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BRIEF COMMUNICATIONS

On Interpolation in L_p Spaces

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1. INTRODUCTION

Suppose that X is a symmetric space of measurable functions on $[0,1]$. If, for some $1 \le p < q \le \infty$, Boyd indices of the space X satisfy the inequalities

$$\frac{1}{q} < \alpha(X) \le \beta(X) < \frac{1}{p},$$

then X is an interpolation space between the spaces L_p and L_q. This means that any bounded linear operator T in L_p and L_q is also bounded in the space X. This theorem was proved by Boyd [1] in 1967 under the additional assumption that X possesses the Fatou property (see also [2] and [3, Theorem 5.16]). For the case in which the space X is separable or possesses the Fatou property, see [4, Theorem 2.b.11]. Combining estimates for the Calderón maximum operator and Semenov's Lemma [5, Chap. 2, Lemma 4.7], we can prove Boyd's theorem for arbitrary symmetric spaces. This result is also contained implicitly in Theorem 6.12 from [5, Chap. 2].

In the case $q = \infty$, the assumption in Boyd's theorem can be weakened. Namely, the fact that X is an interpolation space between L_p and L_∞ is a consequence solely of the one-sided estimate

$$\beta(X) < \frac{1}{p}, \qquad 1 \le p < \infty$$

(see [6, Theorem 4.6], where this was proved even for Lipschitzian operators).

In connection with this, Semenov posed the question if a similar result is not valid for the other end of the scale of symmetric spaces, i.e., for interpolation between the spaces L_1 and L_p. Namely,

Suppose that X is a symmetric space on $[0,1]$ with the Fatou property or an absolutely continuous norm such that the lower Boyd index satisfies $\alpha(X) > 1/p$, $1 < p < \infty$. Is then X an interpolation space between L_1 and L_p?

The answer to this question turns out to be positive. We have succeeded in proving an even stronger result: *If a symmetric space X is an interpolation space between L_1 and L_∞ and $\alpha(X) > 1/p$, then X is also an interpolation space between L_1 and L_p.*

In the last section of the paper, we present a new interpolation theorem for operators of strong type $(1,1)$ and of weak type (p,p), $1 < p < \infty$.

Let us first recall the definitions and notation needed in the paper.

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2. DEFINITIONS AND NOTATION

The Banach space X of measurable functions on $[0,1]$, is said to be *symmetric* if the conditions $x^*(t) \le y^*(t)$ for $t \in [0,1]$ and $y \in X$ yield $x \in X$ and $\|x\|_X \le \|y\|_X$. Here and elsewhere, x^* is a nonincreasing rearrangement of the function $|x|$. For any symmetric space X on $[0,1]$, the continuous embeddings $L_\infty[0,1] \subset X \subset L_1[0,1]$ hold.

For each $s > 0$, the *dilation operator*

$$\sigma_s x(t) = x(t/s)\chi_{[0,1]}(t/s), \qquad t \in [0,1]$$

is defined and is bounded in any symmetric space X (χ_E is the characteristic function of the set $E \subset [0,1]$). Moreover, $\|\sigma_s\|_{X\to X} \le \max(1,s)$. The important characteristics of a symmetric space X are the *lower and upper Boyd indices*:

$$\alpha_X = \lim_{s\to 0} \frac{\ln \|\sigma_s\|_{X\to X}}{\ln s}, \qquad \beta_X = \lim_{s\to\infty} \frac{\ln \|\sigma_s\|_{X\to X}}{\ln s}.$$

The symmetric space X with norm $\|\cdot\|_X$ possesses

(a) the *Fatou property* if for any sequence $\{x_n\} \subset X$ such that $x_n \ge 0$, $x_n \nearrow x$ and $\sup_n \|x_n\|_X < \infty$, we have $x \in X$ and $\|x_n\|_X \nearrow \|x\|_X$;

(b) an *absolutely continuous* norm if for an arbitrary $x \in X$ and any sequence $\{x_n\}$ of measurable functions on $[0,1]$ such that $|x| \ge x_n \ge 0$ and $x_n \downarrow 0$, we obtain $\|x_n\|_X \to 0$.

Note that a symmetric space X is separable if and only if X has an absolutely continuous norm. It follows from the Calderón–Mityagin Theorem [5, Chap. 2, Theorem 4.3] that a symmetric space X possessing the Fatou property or being separable is an *interpolation* space between the spaces L_1 and L_∞. This means that any linear operator T bounded in L_1 and L_∞ is bounded in X and

$$\|T\|_{X\to X} \le C\max(\|T\|_{L_1\to L_1},\ \|T\|_{L_\infty\to L_\infty})$$

for some $C \ge 1$.

The most important examples of symmetric spaces—$L_{p,q}$, Lorentz, Marcinkiewicz, and Orlicz spaces–possess the Fatou property. The spaces M^0_φ and E_Φ of closure of L_∞ in the Marcinkiewicz space M_φ and the Orlicz space L_Φ, respectively, are typical separable symmetric spaces. For the definitions of these spaces and also for other general properties of symmetric spaces, see the monographs [3–5].

3. INTERPOLATION SPACES BETWEEN L_1 AND L_P, $1 < P < \infty$

Theorem 1. *Suppose that $1 < p < \infty$. If a* symmetric space *X is an interpolation space between L_1 and L_∞ and $\alpha(X) > 1/p$, then X is an interpolation space also between L_1 and L_p.*

Scheme of the proof. If $T\colon L_1 \to L_1$ is a linear bounded operator such that $T = T_{|L_p}\colon L_p \to L_p$ is also bounded, then

$$K(t^{1-1/p}, Tx\,;\, L_1, L_p) \le C_1 K(t^{1-1/p}, x\,;\, L_1, L_p) \le C_1 K(t^{1-1/p}, x\,;\, L_1, L_{p,1})$$

for any $x \in L_1$ and all $0 < t \le 1$, where $L_{p,1}$ is the Lorentz space with norm

$$\|x\|_{p,1} = \int_0^1 t^{1/p-1} x^*(t)\, dt.$$

Hence, by Holmstedt's formula [7, Theorems 4.1 and 4.2], we obtain the estimate

$$\int_0^t (Tx)^*(s)\, ds \le C_2\biggl(\int_0^t x^*(s)\, ds + t^{1-1/p}\int_t^1 s^{1/p-1} x^*(s)\, ds\biggr) \tag{1}$$

for any $x \in L_1$ and all $0 < t \le 1$. By Fubini's theorem, for the linear operator

$$Mx(t) = t^{-1/p} \int_t^1 s^{1/p-1} x(s)\, ds, \qquad 0 < t \le 1, \tag{2}$$

we find that

$$\int_0^t (Mx^*)^*(s)\, ds = \frac{p}{p-1}\biggl(\int_0^t x^*(u)\, du + t^{1-1/p}\int_t^1 u^{1/p-1}x^*(u)\, du\biggr)$$
$$\approx K(t^{1-1/p}, x\,;\, L_1, L_{p,1}).$$

Therefore, by (1), we have

$$\int_0^t (Tx)^*(s)\, ds \le C_3 \int_0^t (Mx^*)^*(s)\, ds \tag{3}$$

for all $0 < t \le 1$.

Next, we prove that

$$\|Mx^*\| \le C_4 \|x\|, \qquad x \in X, \tag{4}$$

for an arbitrary symmetric space X such that $\alpha(X) > 1/p$. Note that the estimate (4) was obtained earlier by Boyd (see [8 Theorem 1; 2, Lemma 2] and [3, Theorem 5.15]) for the case in which the symmetric space X possesses the Fatou property. He proved that the operator M is bounded in the space X if and only if the lower Boyd index satisfies $\alpha(X) > 1/p$. We obtain (4) in the general case.

Further, by the Calderón–Mityagin Theorem [5, Chap. 2, Theorem 4.3], a symmetric space X is an interpolation space between the spaces L_1 and L_∞ if and only if for a constant $B > 0$ the following condition is satisfied: if $x \in X$, $y \in L_1$, and

$$\int_0^t y^*(s)\, ds \le \int_0^t x^*(s)\, ds \qquad \text{for all} \quad 0 < t \le 1,$$

then $y \in X$ and $\|y\|_X \le B\|x\|_X$. Hence the estimates (3), (4) imply the following: if $x \in X$, then $Tx \in X$ and $\|Tx\|_X \le C_5 \|x\|_X$. Thus, X is an interpolation space between L_1 and L_p. $\square$

Remark 1. The condition for X to be an interpolation space between L_1 and L_∞ in Theorem 1 cannot be dropped. Let us present an example showing this.

Consider a modification of the Russu space (see [9, Theorem 1] or [5, Chap. 2, Lemma 5.5]). Suppose that ψ is an increasing concave function on $(0, 1]$ such that

$$\psi(0^+) = 0, \qquad \lim_{t\to 0^+} \frac{\psi(2t)}{\psi(t)} = 1$$

and its upper dilation index is

$$\delta_\psi = \lim_{t\to\infty} \frac{\ln \bar{\psi}(t)}{\ln t} = 0,$$

where

$$\bar{\psi}(t) = \sup_{0<s\le 1,\ 0<st\le 1} \frac{\psi(st)}{\psi(s)}\,.$$

These conditions are satisfied, for example, by the function $\psi(t) = \ln^{-1}(e^2/t)$.

In the Marcinkiewicz space M_ψ,

$$\|x\|_{M_\psi} = \sup_{0<t\le 1} \frac{1}{\psi(t)} \int_0^t x^*(s)\,ds$$

we consider the linear subspace

$$\widetilde{G} = \left\{ x \in L_1 \colon \sup_{0<t\le 1} \frac{x^*(t)}{\psi'(t)} < \infty \right\}$$

and by G denote the closure of $\widetilde{G}$ in M_ψ. It is easy to verify that G is a symmetric space on $[0,1]$. Since

$$\|\sigma_t\|_{G\to G} = \|\sigma_t\|_{M_\psi\to M_\psi} = t\bar{\psi}\left(\frac{1}{t}\right),$$

it follows that

$$\alpha(G) = \alpha(M_\psi) = 1 - \delta_\psi = 1 > \frac{1}{p} \qquad \text{for any} \quad 1 < p < \infty.$$

However, the space G is not an interpolation space between L_1 and L_∞ (see [9, Theorem 2] or [5, Chap. 2, Lemma 5.5]) and, especially between L_1 and L_p.

Remark 2. Arguing in the same way as in the proof of Theorem 1, we can find a more general result: *Suppose that* $1 \le r < p < \infty$. *If* X *is an interpolation space between* L_r *and* L_∞ *and* $\alpha(X) > 1/p$, *then* X *is an interpolation space between* L_r *and* L_p. We must only verify additionally that, for a constant $B > 0$ independent of $x \in L_r$ and $0 < t \le 1$, the following inequalities are satisfied:

$$\int_0^t (Tx)^*(s)^r\,ds \le B\left[\int_0^t x^*(s)^r\,ds + tMx^*(t)^r\right] \le B\int_0^t [x^*(s) + Mx^*(s)]^r\,ds,$$

and then use the K-monotonicity of the pair (L_r, L_∞) [10, Theorem 2].

4. INTERPOLATION OF OPERATORS OF STRONG TYPE $(1,1)$ AND OF WEAK TYPE (P,P)

A linear operator T defined on $L_1[0,1]$ is called an operator *of weak type* (p,p), $1 \le p < \infty$ if T boundedly acts from $L_{p,1}$ to $L_{p,\infty}$, where the Marcinkiewicz space $L_{p,\infty}$ consists of all measurable (on $[0,1]$) functions $x(t)$ for which the quasinorm

$$\|x\|_{p,\infty} = \sup_{t\in[0,1]} t^{1/p} x^*(t)$$

is finite. The bounded linear operator $T\colon L_p \to L_p$ is called an operator *of strong type* (p,p). We can easily see that each operator of strong type (p,p) is also one of weak type (p,p), while the converse is not true, since $L_{p,1} \subset L_p \subset L_{p,\infty}$.

In 1969, Boyd [2] proved that in order that any linear operator T of weak type (p,p) and (q,q), $1 \le p < q < \infty$, be bounded in a symmetric space X, the following condition is necessary and sufficient:

$$\frac{1}{q} < \alpha(X) \le \beta(X) < \frac{1}{p}.$$

He obtained this result under the additional assumption that X possesses the Fatou property (see also [3 Theorem 5.16; 4, Theorems 2.b.11 and 2.b.13]). Nevertheless, Boyd's theorem remains also valid for arbitrary symmetric spaces (see the introduction or Theorem 6.12 in [5, Chap. 2]). In the case $q = \infty$, i.e., when the class of all linear operators T of strong type (∞,∞) and of weak type (p,p), $1 \le p < \infty$ is considered, the boundedness of any such operator in X is equivalent to the one-sided estimate $\beta(X) < 1/p$ (see [6, Theorem 4.6] and also [11, Remark 5.9(a)]). A similar result can be proved also for the class of linear operators of weak type (p,p), $1 < p < \infty$, bounded in L_1.

Theorem 2. *Suppose that* $1 < p < \infty$. *In order that any linear operator* T *of strong type* $(1,1)$ *and simultaneously of weak type* (p,p) *be bounded in a symmetric space* X, *the following conditions are necessary and sufficient*: X *is an interpolation space between* L_1 *and* L_∞ *and* $\alpha(X) > 1/p$.

Scheme of the proof. *Sufficiency.* This follows from the proof of Theorem 1, since estimate (1) is valid as before. We must only verify additionally that, given the condition $\alpha(X) > 1/p$, we have $L_{p,\infty} \subset X$.

Necessity. The operator M is of strong type $(1,1)$ and of weak type (p,p). Hence by assumption, M is bounded in X. If

$$0 < \varepsilon < \frac{1}{\|M\|_{X \to X}},$$

then the operator $(I_X - \varepsilon M)^{-1}$ exists and is bounded in X as well (I_X is the identity operator). In addition,

$$(I_X - \varepsilon M)^{-1} x(t) = \sum_{n=0}^{\infty} \varepsilon^n M^n x(t),$$

where the series is convergent in the operator norm and M^n denotes the nth power of M. Since

$$M^n x(t) = t^{-1/p} \int_t^1 s^{1/p-1} \frac{\ln^{n-1}(s/t)}{(n-1)!} x(s)\, ds,$$

it follows that the operator $\widetilde{M} x(t) = M(I_X - \varepsilon M)^{-1} x(t)$, which is bounded in X as well, is of the following form:

$$\widetilde{M} x(t) = t^{-1/p} \int_t^1 s^{1/p-1} \left(\frac{s}{t}\right)^{\varepsilon} x(s)\, ds.$$

Then

$$\widetilde{M} x^*(t) \ge \sigma_\tau x^*(t) \frac{p}{1+\varepsilon p} \tau^{-1/p-\varepsilon}(1 - \tau^{1/p+\varepsilon}).$$

Hence

$$\|\sigma_\tau x\| \le \left(\frac{1}{p} + \varepsilon\right) \frac{\tau^{1/p+\varepsilon}}{1 - \tau^{1/p+\varepsilon}} \|\widetilde{M}\|_{X \to X} \|x\|;$$

this yields

$$\alpha(X) \ge \frac{1}{p} + \varepsilon > \frac{1}{p}.$$

It remains to show that a symmetric space X is an interpolation space between L_1 and L_∞. If the linear operator $T\colon L_1 \to L_1$ and $T = T_{|L_\infty}\colon L_\infty \to L_\infty$, then, by the Riesz–Torin theorem, T is of strong type and hence also of weak type (p,p). But in that case, by assumption, the operator T is also bounded in X, and the proof is complete. $\square$

Remark 3. Theorem 2 can be generalized as follows (see Remark 2): *Suppose that* $1 \le r < p < \infty$. *In order that any linear operator* T *of strong type* (r,r) *and simultaneously of weak type* (p,p) *be bounded in a symmetric space* X, *it is necessary and sufficient that the following conditions be satisfied:* X *is an interpolation space between* L_r *and* L_∞ *and* $\alpha(X) > 1/p$.

Remark 4. Theorems 1 and 2 are also valid for a symmetric space on the semiaxis $(0,\infty)$. In this case, it is only necessary to verify additionally that the condition $\alpha(X) > 1/p$ yields $L_1 \cap L_{p,\infty} \subset X \subset L_1 + L_{p,1}$. In addition, Theorems 1 and 2 are valid for quasilinear operators.

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