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BRIEF COMMUNICATIONS

Some New Extrapolation Estimates for the Scale of L_p-Spaces

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Abstract. Using new extrapolation estimates for the $\mathscr{K}$- and $\mathscr{J}$-functionals of couples of limit spaces of the L_p-scale $(1 < p < \infty)$, we introduce a class of extrapolation functors. A characterization of this class via the real interpolation method permits one to obtain new equivalent expressions for the norms in symmetric spaces "close" to L_∞ and L_1, which depend only on the L_p-norms of a function.

Key words: Yano's extrapolation theorem, Peetre's $\mathscr{K}$- and $\mathscr{J}$-functionals, Banach couple, rearrangement invariant space, Orlicz space, Zygmund space, extrapolation functor, real interpolation method.

1. Introduction. In 1951, Yano proved the following theorem [1; 2, Ch. 12]. Suppose that a linear operator T defined on $\bigcup_{p>1} L_p[0,1]$ satisfies the inequalities

$$\left(\int_0^1 |Tf(s)|^p\,ds\right)^{1/p} \leqslant \frac{C}{(p-1)^\alpha}\left(\int_0^1 |f(s)|^p\,ds\right)^{1/p}$$

for some $\alpha > 0$ and for all p in some right neighborhood of the point 1 with a constant $C > 0$ independent of p and f. Then T is a bounded operator from the Zygmund space $L(\log L)^\alpha$ into L_1. There is also a dual statement: if T is a linear operator bounded in L_p for all sufficiently large p and $\|T\|_{L_p\to L_p} = O(p^\alpha)$ $(p\to\infty)$ for some $\alpha > 0$, then T is bounded from L_∞ into the space $\operatorname{Exp} L^{1/\alpha}$.

More recently, in the 1990s, foundations of general extrapolation theory were laid down by Jawerth and Milman [3–5], who suggested general approaches to the analysis of similar situations. The main goal of this theory is to study the natural limit spaces associated with interpolation scales and estimates of the norms of operators in these spaces. An extensive bibliography on extrapolation theory and its numerous applications can also be found in [3–5].

Using the sum and the intersection of families of Banach spaces as extrapolation functors that result in a construction of the limit spaces, Jawerth and Milman proved some relations that imply extrapolation theorems for operators acting on appropriate spaces. In particular, they obtained the following description of the extrapolation spaces occurring in Yano's theorem (e.g., see [5, p. 22–23]):

$$\Delta_{1<p<\infty}(p^{-\alpha}L_p) = \operatorname{Exp} L^{1/\alpha} \qquad (\alpha > 0) \tag{1}$$

and

$$\Sigma_{1<p<\infty}((p-1)^{-\alpha}L_p) = L(\log L)^\alpha \qquad (\alpha > 0). \tag{2}$$

The main purpose of this note is to show that, at least in the case of the scale of L_p-spaces, extrapolation permits one to recover not only individual limit spaces but also the interpolation functionals corresponding to natural Banach couples of such spaces. This allows one to introduce a class of extrapolation functors equal in strength to the real interpolation method. We also prove more general relations than (1) and (2) and give a new description of symmetric spaces close to L_∞ and L_1.

We deal only with spaces of functions defined on $[0,1]$. Generalizations and strengthenings of Yano's theorem to the case of functions defined on a space with a σ-finite measure can be found in [6, 7].

2. Exponential Orlicz spaces and the scale of L_p-spaces. A Banach function space X on $[0,1]$ is called a *symmetric space* if it follows from $g^*(t) \leqslant f^*(t)$ and $f(t) \in X$ that $g(t) \in X$ and $\|g\|_X \leqslant \|f\|_X$. (Here $f^*(t)$ stands for the nonincreasing rearrangement of $|f(t)|$ [8, p. 83].)

The continuous embeddings $L_\infty \subset X \subset L_1$ hold for any symmetric space X [8, pp. 124–126], where, as usual, the L_∞-norm is defined by $\|f\|_\infty = \operatorname{ess\,sup}_{s\in[0,1]} |f(s)|$.

Orlicz spaces are an important example of symmetric spaces. If $N(t)$ is an Orlicz function, i.e., an increasing convex function on $[0,\infty)$ with $N(0) = 0$, then the norm of the Orlicz space L_N is defined as follows:

$$\|f\|_{L_N} = \inf\left\{u > 0 : \int_0^1 N\left(\frac{|f(t)|}{u}\right) dt \leqslant 1\right\}.$$

For $N(t) = t^p$ $(1 \leqslant p < \infty)$, we obtain the usual L_p-spaces with $\|f\|_p = (\int_0^1 |f(t)|^p\,dt)^{1/p}$. In what follows, we are interested in the exponential Orlicz spaces $\operatorname{Exp} L^\Phi = L_{N_\Phi}$, where $N_\Phi(u) = e^{\Phi(u)} - 1$ and $\Phi(u)$ is an Orlicz function. The norm of $\operatorname{Exp} L^\Phi$ is equivalent [9, 10] to the norm

$$\|f\|_{M(\varphi)} = \sup_{0<t\leqslant 1} \frac{\int_0^t f^*(s)\,ds}{\varphi(t)}$$

of the Marcinkiewicz space $M(\varphi)$ corresponding to the concave function $\varphi(u) = u\Phi^{-1}(\ln(1+1/u))$. In particular, it follows that $\operatorname{Exp} L^\Phi = \Lambda(\varphi)^*$, where $\Lambda(\varphi)$ is the Lorentz space with norm $\|f\|_{\Lambda(\varphi)} = \int_0^1 f^*(s)\,d\varphi(s)$ [8, p. 152–154]. For $\Phi(u) = u^\alpha$ $(\alpha \geqslant 1)$, we obtain the well-known Zygmund spaces $\operatorname{Exp} L^\alpha$ and $L(\log L)^{1/\alpha}$, respectively [11].

Let X_k $(k = 1, 2, \dots)$ be Banach spaces linearly and continuously embedded in a linear topological space $\mathscr{T}$. Then their intersection $\Delta_{k=1}^\infty X_k$ is defined as the Banach space of all $x \in \bigcap_{k=1}^\infty X_k$ such that $\|x\| = \sup_{k=1,2,\dots} \|x\|_{X_k} < \infty$. Suppose in addition that there exists a Banach space X_0 embedded in $\mathscr{T}$ such that $X_k \subset X_0$ $(k = 1, 2, \dots)$ with constant 1. Then we can define the *sum* $\sum_{k=1}^\infty X_k$ as the set of all $x \in X_0$ representable in the form $x = \sum_{k=1}^\infty x_k$ $(x_k \in X_k)$, where $\sum_{k=1}^\infty \|x_k\|_{X_k} < \infty$. This set is a Banach space with respect to the norm $\|x\| = \inf \sum_{k=1}^\infty \|x_k\|_{X_k}$, where the infimum is taken over all such representations of x. Finally, if X is a Banach space and $b > 0$, then bX is defined as the space X equipped with the norm $\|x\|_{bX} = b\,\|x\|_X$.

Theorem 1. *For an arbitrary Orlicz function Φ, one has*

$$\Delta_{j=1}^\infty\left(\frac{1}{\Phi^{-1}(2^j)}\,L_{2^j}\right) = \operatorname{Exp} L^\Phi, \tag{3}$$

$$\Sigma_{j=1}^\infty(\Phi^{-1}(2^j)\,L_{1+2^{-j}}) = \Lambda(\varphi) \tag{4}$$

up to equivalence of norms.

For $\Phi(u) = u^{1/\alpha}$ $(0 < \alpha \leqslant 1)$, relations (3) and (4) are the discrete counterparts of (1) and (2), respectively.

3. An expression of the interpolation functionals of the Banach couples $(L_\infty, \operatorname{Exp} L^\Phi)$ and $(L_1, \Lambda(\varphi))$ via L_p-norms. In interpolation theory of operators [8, 12, 13], a special role is played by the Peetre $\mathscr{K}$- and $\mathscr{J}$-functionals, defined for any Banach couple (X_0, X_1) and any $t > 0$ by the formulas

$$\mathscr{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\},$$
$$\mathscr{J}(t, x; X_0, X_1) = \max\{\|x\|_{X_0},\ t\|x\|_{X_1}\}.$$

Relations (3) and (4) show that $\operatorname{Exp} L^\Phi$ and $\Lambda(\varphi)$ are limit spaces of the scale of L_p-spaces. The same is true of L_∞ and L_1, since

$$\Delta_{j=1}^{\infty} L_{2^j} = L_\infty \quad \text{and} \quad \Sigma_{j=1}^{\infty} L_{1+2^{-j}} = L_1. \tag{5}$$

There is a similar extrapolation description of the interpolation functionals of the Banach couples $(L_\infty, \operatorname{Exp} L^\Phi)$ and $(L_1, \Lambda(\varphi))$. (In what follows, the symbol $\asymp$ means the existence of a two-sided estimate with constants that depend only on the Orlicz function Φ.)

Theorem 2. *If Φ is an arbitrary Orlicz function, then*

$$\mathscr{K}(2^k, f; L_\infty, \operatorname{Exp} L^\Phi) \asymp \|f\|_{\Delta_{j=k}^{\infty}(2^{k-j} L_{\Phi(2^j)})} = \sup_{j \geqslant k} (2^{k-j} \|f\|_{\Phi(2^j)})$$

for any $f \in \operatorname{Exp} L^\Phi$ and $k = 1, 2, \ldots$.

Theorem 3. *If Φ is an arbitrary Orlicz function, then $\mathscr{J}(2^{-k}, g; L_1, \Lambda(\varphi)) \asymp \|g\|_{U_k}$, where $U_k = \Sigma_{j \geqslant k}(2^{j-k} L_{r_j})$ and $r_j = 1 + [\Phi(2^j)]^{-1}$, for any $g \in \Lambda(\varphi)$ and $k = 1, 2, \ldots$.*

Note that the embeddings $L_\infty \subset \operatorname{Exp} L^\Phi$ and $\Lambda(\varphi) \subset L_1$ with constant 1 imply that

$$\mathscr{K}(t, f; L_\infty, \operatorname{Exp} L^\Phi) = t\,\|f\|_{\operatorname{Exp} L^\Phi} \quad (0 < t \leqslant 1) \quad \text{and} \quad \mathscr{J}(t, g; L_1, \Lambda(\varphi)) = t\,\|g\|_{\Lambda(\varphi)} \quad (t \geqslant 1).$$

4. Extrapolation functors on the scale of L_p-spaces and the real interpolation method. If E is a Banach lattice of two-sided sequences of real numbers, then we define $E(2^{-k})$ as the lattice that consists of all $a = (a_k)_{k=-\infty}^{\infty}$ such that $(a_k 2^{-k})_{k=-\infty}^{\infty} \in E$ and $\|a\|_{E(2^{-k})} = \|(a_k 2^{-k})\|_E$.

Furthermore, $\Delta(\vec{l}_\infty) = l_\infty \cap l_\infty(2^{-k})$, $\Sigma(\vec{l}_\infty) = l_\infty + l_\infty(2^{-k})$, $\Delta(\vec{l}_1) = l_1 \cap l_1(2^{-k})$, $\Sigma(\vec{l}_1) = l_1 + l_1(2^{-k})$, and $e^k = (e^k_j)$, $e^k_k = 1$, $e^k_j = 0$ $(j \neq k)$, $k = 0, \pm 1, \pm 2, \ldots$.

Definition 1. Suppose that Φ is an Orlicz function and F is a Banach lattice of two-sided sequences such that $\Delta(\vec{l}_\infty) \subset F \subset \Sigma(\vec{l}_\infty)$. The set $\mathscr{L}^{\mathscr{K}}_{\Phi,F}$ consists of all measurable functions f on $[0,1]$ such that the sequence $a_f = \Sigma_{k=0}^{\infty} \|f\|_{\Phi(2^k)} e^k$ belongs to F.

The norm $\|f\|_{\mathscr{L}^{\mathscr{K}}_{\Phi,F}} = \|a_f\|_F$ makes $\mathscr{L}^{\mathscr{K}}_{\Phi,F}$ a symmetric space. It follows from (3) and (5) that $\mathscr{L}^{\mathscr{K}}_{\Phi,l_\infty} = L_\infty$ and $\mathscr{L}^{\mathscr{K}}_{\Phi,l_\infty(2^{-k})} = \operatorname{Exp} L^\Phi$. Consequently, $L_\infty \subset \mathscr{L}^{\mathscr{K}}_{\Phi,F} \subset \operatorname{Exp} L^\Phi$.

Definition 2. Suppose that Φ is an Orlicz function and G is a Banach lattice of two-sided sequences such that $\Delta(\vec{l}_1) \subset G \subset \Sigma(\vec{l}_1)$. By $\mathscr{L}^{\mathscr{J}}_{\Phi,G}$ we denote the set of all measurable functions g on $[0,1]$ possessing a representation of the form $g = \sum_{k=1}^{\infty} g_k$ (convergence in L_1), where $g_k \in L_{r_k}$, $r_k = 1 + [\Phi(2^k)]^{-1}$, and $\sum_{k=1}^{\infty} \|g_k\|_{r_k} e^{-k} \in G$.

Then $\mathscr{L}^{\mathscr{J}}_{\Phi,G}$ is a symmetric space with respect to the norm

$$\|g\|_{\mathscr{L}^{\mathscr{J}}_{\Phi,G}} = \inf \left\| \sum_{k=1}^{\infty} \|g_k\|_{r_k} e^{-k} \right\|_G,$$

where the supremum is taken over all such representations of g. Since relations (4) and (5) imply that $\mathscr{L}^{\mathscr{J}}_{\Phi,l_1} = L_1$ and $\mathscr{L}^{\mathscr{J}}_{\Phi,l_1(2^{-k})} = \Lambda(\varphi)$, we have $\Lambda(\varphi) \subset \mathscr{L}^{\mathscr{J}}_{\Phi,G} \subset L_1$.

These constructions of the spaces $\mathscr{L}^{\mathscr{K}}_{\Phi,F}$ and $\mathscr{L}^{\mathscr{J}}_{\Phi,G}$ can be treated as extrapolation functors on the scale of L_p-spaces [3–5]. One can find the values of these functors by studying the corresponding functors of the real interpolation method. Let E be a Banach lattice of two-sided sequences. If (X_0, X_1) is an arbitrary Banach couple, then the space $(X_0, X_1)^{\mathscr{K}}_E$ consists of all $x \in X_0 + X_1$ such that $\|x\| = \|(\mathscr{K}(2^j, x; X_0, X_1))_j\|_E < \infty$. The space $(X_0, X_1)^{\mathscr{J}}_E$ consists of all $x \in X_0 + X_1$ representable in the form $x = \sum_{j=-\infty}^{\infty} u_j$ (convergence in $X_0 + X_1$), where $u_j \in X_0 \cap X_1$. The norm of $(X_0, X_1)^{\mathscr{J}}_E$ is given by

$$\inf_{\{u_j\}} \|(\mathscr{J}(2^j, u_j; X_0, X_1))_j\|_E,$$

where the infimum is taken over all such sequences $\{u_j\}_{j=-\infty}^{\infty}$.

Suppose that $E \supset \Delta(\vec{l}_\infty)$ (respectively, $\{0\} \neq E \subset \Sigma(\vec{l}_1)$). Then the mapping $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathscr{K}}$ (respectively, $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathscr{J}}$) is an interpolation functor [13]. In particular, $(X_0, X_1)_E^{\mathscr{K}}$ is an interpolation space with respect to the Banach couple (X_0, X_1): if a linear operator T is bounded in X_0 and X_1, then it is also bounded in $(X_0, X_1)_E^{\mathscr{K}}$). (The same assertion holds for the space $(X_0, X_1)_E^{\mathscr{J}}$.) The set of all such functors is called the *real* $\mathscr{K}$- (respectively, $\mathscr{J}$-)*method*. We point out that E can always be assumed to be an interpolation lattice with respect to the couple $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$ (respectively, $\vec{l}_1 = (l_1, l_1(2^{-k}))$) [13, Corollaries 3.3.10, 3.4.6]. Such E will be called a *parameter* of the $\mathscr{K}$- (respectively $\mathscr{J}$-)method of interpolation.

Theorem 4. *For an arbitrary Orlicz function* Φ *and any parameter* F *of the* $\mathscr{K}$*-method, one has* $\mathscr{L}_{\Phi,F}^{\mathscr{K}} = (L_\infty, \operatorname{Exp} L^\Phi)_F^{\mathscr{K}}$ *(up to equivalence of norms).*

As was already noted in the Introduction, $\operatorname{Exp} L^\Phi = M(\varphi)$, where $\varphi(u) = u\Phi^{-1}(\ln(1 + 1/u))$. Hence the interpolation for the couple $(L_\infty, \operatorname{Exp} L^\Phi)$ can be described by the real $\mathscr{K}$-method [14]. This means that each interpolation space X for this couple is representable as a space of the real $\mathscr{K}$-method, that is, $X = (L_\infty, \operatorname{Exp} L^\Phi)_F^{\mathscr{K}}$ for an appropriate parameter F of $\mathscr{K}$-method. Therefore, we arrive at the following assertion.

Corollary 1. *Let* Φ *be an Orlicz function. If* X *is an interpolation space with respect to the couple* $(L_\infty, \operatorname{Exp} L^\Phi)$*, then there exists a parameter* F *of the* $\mathscr{K}$*-method such that* $X = \mathscr{L}_{\Phi,F}^{\mathscr{K}}$.

Similar results hold for the functor $\mathscr{L}_{\Phi,G}^{\mathscr{J}}$ and the couple $(L_1, \Lambda(\varphi))$.

Theorem 5. *For an arbitrary Orlicz function* Φ *and any parameter* G *of the* $\mathscr{J}$*-method, one has* $\mathscr{L}_{\Phi,G}^{\mathscr{J}} = (L_1, \Lambda(\varphi))_G^{\mathscr{J}}$ *(up to equivalence of norms).*

Corollary 2. *Let* Φ *be an Orlicz function. If* Y *is an interpolation space with respect to the couple* $(L_1, \Lambda(\varphi))$*, then there exists a parameter* G *of the* $\mathscr{J}$*-method such that* $Y = \mathscr{L}_{\Phi,G}^{\mathscr{J}}$.

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