$ we have

$$C^{-1}\sqrt{p} \leqslant B_p \leqslant C\sqrt{p}$$

with some constant $C>0$.

In 1993, Hitczenko [9] proved inequalities similar to (1) but having one considerable advantage: the constants in them are independent of p. In these inequalities the l_2-norm of the sequence of coefficients $a=(a_k)_{k=1}^\infty$ must be replaced by its $\mathcal{K}$-functional $\mathcal{K}_{1,2}(t,a)=\mathcal{K}(t,a;l_1,l_2)$ for the Banach couple (l_1,l_2). Let us present the precise statement: there exists a positive constant c such that for all $p\geqslant 1$ and $a=(a_k)_{k=1}^\infty\in l_2$,

$$c\mathcal{K}_{1,2}(\sqrt{p}, a) \leqslant \biggl\|\sum_{k=1}^\infty a_k r_k\biggr\|_p \leqslant \mathcal{K}_{1,2}(\sqrt{p}, a). \tag{2}$$

Here and throughout $\|f\|_p = \|f\|_{L_p[0,1]}$.

Later, the present author showed [10], [11] that

$$C_1^{-1}\mathcal{K}_{1,2}(t,a) \leqslant \mathcal{K}\biggl(t, \sum_{k=1}^\infty a_k r_k; L_\infty, \operatorname{Exp} L^2\biggr) \leqslant C_1\mathcal{K}_{1,2}(t,a), \tag{3}$$

with constant $C_1 > 0$ independent of $a = (a_k)_{k=1}^\infty \in l_2$ and $t > 0$. It follows by relations (2) and (3) that for some $C > 0$ and all $a = (a_k)_{k=1}^\infty \in l_2$ and $p \geqslant 1$ we have

$$C^{-1}\biggl\|\sum_{k=1}^\infty a_k r_k\biggr\|_p \leqslant \mathcal{K}\biggl(\sqrt{p}\,, \sum_{k=1}^\infty a_k r_k; L_\infty, \operatorname{Exp} L^2\biggr) \leqslant C\biggl\|\sum_{k=1}^\infty a_k r_k\biggr\|_p.$$

If in place of the Rademacher sum one substitutes in the last relation an arbitrary measurable function f, then the right-hand inequality fails (see Remark 1). However, if one replaces $\|f\|_p$ by a quantity with a more regular behaviour in p, then one obtains correct inequalities:

$$C^{-1}\sqrt{p}\sup_{q\geqslant p}\frac{\|f\|_q}{\sqrt{q}} \leqslant \mathcal{K}(\sqrt{p}\,, f; L_\infty, \operatorname{Exp} L^2) \leqslant C\sqrt{p}\sup_{q\geqslant p}\frac{\|f\|_q}{\sqrt{q}}\,.$$

The proof of slightly more general inequalities (for $\mathcal{K}(p, f; L_\infty, \operatorname{Exp} L^\beta)$) is the main subject of our § 2 (in § 1 we present definitions and notation required in what follows). On this basis, in § 3 using the real $\mathcal{K}$-interpolation method we find equivalent relations for the norms of symmetric spaces 'close' to L_∞ involving only the L_p-norms of the functions (more precisely, their asymptotic behaviour as $p \to \infty$). Finally, in § 4 we obtain dual results on the $\mathcal{J}$-functional for the couple $(L_1, L(\log L)^{1/\beta})$ and on spaces 'close' to another 'end-point' of the symmetric space scale, the space L_1.

§ 1. Definitions, notation, and auxiliary information

An extended exposition of the theory of symmetric function spaces and operator interpolation theory can be found, for instance, in the monographs [12]–[15].

A major role in various analytic questions is played by the Zygmund spaces $L(\log L)^\alpha$ and $\operatorname{Exp} L^\beta$ (see, for instance, [16]). They consist of all measurable functions $f\colon [0,1] \to \mathbb{R}$ with finite quasinorms

$$\|f\|_{L(\log L)^\alpha} = \int_0^1 \log_2^\alpha\biggl(\frac{2}{t}\biggr) f^*(t)\,dt \quad\text{and}\quad \|f\|_{\operatorname{Exp} L^\beta} = \sup_{0<t\leqslant 1}\log_2^{-1/\beta}\biggl(\frac{2}{t}\biggr) f^*(t),$$

respectively. Here $\alpha, \beta > 0$ and $f^*(t)$ is a non-increasing rearrangement of the function $|f(t)|$ ([14], Chapter 2, § 2). In each of these spaces one can introduce quasinorms making it symmetric (or rearrangement invariant).

Recall that a Banach space X of measurable functions on $[0,1]$ is said to be *symmetric* if the following conditions hold:

(a) if $y = y(t) \in X$ and $|x(t)| \leqslant |y(t)|$, then $x = x(t) \in X$ and $\|x\| \leqslant \|y\|$;

(b) if $y = y(t) \in X$ and $x^*(t) = y^*(t)$, then $x \in X$ and $\|x\| = \|y\|$.

An important, and the simplest, example of symmetric spaces are the L_p-spaces $(1 \leqslant p \leqslant \infty)$ with the usual norm:

$$\|x\|_p = \biggl(\int_0^1 |x(t)|^p\,dt\biggr)^{1/p} \quad \text{for } p < \infty$$

and

$$\|x\|_\infty = \operatorname*{ess\,sup}_{0\leqslant t\leqslant 1} |x(t)| \quad \text{for } p = \infty.$$

Orlicz spaces are their natural generalization. Let $N(t)$ be an increasing convex function on $[0,\infty)$, $N(0)=0$. Then the Orlicz space L_N consists of all measurable functions $x = x(t)$ on $[0,1]$ such that the norm

$$\|x\|_{L_N} = \inf\biggl\{u > 0 : \int_0^1 N\biggl(\frac{|x(t)|}{u}\biggr)\,dt \leqslant 1\biggr\}$$

is finite. In particular, if $N(t) = t^p$ $(1 \leqslant p < \infty)$, then we obtain the L_p-spaces.

Other examples of symmetric spaces are Lorentz and Minkowski spaces. If $\varphi(t)$ is an increasing concave function on $[0,1]$, then the Marcinkiewicz space $M(\varphi)$ consists of all measurable functions $x = x(s)$ on $[0,1]$ such that

$$\|x\|_{M(\varphi)} = \sup_{0<t\leqslant 1}\biggl(\varphi^{-1}(t)\int_0^t x^*(s)\,ds\biggr) < \infty,$$

and the Lorentz space $\Lambda_p(\varphi)$ $(1 \leqslant p < \infty)$ consists of all $x = x(s)$ such that

$$\|x\|_{\Lambda_p(\varphi)} = \biggl(\int_0^1 (x^*(s))^p\,d\varphi(s)\biggr)^{1/p} < \infty.$$

Note that we can introduce in $\operatorname{Exp} L^\beta$ equivalent norms making it an Orlicz space and a Marcinkiewicz space: it coincides with the space L_{N_β}, where $N_\beta(t) = e^{t^\beta} - 1$ if $\beta \geqslant 1$ and $N_\beta(t) = \exp(t + t_\beta^{1/\beta})^\beta - \exp t_\beta$, $t_\beta = 1/\beta - 1$ if $0 < \beta < 1$, and it also coincides with the space $M(\varphi_\beta)$, where $\varphi_\beta(t) = t\log_2^{1/\beta}(e^{\frac{1+\beta}{\beta}}/t)$ (see [17], [18], and also [14], Theorem 5.3). In a similar way, $L(\log L)^\alpha$ coincides both with an Orlicz space L_{M_α}, where $M_\alpha(t)$ is equivalent at infinity to the function $t\log_2^\alpha t$, and with the Lorentz space $\Lambda_1(\varphi_{1/\alpha})$ [18]. Hence, in particular, in view of the duality between Lorentz and Marcinkiewicz spaces ([14], Theorem 5.2) we have $(L(\log L)^\alpha)^* = \operatorname{Exp} L^{1/\alpha}$ and $L(\log L)^\alpha$ is a subspace of $(\operatorname{Exp} L^{1/\alpha})^*$ (the corresponding quasinorms are equivalent).

If (X_0, X_1) is a Banach couple (that is, X_0 and X_1 are continuously and linearly embedded in some Hausdorff linear topological space), then one defines in the natural fashion their intersection $X_0 \cap X_1$ and sum $X_0 + X_1$, with norms

$$\|x\| = \max_{i=0,1}\|x\|_{X_i} \text{ and } \|x\| = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1,\ x_i \in X_i,\ i = 0,1\},$$

respectively.

Let (X_0, X_1) and (Y_0, Y_1) be Banach couples. Then a triple of spaces (X_0, X_1, X), $X_0 \cap X_1 \subset X \subset X_0 + X_1$, is called an *interpolation* triple with respect to another triple (Y_0, Y_1, Y), $Y_0 \cap Y_1 \subset Y \subset Y_0 + Y_1$, if each linear operator T defined on $X_0 + X_1$ and bounded as an operator from X_0 into Y_0 and from X_1 into Y_1 is also bounded from X into Y. If $X_i = Y_i$ $(i = 0,1)$ and $X = Y$, then one says that X is an *interpolation space* with respect to the couple (X_0, X_1).

An important method for finding interpolation spaces is the real interpolation method based on the use of the Peetre $\mathcal{K}$- and $\mathcal{J}$-functionals:

$$\mathcal{K}(t,x;X_0,X_1)=\inf\{\|x_0\|_{X_0}+t\|x_1\|_{X_1}:x=x_0+x_1,\ x_0\in X_0,\ x_1\in X_1\}$$

and

$$\mathcal{J}(t,x;X_0,X_1)=\max\{\|x\|_{X_0},\ t\|x\|_{X_1}\}.$$

For fixed positive t the first functional is a norm in the sum X_0+tX_1 and the second is a norm in the intersection $X_0\cap tX_1$ (if X is a Banach space and $a>0$, then the space aX contains the same elements as X, but with the norm $\|x\|_{aX}=a\|x\|_X$). If we fix $x\in X_0+X_1$ (respectively, $x\in X_0\cap X_1$), then $\mathcal{K}(t,x;X_0,X_1)$ (respectively, $\mathcal{J}(t,x;X_0,X_1)$) is an increasing concave function of t.

We recall a well-known result on the duality between $\mathcal{K}$- and $\mathcal{J}$-functionals ([12], § 3.7). Let (X_0,X_1) be a Banach couple such that $X_0\cap X_1$ is dense in X_0 and X_1. If the sum X_0+X_1 is endowed with the norm $\mathcal{K}(t,x;X_0,X_1)$ $(t>0)$, then its dual space is isometric to the intersection $X_0^*\cap X_1^*$ endowed with the norm $\mathcal{J}(t^{-1},x^*;X_0^*,X_1^*)$. Using this fact, the already mentioned duality between Lorentz and Marcinkiewicz spaces, and the formula for the $\mathcal{J}$-functional for the corresponding couple of Lorentz spaces ([14], Theorem 5.10) one easily obtains the following result on the $\mathcal{K}$-functional for a couple of Marcinkiewicz spaces:

$$\mathcal{K}(t,x;M(\psi),M(\varphi))\asymp\sup_{0<u\leqslant 1}[\max(\psi(u),t^{-1}\varphi(u))]^{-1}\int_0^u x^*(s)\,ds \tag{4}$$

with constants independent of $x\in M(\psi)+M(\varphi)$ and $t>0$ (see also [19]). Here and throughout, an expression of the type $F_1\asymp F_2$ means that $cF_1\leqslant F_2\leqslant CF_1$ for some positive c and C; usually, the constants c and C are independent of all or of some arguments of the functions F_1 and F_2.

Let E be a Banach lattice of two-sided number sequences $a=(a_j)_{j=-\infty}^{\infty}$. If (X_0,X_1) is an arbitrary Banach couple, then the space of the $\mathcal{K}$-method $(X_0,X_1)_E^{\mathcal{K}}$ consists of all $x\in X_0+X_1$ such that

$$\|x\|=\|(\mathcal{K}(2^j,x;X_0,X_1))_j\|_E<\infty.$$

The space of the $\mathcal{J}$-method $(X_0,X_1)_E^{\mathcal{J}}$ contains all $x\in X_0+X_1$ admitting the representation

$$x=\sum_{j=-\infty}^{\infty}u_j\quad\text{(the convergence in } X_0+X_1\text{)}, \tag{5}$$

where $u_j\in X_0\cap X_1$. The norm in $(X_0,X_1)_E^{\mathcal{J}}$ is set to be

$$\inf_{\{u_j\}}\|(\mathcal{J}(2^j,u_j;X_0,X_1))_j\|_E,$$

where we take the infimum over all sequences $\{u_j\}_{j=-\infty}^{\infty}$ such that (5) holds.

If E is a Banach lattice of two-sided sequences and $\alpha = (\alpha_k)_{k=-\infty}^{\infty}$ is a non-negative number sequence, then $E(\alpha_k)$ is the space of all sequences $a = (a_k)_{k=-\infty}^{\infty}$ such that $(a_k\alpha_k)_{k=-\infty}^{\infty} \in E$ and $\|a\|_{E(\alpha_k)} = \|(a_k\alpha_k)\|_E$. Let $E \supset \Delta(\vec{l}_\infty) = l_\infty \cap l_\infty(2^{-k})$ (respectively, $\{0\} \neq E \subset \Sigma(\vec{l}_1) = l_1 + l_1(2^{-k})$). Then the map $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathcal{K}}$ (respectively, $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathcal{J}}$) defines an interpolation functor: namely, for arbitrary Banach couples (X_0, X_1) and (Y_0, Y_1) the triple $(X_0, X_1, (X_0, X_1)_E^{\mathcal{K}})$ is an interpolation triple with respect to $(Y_0, Y_1, (Y_0, Y_1)_E^{\mathcal{K}})$ (a similar definition holds in the case of the $\mathcal{J}$-method). The set of all such functors is called the *real* $\mathcal{K}$- (respectively, $\mathcal{J}$-) *method of interpolation*. It is important to note that in the case of the $\mathcal{K}$- (or $\mathcal{J}$-) method its parameter E can always be assumed to be an interpolation space with respect to the couple $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$ (respectively, with respect to $\vec{l}_1 = (l_1, l_1(2^{-k}))$; [13], Corollaries 3.3.10 and 3.4.6).

In particular, for $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$ we obtain the classical interpolation spaces

$$(X_0, X_1)_{\theta,p} = (X_0, X_1)_{l_p(2^{-k\theta})}^{\mathcal{K}} = (X_0, X_1)_{l_p(2^{-k\theta})}^{\mathcal{J}},$$

the properties of which are studied in detail in the monograph [12].

§ 2. The $\mathcal{K}$-functional for the couple $(L_\infty, \operatorname{Exp} L^\beta)$

Lemma 1. *For arbitrary $\beta > 0$,*

$$\mathcal{K}(t, f; L_\infty, \operatorname{Exp} L^\beta) \asymp t \sup_{0<u\leqslant\min(1,2^{1-t^\beta})} \left(f^*(u) \log_2^{-1/\beta}\left(\frac{2}{u}\right)\right)$$

with constants independent of $f \in \operatorname{Exp} L^\beta$ and $t > 0$.

Proof. By relation (4) and in view of the equalities $L_\infty = M(\varphi_0)$, $\varphi_0(u) = u$ and $\operatorname{Exp} L^\beta = M(\varphi_\beta)$, $\varphi_\beta(u) = t \log_2^{1/\beta}(e^{\frac{1+\beta}{\beta}}/u)$ (see § 1), we obtain

$$\begin{gathered}\mathcal{K}(t, f; L_\infty, \operatorname{Exp} L^\beta) \asymp \|f\|_{M(\psi_t)}, \\ \text{where} \quad \psi_t(u) = u \max\left\{1, t^{-1} \log_2^{1/\beta}\left(\frac{e^{(1+\beta)/\beta}}{u}\right)\right\},\end{gathered} \tag{6}$$

with constant depending only on $\beta > 0$.

Let $\mathcal{M}_\psi(s)$ be the dilation function of the function ψ, that is,

$$\mathcal{M}_\psi(s) = \sup_{0<u\leqslant\min(1,1/s)} \frac{\psi(su)}{\psi(u)}.$$

If $\chi_\beta(u) = \log_2^{1/\beta}(e^{(1+\beta)/\beta}/u)$, then direct calculations show that

$$\mathcal{M}_{\chi_\beta}\left(\frac{1}{2}\right) = \left(\frac{\beta \ln 2}{\beta+1} + 1\right)^{1/\beta} < 2. \tag{7}$$

We now find an estimate of $\mathcal{M}_{\psi_t}(1/2)$ using the equality

$$\mathcal{M}_{\psi_t}\left(\frac{1}{2}\right) = \frac{1}{2} \sup_{0<u\leqslant 1} F_t(u), \quad \text{where} \quad F_t(u) = \frac{\max(1, t^{-1}\chi_\beta(u/2))}{\max(1, t^{-1}\chi_\beta(u))}. \tag{8}$$

We consider three cases:

(a) $\chi_\beta(u) \geqslant t$; then

$$F_t(u) = \frac{\chi_\beta(u/2)}{\chi_\beta(u)} \leqslant \mathcal{M}_{\chi_\beta}\left(\frac{1}{2}\right);$$

(b) $\chi_\beta(u) < t \leqslant \chi_\beta(u/2)$; in that case we also have

$$F_t(u) = \frac{\chi_\beta(u/2)}{t} \leqslant \frac{\chi_\beta(u/2)}{\chi_\beta(u)} \leqslant \mathcal{M}_{\chi_\beta}\left(\frac{1}{2}\right);$$

(c) $\chi_\beta(u/2) < t$; then $F_t(u) = 1 \leqslant \mathcal{M}_{\chi_\beta}(1/2)$.

Thus, in view of (7) and (8), $\sup_{t>0} \mathcal{M}_{\psi_t}(1/2) \leqslant 2^{-1}\mathcal{M}_{\chi_\beta}(1/2) < 1$. Hence proceeding as in the proof of Lemma 1.4 in [14] we obtain that for all $0 < s \leqslant 1$,

$$\int_0^s \frac{\psi_t(u)}{u}\,du \leqslant C\psi_t(s),$$

with positive constant C depending only on β. Hence, by the definition of the norm in a Marcinkiewicz space,

$$\begin{aligned}\|f\|_{M(\psi_t)} &\leqslant \sup_{0<s\leqslant 1} \frac{1}{\psi_t(s)} \int_0^s \frac{\psi_t(u)}{u}\,du \sup_{0<u\leqslant 1}\left\{f^*(u)\frac{u}{\psi_t(u)}\right\}\\ &\leqslant C \sup_{0<u\leqslant 1}\left\{f^*(u)\frac{u}{\psi_t(u)}\right\}.\end{aligned}$$

Since the reverse inequality

$$\|f\|_{M(\psi_t)} \geqslant \sup_{0<u\leqslant 1}\left\{f^*(u)\frac{u}{\psi_t(u)}\right\}$$

is obvious, it follows by (6) that

$$\begin{aligned}\mathcal{K}(t, f; L_\infty, \operatorname{Exp} L^\beta) &\asymp \sup_{0<u\leqslant 1}\left\{f^*(u)\frac{u}{\psi_t(u)}\right\}\\ &\asymp \sup_{0<u\leqslant 1}\left\{f^*(u)\min\left(1, t\log_2^{-1/\beta}\left(\frac{2}{u}\right)\right)\right\}\\ &= t \sup_{0<u\leqslant \min(1,2^{1-t^\beta})}\left(f^*(u)\log_2^{-1/\beta}\left(\frac{2}{u}\right)\right)\end{aligned}$$

with constants depending only on β.

Theorem 1. *For arbitrary $\beta > 0$,*

$$\mathcal{K}(p, f; L_\infty, \operatorname{Exp} L^\beta) \asymp p \sup_{q\geqslant p} \frac{\|f\|_{q^\beta}}{q} \tag{9}$$

with constants independent of $f \in \operatorname{Exp} L^\beta$ and $p \geqslant 1$.

Proof. We set $k_\beta = \max(1, 2/\beta)$. Then for each $q \geqslant 1$,

$$\|f\|_q = \left(\int_0^1 (f^*(s))^q\,ds\right)^{1/q} \leqslant \left(2^{k_\beta q-1}\int_0^{2^{1-k_\beta q}} (f^*(s))^q\,ds\right)^{1/q}$$
$$\leqslant 2^{k_\beta}\left(\int_0^{2^{1-k_\beta q}} \log_2^{q/\beta}\left(\frac{2}{u}\right)du\right)^{1/q} \sup_{0<u\leqslant 2^{1-q}}\left(f^*(u)\log_2^{-1/\beta}\left(\frac{2}{u}\right)\right). \tag{10}$$

We now find an estimate of the integral

$$I(\beta, q) = \int_0^{2^{1-k_\beta q}} \log_2^{q/\beta}\left(\frac{2}{u}\right)du.$$

After a change of variable we obtain

$$I(\beta, q) = \frac{2\beta\ln 2}{q}\int_{(k_\beta q)^{q/\beta}}^\infty x^{\beta/q}2^{-x^{\beta/q}}\,dx = 2\ln 2 I_1(\beta, q), \tag{11}$$

where $I_1(\beta, q) = \displaystyle\int_{k_\beta q}^\infty v^{q/\beta}2^{-v}\,dv$. Integrating by parts we find an estimate of $I_1(\beta, q)$:

$$I_1(\beta, q) = \frac{1}{\ln 2}\left((k_\beta q)^{q/\beta}2^{-k_\beta q} + \frac{q}{\beta}\int_{k_\beta q}^\infty v^{q/\beta-1}2^{-v}\,dv\right)$$
$$\leqslant \frac{1}{\ln 2}\left[(k_\beta q)^{q/\beta}2^{-k_\beta q} + \frac{1}{\beta k_\beta}I_1(\beta, q)\right] \leqslant \frac{1}{\ln 2}\left[(k_\beta q)^{q/\beta}2^{-k_\beta q} + \frac{1}{2}I_1(\beta, q)\right]$$

since $\beta k_\beta \geqslant 2$. Hence

$$I_1(\beta, q) \leqslant \frac{2}{2\ln 2 - 1}(k_\beta q)^{q/\beta}2^{-k_\beta q}.$$

It now follows by (11) that

$$I(\beta, q) \leqslant \frac{4\ln 2}{2\ln 2 - 1}(k_\beta q)^{q/\beta}2^{-k_\beta q},$$

and therefore by (10) there exists $C(\beta) > 0$ such that for all $q \geqslant 1$,

$$\|f\|_q \leqslant C(\beta)q^{1/\beta}\sup_{0<u\leqslant 2^{1-q}}\left(f^*(u)\log_2^{-1/\beta}\left(\frac{2}{u}\right)\right).$$

Then, however, by Lemma 1,

$$\|f\|_q \leqslant C(\beta)\mathcal{K}(q^{1/\beta}, f; L_\infty, \operatorname{Exp} L^\beta)$$

with some (not necessarily the same) positive constant $C(\beta)$ independent of f and $q \geqslant 1$.

Now if $q \geqslant p$, then it follows by the last inequality and the concavity of the $\mathcal{K}$-functional that

$$\frac{\|f\|_q}{q^{1/\beta}} \leqslant C(\beta)\frac{\mathcal{K}(q^{1/\beta}, f; L_\infty, \operatorname{Exp} L^\beta)}{q^{1/\beta}} \leqslant C(\beta)\frac{\mathcal{K}(p^{1/\beta}, f; L_\infty, \operatorname{Exp} L^\beta)}{p^{1/\beta}}.$$

Hence

$$p^{1/\beta}\sup_{q\geqslant p}\frac{\|f\|_q}{q^{1/\beta}} \leqslant C(\beta)\mathcal{K}(p^{1/\beta}, f; L_\infty, \operatorname{Exp} L^\beta)$$

or after a change of variable,

$$p\sup_{q\geqslant p}\frac{\|f\|_{q^\beta}}{q} \leqslant C(\beta)\mathcal{K}(p, f; L_\infty, \operatorname{Exp} L^\beta).$$

For the proof of the reverse inequality, in view of Lemma 1 it is sufficient to demonstrate that for all u, $0 < u \leqslant 2^{1-p}$ $(p \geqslant 1)$ we have

$$f^*(u) \leqslant C_1(\beta)\sup_{q\geqslant p}\frac{\|f\|_q}{q^{1/\beta}}\log_2^{1/\beta}\left(\frac{2}{u}\right) \tag{12}$$

with $C_1(\beta)$ dependent only on β.

It follows from the description of $\operatorname{Exp} L^\beta$ as an Orlicz space (see § 1) and the expansion of $N_\beta(t) = e^{t^\beta} - 1$ in a Taylor series that

$$\|f\|_{\operatorname{Exp} L^\beta} \asymp \sup_{q\geqslant 1}\frac{\|f\|_q}{q^{1/\beta}}.$$

Hence

$$f^*(u) \leqslant C_2(\beta)\sup_{q\geqslant 1}\frac{\|f\|_q}{q^{1/\beta}}\log_2^{1/\beta}\left(\frac{2}{u}\right) \quad \text{for } 0 < u \leqslant 1. \tag{13}$$

Note that we can assume that $f = f^*$ and $\{s \in [0,1] : f(s) \neq 0\} \subset [0, 2^{1-p}]$ in the proof of (12). Hence if $1 \leqslant q \leqslant p$, then by Hölder's inequality

$$\|f\|_q \leqslant 2^{\frac{(1-p)(p-q)}{pq}}\|f\|_p \leqslant 2^{-\frac{p}{q}}\|f\|_p. \tag{14}$$

Since for $x \geqslant 1$ we have $x^{1/\beta} \leqslant C_3(\beta)2^x$, it follows by (14) that

$$\frac{\|f\|_q}{q^{1/\beta}} \leqslant 2\max_{1\leqslant q\leqslant p}\frac{2^{-p/q}p^{1/\beta}}{q^{1/\beta}}\frac{\|f\|_p}{p^{1/\beta}} \leqslant 2C_3(\beta)\frac{\|f\|_p}{p^{1/\beta}}$$

for $1 \leqslant q \leqslant p$. Thus, (13) yields (12).

Remark 1. The right-hand side of (9) cannot be replaced by $\|f\|_{p^\beta}$. In fact, consider the functions $h_u(t) = 1$ for $t \in [0,u]$, and $h_u(t) = 0$ for $t \notin [0,u]$ $(0 < u \leqslant 2^{1-p})$. Then $\|h_u\|_p = u^{1/p}$ and by Lemma 1,

$$\mathcal{K}(p^{1/\beta}, h_u; L_\infty, \operatorname{Exp} L^\beta) \asymp p^{1/\beta}\log_2^{-1/\beta}\left(\frac{2}{u}\right).$$

Hence

$$\sup_{0<u\leqslant 2^{1-p}}\frac{\mathcal{K}(p^{1/\beta}, h_u; L_\infty, \operatorname{Exp} L^\beta)}{\|h_u\|_p} \asymp p^{1/\beta}\sup_{0<u\leqslant 2^{1-p}}\log_2^{-1/\beta}\left(\frac{2}{u}\right)u^{-1/p} = \infty.$$

Theorem 1 enables one to find a new description of interpolation spaces relative to the Banach couple $(L_\infty, \operatorname{Exp} L^\beta)$.

§ 3. Description of symmetric spaces 'close' to L_∞

In what follows we denote by e^k $(k = 0, \pm 1, \pm 2, \dots)$ the standard unit basis vectors in the space of two-sided number sequences, that is, $e^k = (e^k_j)$, $e^k_k = 1$, $e^k_j = 0$ $(j \neq k)$.

Definition 1. Let F be an intermediate Banach lattice with respect to the Banach couple $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$, which means that there exist continuous embeddings

$$\Delta(\vec{l}_\infty) \subset F \subset \Sigma(\vec{l}_\infty). \tag{15}$$

Let $\mathcal{L}^{\mathcal{K}}_{\beta,F}$ be the set of measurable functions f on $[0,1]$ such that the sequence

$$a_f = \sum_{k=0}^{\infty} \|f\|_{2^{\beta k}} e^k$$

belongs to F.

It is clear from the definition that $\mathcal{L}^{\mathcal{K}}_{\beta,F}$ is a symmetric space with norm

$$\|f\|_{\mathcal{L}^{\mathcal{K}}_{\beta,F}} = \|a_f\|_F.$$

In particular, $\mathcal{L}^{\mathcal{K}}_{\beta,l_\infty} = L_\infty$ and $\mathcal{L}^{\mathcal{K}}_{\beta,l_\infty(2^{-k})} = \operatorname{Exp} L^\beta$ (the last equality follows from the expansion of $N_\beta(u) = e^{u^\beta} - 1$ in a Taylor series). Hence

$$L_\infty \subset \mathcal{L}^{\mathcal{K}}_{\beta,F} \subset \operatorname{Exp} L^\beta.$$

Moreover, it will follow from Theorem 2, our main result in this section, that $\mathcal{L}^{\mathcal{K}}_{\beta,F}$ is an interpolation space relative to the couple $(L_\infty, \operatorname{Exp} L^\beta)$ if F is an interpolation lattice with respect to the couple $\vec{l}_\infty$.

We start with an auxiliary result. Consider two projections in spaces of two-sided number sequences:

$$P_+ a = \sum_{j=0}^{\infty} a_j e^j \quad \text{and} \quad P_- a = \sum_{j=-\infty}^{-1} a_j e^j, \quad \text{where} \quad a = (a_j)_{j=-\infty}^{\infty}.$$

Lemma 2. *Let F be a Banach lattice of two-sided number sequences such that* (15) *holds. Then there exists a positive constant C such that for each sequence $a = (a_j)_{j=-\infty}^{\infty}$ satisfying the condition $|a_j| \leqslant 2^j |a_0|$ $(j = -1, -2, \dots)$,*

$$\|a\|_F \leqslant C \|P_+ a\|_F.$$

Proof. Since $a = P_- a + P_+ a$, it follows that

$$\|a\|_F \leqslant \|P_- a\|_F + \|P_+ a\|_F. \tag{16}$$

By the assumptions of the lemma,

$$\|P_- a\|_F \leqslant |a_0| \left\| \sum_{j=-\infty}^{-1} 2^j e^j \right\|_F \leqslant C_1 |a_0| \left\| \sum_{j=-\infty}^{-1} 2^j e^j \right\|_{\Delta(\vec{l}_\infty)} = C_1 |a_0|;$$

moreover,

$$\|P_+ a\|_F \geqslant |a_0| \, \|e^0\|_F \geqslant |a_0| C_2^{-1} \|e^0\|_{\Sigma(\vec{l}_\infty)} = C_2^{-1} |a_0|,$$

where C_1 and C_2 are the constants of the embeddings in (15). Hence it follows by (16) that

$$\|a\|_F \leqslant (1 + C_1 C_2) \|P_+ a\|_F,$$

which proves the lemma.

Theorem 2. *Assume that $\beta > 0$ and let F be an interpolation Banach lattice with respect to the couple $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$. Then $\mathcal{L}^{\mathcal{K}}_{\beta,F} = (L_\infty, \operatorname{Exp} L^\beta)^{\mathcal{K}}_F$ (with equivalence of norms).*

Proof. It is sufficient to demonstrate that, with some constants depending only on F, for an arbitrary function f on $[0,1]$ we have the equivalence

$$\|a_f\|_F \asymp \|b_f\|_F, \tag{17}$$

where

$$a_f = \sum_{k=0}^{\infty} \|f\|_{2^{\beta k}} e^k \quad \text{and} \quad b_f = \sum_{k=-\infty}^{\infty} \mathcal{K}(2^k, f; L_\infty, \operatorname{Exp} L^\beta) e^k.$$

First of all, $L_\infty \subset \operatorname{Exp} L^\beta$ with constant 1, therefore for $k = -1, -2, \dots$ we have $(b_f)_k = 2^k (b_f)_0 = 2^k \|f\|_{\operatorname{Exp} L^\beta}$. Hence by Lemma 2,

$$\|b_f\|_F \asymp \|P_+ b_f\|_F. \tag{18}$$

Next, it is well known ([13], Remark 3.3.8) that if F is an interpolation space with respect to the couple $\vec{l}_\infty$, then

$$\|a\|_F \asymp \|\overline{a}\|_F, \quad \text{where} \quad \overline{a} = (\overline{a}_k)_{k=-\infty}^{\infty}, \qquad \overline{a}_k = \sup_{j=0,\pm1,\dots} \{\min[1, 2^{k-j}] \, |a_j|\}.$$

Hence

$$\|a_f\|_F \asymp \|\overline{a}_f\|_F$$

and by the obvious inequalities $\overline{a}_f \geqslant P_+ \overline{a}_f \geqslant a_f$,

$$\|a_f\|_F \asymp \|P_+ \overline{a}_f\|_F. \tag{19}$$

Since the sequence of a_f increases, it follows that

$$P_+ \overline{a}_f = \sum_{k=0}^{\infty} 2^k \sup_{j \geqslant k} \frac{\|f\|_{2^{\beta j}}}{2^j} e^k$$

and therefore by Theorem 1,

$$\|P_+\overline{a}_f\|_F \asymp \|P_+b_f\|_F$$

with universal constants. Hence relation (17) is a consequence of (18) and (19).

Remark 2. Keeping the notation of Theorem 2 we shall now show another way of proving the embedding $X \subset \mathcal{L}^{\mathcal{K}}_{\beta,F}$ by interpolation arguments. In fact, let T be an arbitrary linear operator bounded in both L_∞ and $\operatorname{Exp} L^\beta$. Then the sublinear operator

$$Qf = \sum_{k=0}^{\infty} \|Tf\|_{2^{\beta k}} e^k$$

is bounded from L_∞ into l_∞ and from $\operatorname{Exp} L^\beta$ into $l_\infty(2^{-k})$. Since the functors of the real method interpolate sublinear operators and $F = (l_\infty, l_\infty(2^{-k}))^{\mathcal{K}}_F$ ([13], Lemma 3.3.7), Q is bounded from $X = (L_\infty, \operatorname{Exp} L^\beta)^{\mathcal{K}}_F$ into F, that is, in our notation,

$$\|Tf\|_{\mathcal{L}^{\mathcal{K}}_{\beta,F}} = \|a_f\|_F \leqslant C\|f\|_X.$$

Thus, the triple of spaces $(L_\infty, \operatorname{Exp} L^\beta, X)$ is an interpolation triple with respect to $(L_\infty, \operatorname{Exp} L^\beta, \mathcal{L}^{\mathcal{K}}_{\beta,F})$. In particular, if we take for T the identity operator, then we obtain the embedding $X \subset \mathcal{L}^{\mathcal{K}}_{\beta,F}$.

Interpolation in the couple $(L_\infty, \operatorname{Exp} L^\beta)$ is described by the linear $\mathcal{K}$-method [19] (see also [10], Proposition 1). This means that each interpolation space X with respect to this couple can be represented in the form $X = (L_\infty, \operatorname{Exp} L^\beta)^{\mathcal{K}}_F$, where F is a Banach lattice of two-sided number sequences. Here, as mentioned in § 1, we can assume that F is an interpolation space with respect to $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$. Hence by Theorem 2 we obtain the following result.

Corollary 1. *Assume that* $\beta > 0$. *Then for each interpolation space* X *with respect to the couple* $(L_\infty, \operatorname{Exp} L^\beta)$ *there exists a Banach lattice* F *such that* $X = \mathcal{L}^{\mathcal{K}}_{\beta,F}$ (*with equivalence of norms*).

Remark 3. We point out that many recent authors investigating function spaces 'close' to L_∞ used the asymptotic behaviour of the L_p-norms of a function as $p \to \infty$. For instance, in [20] the author studies the convergence of Walsh series in the spaces $G_{p,\alpha}$ $(p > 1,\ \alpha \geqslant 1)$, one of the equivalent norms in which has the following form:

$$\|f\|_{p,\alpha} = \left(\int_1^\infty \left(\frac{\|f\|_x}{x^\alpha}\right)^p dx\right)^{1/p}.$$

In particular, [20] investigates the position of these spaces on the scale of Orlicz exponential spaces. Namely, it is shown that for $p > 1$, $q > 0$, and $\alpha \geqslant 1$ such that $p^{-1} + q^{-1} = \alpha$ and each $r > q$,

$$L_{N_r} \subset G_{p,\alpha} \subset L_{N_q},$$

where as before, $N_r(t) = e^{t^r} - 1$ $(r \geqslant 1)$ and $N_r(t) = \exp(t + t_r^{1/r})^r - \exp t_r$, $t_r = 1/r - 1$ $(0 < r < 1)$.

Using Theorem 2 and the results of [10] it is easy to show that the $G_{p,\alpha}$ are Lorentz spaces. In fact, fix $p>1$ and $\alpha\geqslant 1$ and consider arbitrary $\beta\in(0,p/(\alpha p-1))$. An immediate verification shows that $G_{p,\alpha}=\mathcal{L}^{\mathcal{K}}_{\beta,l_p(2^{-\theta k})}$, where $\theta=\beta(\alpha p-1)/p$, $\theta\in(0,1)$. Since $l_p(2^{-\theta k})$ is an interpolation space with respect to the couple $\vec{l}_\infty$ ([12], Theorem 5.6.1), it follows by Theorem 2 that

$$G_{p,\alpha}=(L_\infty,\operatorname{Exp}L^\beta)^{\mathcal{K}}_{l_p(2^{-\theta k})}.$$

However, the last space (see [10], Example 2) coincides (with equivalence of norms) with the Lorentz space $\Lambda_p(\varphi)$, where $\varphi(s)=\log_2^{1-\alpha p}(2/s)$.

One can deduce from Theorem 2 a whole 'series' of results similar to Yano's theorem. Using the last observation we present one of them as an example.

Theorem 3. *Let T be a linear operator bounded in L_p for each $p\geqslant 1$ and let*

$$\int_1^\infty\left(\frac{\|T\|_{L_x\to L_x}}{x^\alpha}\right)^q dx<\infty$$

for some $\alpha\geqslant 1$ and $q>1$. Then T is a bounded operator from L_∞ into the Lorentz space $\Lambda_q(\varphi)$, where $\varphi(s)=\log_2^{1-\alpha q}(2/s)$.

Proof. By Remark 3 and the assumptions of the theorem,

$$\begin{aligned}\|Tf\|_{\Lambda_q(\varphi)}&\asymp\left\{\int_1^\infty\left(\frac{\|Tf\|_x}{x^\alpha}\right)^q dx\right\}^{1/q}\\&\leqslant\left\{\int_1^\infty\left(\frac{\|T\|_{L_x\to L_x}}{x^\alpha}\right)^q dx\right\}^{1/q}\sup_{x\geqslant 1}\|f\|_x=C\|f\|_\infty.\end{aligned}$$

In the last section of this paper we establish dual results on symmetric spaces 'close' to L_1. The key role here is played by the $\mathcal{J}$-method of real interpolation.

§ 4. Description of spaces 'close' to L_1

We recall the definition of the intersection and the sum of a countable family of Banach spaces. Let X_k $(k=1,2,\dots)$ be Banach spaces continuously linearly embedded in a topological linear space $\mathcal{T}$. Then their intersection is the Banach space $\Delta_{k=1}^\infty X_k$ of all $x\in\bigcap_{k=1}^\infty X_k$ such that $\|x\|=\sup_{k=1,2,\dots}\|x\|_{X_k}<\infty$. In the construction of the sum $\sum_{k=1}^\infty X_k$ we assume additionally that there exists a Banach space X_0 embedded in $\mathcal{T}$ such that $X_k\subset X_0$ with constant 1 for all $k=1,2,\dots$. Then we denote by $\sum_{k=1}^\infty X_k$ the set of $x\in X_0$ representable in the following form:

$$x=\sum_{k=1}^\infty x_k\quad(x_k\in X_k),\quad\text{where}\quad\sum_{k=1}^\infty\|x_k\|_{X_k}<\infty.\tag{20}$$

Now, the introduction in the linear space $\sum_{k=1}^\infty X_k$ of the norm

$$\|x\|=\inf\sum_{k=1}^\infty\|x_k\|_{X_k},$$

where the infimum is taken over all representations (20), makes it a Banach space.

Lemma 3. *If the intersection* $\Delta_{k=1}^{\infty} X_k$ *is dense in* X_k *for each* $k = 1, 2, \ldots$, *then its dual space* $(\Delta_{k=1}^{\infty} X_k)^*$ *coincides (isometrically) with* $\sum_{k=1}^{\infty} X_k^*$.

Since this is in effect a known result (for instance, Proposition 2.4.6 in the monograph [13] is very close), we leave out its proof.

Lemma 4. *A Banach lattice* G *of two-sided number sequences is an interpolation lattice with respect to the couple* $\vec{l}_1 = (l_1, l_1(2^{-k}))$ *if and only if there exists* $c > 0$ *such that*

$$a \in \Sigma(\vec{l}_1), \quad a = \sum_{k=-\infty}^{\infty} a^k \quad (\textit{convergence in } \Sigma(\vec{l}_1)), \quad a^k = (a_i^k) \in \Delta(\vec{l}_1)$$

yields

$$\left\| \left(\sum_{i=-\infty}^{\infty} \max(1, 2^{k-i}) |a_i^k| \right)_k \right\|_G \geqslant c \|a\|_G. \tag{21}$$

Proof. It is well known ([13], Corollary 3.4.7) that G is an interpolation space with respect to $\vec{l}_1$ if and only if

$$(l_1, l_1(2^{-k}))_G^{\mathcal{J}} = G. \tag{22}$$

If (22) holds, then for the proof of (21) it is sufficient to use the definition of the norm in the space $(l_1, l_1(2^{-k}))_G^{\mathcal{J}}$ and the asymptotic equality

$$\mathcal{J}(t, b; l_1, l_1(2^{-k})) \asymp \sum_{i=-\infty}^{\infty} \max(1, t2^{-i}) |b_i| \tag{23}$$

with constants independent of $b = (b_i)_{i=-\infty}^{\infty} \in \Delta(\vec{l}_1)$ and $t > 0$.

We now claim that (21) yields (22). First of all, we observe that

$$(l_1, l_1(2^{-k}))_G^{\mathcal{J}} \supset G$$

for each lattice G intermediate relative to the couple $\vec{l}_1$. For if $a = (a_j)_{j=-\infty}^{\infty} \in G$, then we have the representation $a = \sum_{k=-\infty}^{\infty} a_k e^k$, where the e^k are the standard basis vectors. We have $a_k e^k \in \Delta(\vec{l}_1)$ and $\mathcal{J}(2^k, a_k e^k; l_1, l_1(2^{-k})) = |a_k|$. Hence $\|a\|_{(l_1, l_1(2^{-k}))_G^{\mathcal{J}}} \leqslant \|a\|_G$. We point out in conclusion that in view of relation (23) the reverse embedding in (22) is an immediate consequence of condition (21).

Theorem 4. *The following asymptotic equality holds for arbitrary positive* $\beta > 0$ *with constants independent of* $g \in L(\log L)^{1/\beta}$ *and* $k = 1, 2, \ldots$:

$$\mathcal{J}(2^{-k}, g; L_1, L(\log L)^{1/\beta}) \asymp \|g\|_{U_k}, \tag{24}$$

where $U_k = \sum_{j \geqslant k} 2^{j-k} L_{r_j}$, $r_j = 1 + 2^{-\beta j}$.

Proof. By Theorem 1,

$$\mathcal{K}(2^k, f; L_\infty, \operatorname{Exp} L^\beta) \asymp \|f\|_{V_k},$$

where $V_k = \Delta_{j\geqslant k} 2^{k-j} L_{2^{\beta j}}$. The constants in this equivalence are independent of $k = 1, 2, \ldots$ and $f \in \operatorname{Exp} L^\beta$. Hence passing to the dual spaces, in view of the duality between $\mathcal{K}$- and $\mathcal{J}$-functionals (see § 1) we obtain

$$\mathcal{J}(2^{-k}, g; L_\infty^*, (\operatorname{Exp} L^\beta)^*) \asymp \|g\|_{V_k^*}$$

with constants independent of $k = 1, 2, \ldots$ and $g \in (\operatorname{Exp} L^\beta)^*$. Since $L(\log L)^{1/\beta}$ and L_1 are subspaces of $(\operatorname{Exp} L^\beta)^*$ and L_∞^*, respectively (see § 1), it now follows by Lemma 3 that

$$\mathcal{J}(2^{-k}, g; L_1, L(\log L)^{1/\beta}) \asymp \|g\|_{U_k'}$$

for all $g \in L(\log L)^{1/\beta}$. Here

$$U_k' = \sum_{j\geqslant k} 2^{j-k} L_{s_j},$$

where $k = 1, 2, \ldots$ and $s_j = 1 + 1/(2^{\beta j} - 1)$.

For all $\beta > 0$ and $j = 1, 2, \ldots$ we have

$$u_\beta s_{j+1} < r_j < s_j, \quad \text{where} \quad u_\beta = \frac{2^{2\beta} - 1}{2^\beta}.$$

It now follows by the definition of U_k that $U_k' = U_k$ and $\|g\|_{U_k} \leqslant \|g\|_{U_k'} \leqslant u_\beta^{-1} \|g\|_{U_k}$.

Definition 2. Let G be an intermediate Banach lattice with respect to the Banach couple $\vec{l_1} = (l_1, l_1(2^{-k}))$, which means that there exist continuous embeddings

$$\Delta(\vec{l_1}) \subset G \subset \Sigma(\vec{l_1}). \tag{25}$$

Let $\mathcal{L}_{\beta,G}^{\mathcal{J}}$ be the set of measurable functions g on $[0, 1]$ having a representation

$$g = \sum_{k=1}^\infty g_k \quad (\text{convergence in } L_1), \qquad g_k \in L_{r_k}, \quad r_k = 1 + 2^{-\beta k}, \tag{26}$$

where

$$\sum_{k=1}^\infty \|g_k\|_{r_k} e^{-k} \in G. \tag{27}$$

It is easy to verify that $\mathcal{L}_{\beta,G}^{\mathcal{J}}$ is a symmetric space with norm

$$\|g\|_{\mathcal{L}_{\beta,G}^{\mathcal{J}}} = \inf \left\| \sum_{k=1}^\infty \|g_k\|_{r_k} e^{-k} \right\|_G,$$

where the infimum is taken over all representations (26). Moreover, $\mathcal{L}_{\beta,l_1}^{\mathcal{J}} = L_1$ and $\mathcal{L}_{\beta,l_1(2^{-k})}^{\mathcal{J}} = L(\log L)^{1/\beta}$. Hence for each Banach lattice G with property (25) we obtain

$$L(\log L)^{1/\beta} \subset \mathcal{L}_{\beta,G}^{\mathcal{J}} \subset L_1.$$

We claim that, in addition, $\mathcal{L}_{\beta,G}^{\mathcal{J}}$ is an interpolation space with respect to the couple $(L_1, L(\log L)^{1/\beta})$ if so is G with respect to $\vec{l_1} = (l_1, l_1(2^{-k}))$.

Theorem 5. *Assume that $\beta > 0$ and let G be an interpolation Banach lattice with respect to the couple $\vec{l_1} = (l_1, l_1(2^{-k}))$. Then $\mathcal{L}^{\mathcal{J}}_{\beta,G} = (L_1, L(\log L)^{1/\beta})^{\mathcal{J}}_G$ (with equivalence of norms).*

Proof. Now let $Y = (L_1, L(\log L)^{1/\beta})^{\mathcal{J}}_G$ and let

$$\mathcal{J}(t,h) = \mathcal{J}(t,h; L_1, L(\log L)^{1/\beta})$$

for arbitrary $h \in L(\log L)^{1/\beta}$ and $t > 0$.

First of all, by the definition of space of the $\mathcal{J}$-method

$$\|f\|_Y = \inf \|(\mathcal{J}(2^k, f_k))_k\|_G,$$

where the infimum is taken over all representations

$$f = \sum_{k=-\infty}^{\infty} f_k \quad \text{(convergence in } L_1), \qquad f_k \in L(\log L)^{1/\beta}. \tag{28}$$

If $g \in \mathcal{L}^{\mathcal{J}}_{\beta,G}$, then we have the representation (26) with property (27). In this case we immediately obtain by Theorem 4 that

$$g \in Y \quad \text{and} \quad \|g\|_Y \leqslant C_1 \bigg\| \sum_{k=1}^{\infty} \|g_k\|_{r_k} e^{-k} \bigg\|_G.$$

Hence $\mathcal{L}^{\mathcal{J}}_{\beta,G} \subset Y$ and $\|g\|_Y \leqslant C_1 \|g\|_{\mathcal{L}^{\mathcal{J}}_{\beta,G}}$.

For the proof of the reverse embedding we shall show first that in the calculation of the norm in Y one can content oneself (up to an equivalence) with representations (28) in which $f_k = 0$ for $k = 0, 1, 2, \dots$.

Since $L(\log L)^{1/\beta} \subset L_1$ with constant 1, it follows that

$$\mathcal{J}(2^k, g) = 2^k \|g\|_{L(\log L)^{1/\beta}}$$

for $k = 0, 1, 2, \dots$. Hence if C_2 is the constant of the embedding $G \subset \Sigma(\vec{l_1})$, then

$$\begin{aligned} \bigg\| \sum_{k=0}^{\infty} \mathcal{J}(2^k, f_k) e^k \bigg\|_G &= \bigg\| \sum_{k=0}^{\infty} 2^k \|f_k\|_{L(\log L)^{1/\beta}} e^k \bigg\|_G \\ &\geqslant C_2^{-1} \bigg\| \sum_{k=0}^{\infty} 2^k \|f_k\|_{L(\log L)^{1/\beta}} e^k \bigg\|_{l_1(2^{-k})} = C_2^{-1} \sum_{k=0}^{\infty} \|f_k\|_{L(\log L)^{1/\beta}}. \end{aligned} \tag{29}$$

In place of the fixed representation (28) we now consider another one:

$$f = \sum_{k=-\infty}^{\infty} f'_k, \qquad f'_k = f_k \quad \text{for } k = -2, -3, \dots,$$

$$f'_{-1} = \sum_{i=-1}^{\infty} f_i, \qquad f'_k = 0 \quad \text{for } k = 0, 1, \dots.$$

Then in view of (29) and since G is a Banach lattice, it follows that

$$\begin{aligned}\|(\mathcal{J}(2^k, f_k'))_k\|_G &\leqslant \|(\mathcal{J}(2^k, f_k))_k\|_G + \sum_{k=0}^{\infty} \|f_k\|_{L(\log L)^{1/\beta}} \|e^{-1}\|_G \\ &\leqslant (1 + 2C_2C_3)\|(\mathcal{J}(2^k, f_k))_k\|_G,\end{aligned}$$

where C_3 is the constant of the embedding $\Delta(\vec{l}_1) \subset G$.

Now let $f \in Y$. Then, as shown before, there exists a representation (28) such that $f_k = 0$ for $k = 0, 1, \ldots$ and

$$\|f\|_Y \geqslant C_4^{-1} \left\| \sum_{k=1}^{\infty} \mathcal{J}(2^{-k}, f_{-k}) e^{-k} \right\|_G. \tag{30}$$

By Theorem 4, for each $k = 1, 2, \ldots$ we can find a representation

$$f_{-k} = \sum_{j=k}^{\infty} g_{k,j}, \quad \text{where} \quad g_{k,j} \in L_{r_j}, \tag{31}$$

and in addition,

$$\mathcal{J}(2^{-k}, f_{-k}) \geqslant C_5^{-1} \sum_{j=k}^{\infty} 2^{j-k} \|g_{k,j}\|_{r_j}. \tag{32}$$

Again, since $G \subset \Sigma(\vec{l}_1)$, it follows by (30) and (32) that

$$\begin{aligned}\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \|g_{k,j}\|_{r_j} &\leqslant \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} 2^{j-k} \|g_{k,j}\|_{r_j} \leqslant C_5 \sum_{k=1}^{\infty} \mathcal{J}(2^{-k}, f_{-k}) \\ &\leqslant C_5C_2 \left\| \sum_{k=1}^{\infty} \mathcal{J}(2^{-k}, f_{-k}) e^{-k} \right\|_G \leqslant C_5C_2C_4\|f\|_Y < \infty.\end{aligned} \tag{33}$$

This shows, in particular, that the double series

$$\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} g_{k,j}$$

is absolutely convergent in L_1 and has the same sum as the corresponding repeated series, that is, f. Hence

$$f = \sum_{j=1}^{\infty} g_j \quad (\text{convergence in } L_1), \quad \text{where} \quad g_j = \sum_{k=1}^{j} g_{k,j}.$$

Here, in view of (31), $g_j \in L_{r_j}$ and

$$\|g_j\|_{r_j} \leqslant \sum_{k=1}^{j} \|g_{k,j}\|_{r_j}. \tag{34}$$

Consider now the sequence

$$b_g = \sum_{j=1}^{\infty} \sum_{k=1}^{j} \|g_{k,j}\|_{r_j} e^{-j}. \tag{35}$$

In view of (33), we conclude that

$$b_g = \sum_{k=1}^{\infty} b^k \quad (\text{convergence in } \Sigma(\vec{l}_1)), \quad \text{where} \quad b^k = \sum_{j=k}^{\infty} \|g_{k,j}\|_{r_j} e^{-j}.$$

It follows by (32) that $b^k \in \Delta(\vec{l}_1)$ for each $k = 1, 2, \dots$. Hence by Lemma 4, since G is an interpolation space with respect to the couple $\vec{l}_1$,

$$\begin{aligned}\|b_g\|_G &\leqslant C_6 \bigg\| \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \max(1, 2^{j-k}) \|g_{k,j}\|_{r_j} e^{-k} \bigg\|_G \\ &= C_6 \bigg\| \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} 2^{j-k} \|g_{k,j}\|_{r_j} e^{-k} \bigg\|_G.\end{aligned}$$

Hence it follows by (34), (35), (30), and (32) that

$$\bigg\| \sum_{j=1}^{\infty} \|g_j\|_{r_j} e^{-j} \bigg\|_G \leqslant \|b_g\|_G \leqslant C_4 C_5 C_6 \|f\|_Y,$$

so that $f \in \mathcal{L}^{\mathcal{J}}_{\beta,G}$ and $\|f\|_{\mathcal{L}^{\mathcal{J}}_{\beta,G}} \leqslant C_4 C_5 C_6 \|f\|_Y$.

Interpolation in the couple $(L_1, L(\log L)^{1/\beta})$ is described by the real $\mathcal{K}$-method ([21], Corollary 5.10; see also [22]). In view of results on the connection between $\mathcal{K}$- and $\mathcal{J}$-functors [13], it is also described by the $\mathcal{J}$-method. In other words, if Y is an interpolation space with respect to $(L_1, L(\log L)^{1/\beta})$, then $Y = (L_1, L(\log L)^{1/\beta})^{\mathcal{J}}_G$ for some interpolation Banach lattice G with respect to the couple $\vec{l}_1 = (l_1, l_1(2^{-k}))$ (see § 1). Hence Theorem 5 has the following consequence.

Corollary 2. *Assume that $\beta > 0$. Then for each interpolation space Y with respect to the couple $(L_\infty, \operatorname{Exp} L^\beta)$ there exists a Banach lattice G such that $Y = \mathcal{L}^{\mathcal{J}}_{\beta,G}$ (with equivalence of norms).*

We now present another concrete extrapolation result.

Theorem 6. *Let ε and A be positive quantities such that for $1 < p < 1 + \varepsilon$ a linear operator T is bounded in the space L_p and $\|T\|_{L_p \to L_p} \leqslant A \log_2(1/(p-1))$. Then T can be defined in the Lorentz space $\Lambda_1(\varphi)$ corresponding to the function $\varphi(t) = t \log_2 \log_2(16/t)$ such that it becomes a bounded operator from $\Lambda_1(\varphi)$ into L_1.*

Proof. Assume that $\beta > 0$. Then $\Lambda_1(\varphi)$ is an interpolation space with respect to the couple $(L_1, L(\log L)^{1/\beta})$. Moreover, one can prove the following equality, to be treated as an isomorphism, with constant depending on $\beta > 0$:

$$\Lambda_1(\varphi) = (L_1, L(\log L)^{1/\beta})^{\mathcal{J}}_{l_1(w_k)}, \tag{36}$$

where the 'weight' w_k is $-k$ if $k < 0$ and $w_k = 1$ if $k \geqslant 0$.

In fact, let us show that the corresponding dual spaces coincide (up to an isomorphism). By the duality theorem for the real interpolation method [13],

$$((L_1, L(\log L)^{1/\beta})^{\mathcal{J}}_{l_1(w_k)})^* = (L_\infty, \operatorname{Exp} L^\beta)^{\mathcal{K}}_{l_\infty(u_k)},$$

where $u_k = 1/(k+1)$ if $k \geqslant 0$ and $u_k = 1$ if $k < 0$. Since $(\Lambda_1(\varphi))^* = M(\varphi)$ ([14], Theorem 5.2), (36) is a consequence of the equality

$$M(\varphi) = (L_\infty, \operatorname{Exp} L^\beta)^{\mathcal{K}}_{l_\infty(u_k)},$$

which is proved in [10], Example 1 for $\beta = 2$. The case of arbitrary positive $\beta > 0$ is perfectly similar.

We now select β such that $2^{-\beta} < \varepsilon$. If $g \in \Lambda_1(\varphi)$, then in view of (36) and Theorem 5, there exists a representation

$$g = \sum_{k=1}^{\infty} g_k \quad (\text{convergence in } L_1), \quad g_k \in L_{r_k}, \qquad r_k = 1 + 2^{-\beta k},$$

such that

$$\sum_{k=1}^{\infty} \|g_k\|_{r_k} k < \infty.$$

By the assumptions of the theorem

$$\sum_{k=1}^{\infty} \|Tg_k\|_{r_k} \leqslant \sum_{k=1}^{\infty} \|T\|_{L_{r_k} \to L_{r_k}} \|g_k\|_{r_k} \leqslant A\beta \sum_{k=1}^{\infty} \|g_k\|_{r_k} k.$$

Hence

$$Tg = \sum_{k=1}^{\infty} Tg_k \quad (\text{convergence in } L_1)$$

and in view of the equality $L_1 = \mathcal{L}^{\mathcal{J}}_{\beta, l_1}$ and Definition 2, $\|Tg\|_1 \leqslant C\|g\|_{\Lambda_1(\varphi)}$ with some positive constant C depending only on β.

Remark 4. As is known ([5], Chapter 2), from the values $F_\theta(X_0, X_1)$ at a couple (X_0, X_1) of the functors in a sufficiently broad class $\{F_\theta\}_{0<\theta<1}$ one can recover the $\mathcal{K}$- and $\mathcal{J}$-functionals for this couple. Theorems 1 and 4 solve another problem: they enable one to find (up to an equivalence) similar functionals for couples of limiting spaces of the L_p-scale. As a very special case of relations (9) and (24) one can regard equalities (2.26) and (2.27) presented, for instance, in [5] and giving a description of the 'extreme' extrapolation spaces of this scale, $\operatorname{Exp} L^\beta$ and $L(\log L)^{1/\beta}$, as the intersection and the sum of the L_p-spaces, respectively.

Remark 5. Our results here can be generalized in a natural way to space scales obtained by the real method from an arbitrary Banach couple (X_0, X_1) such that $X_1 \subset X_0$. In particular, one can consider the 'extremal' scales $(X_0, X_1)_{\theta,1}$ and $(X_0, X_1)_{\theta,\infty}$ $(0 < \theta < 1)$.

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Received 8/OCT/02
Translated by IPS(DoM)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX