

On Complementability of Subspaces Generated by Contractions and Shifts of Functions

S. V. Astashkin*, L. Maligranda**, and E. M. Semenov***

Presented by Academician Yu.G. Reshetnyak May 29, 2002

Received July 3, 2002

Suppose that E is an r.i. space on $[0, 1]$ (the definition is given below), $a \in E$, $1 \leq k \leq 2^n$, and $n = 0, 1, \dots$. We set

$$a_{n,k}(t) = \begin{cases} a(2^n t - k + 1) & \text{if } \frac{k-1}{2^n} \leq t \leq \frac{k}{2^n} \\ 0 & \text{for other } t \in [0, 1] \end{cases}$$

and $Q_{n,a} = \text{span}\{a_{n,k}, 1 \leq k \leq 2^n\}$. We denote the set of $a \in E$ such that $Q_{n,a}$ is uniformly complemented in E , i.e., there exist projectors P_n from E onto $Q_{n,a}$ with $\sup_n \|P_n\| < \infty$, by $\mathfrak{N}(E)$. This paper studies the set $\mathfrak{N}(E)$ and the class of r.i. spaces E coinciding with $\mathfrak{N}(E)$. It turns out that $\mathfrak{N}(E)$ is closely related to the space of tensor multipliers acting in E .

Below, we give the necessary definitions.

A Banach space E of measurable functions on $[0, 1]$ is called symmetric or rearrangement invariant (r.i.) if

(i) $|x(t)| \leq |y(t)|$ and $y \in E$ imply $x \in E$ and $\|x\|_E \leq \|y\|_E$;

(ii) the equimeasurability of functions x and y and the inclusion $y \in E$ imply $x \in E$ and $\|x\|_E = \|y\|_E$.

Following [1], we assume that E is separable or dual to a separable space.

Examples of r.i. spaces are L_p with $1 \leq p \leq \infty$ and the Orlicz, Lorentz, and Marcinkiewicz spaces. Let Ω denote the set of increasing concave functions $\varphi(t)$ on $[0, 1]$ such that $\varphi(0) = \varphi(+0) = 0$ and $\varphi(1) = 1$. Each

function $\varphi \in \Omega$ generates a Lorentz space $\Lambda(\varphi)$ and a Marcinkiewicz space $M(\varphi)$ with norms

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t) d\varphi(t),$$

$$\|x\|_{M(\varphi)} = \sup_{0 < s \leq 1} \frac{1}{\varphi(s)} \int_0^s x^*(t) dt,$$

where $x^*(t)$ is the permutation of $|x(t)|$ in decreasing order. The space $\Lambda(\varphi)$ is separable, and $(\Lambda(\varphi))^* = M(\varphi)$.

If $\Phi(t)$ is a convex increasing function on $[0, \infty)$ and

$$\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{\Phi(t)} = 0,$$

then the Orlicz space L_Φ is, by definition, the set of functions such that $\Phi(\varepsilon x) \in L_1$ for some $\varepsilon > 0$ with norm

$$\|x\|_{L_\Phi} = \inf \left\{ \lambda: \lambda > 0, \int_0^1 \Phi\left(\frac{|x(t)|}{\lambda}\right) dt \leq 1 \right\} < \infty.$$

The condition $\Phi \in \Delta_2$ means that $\Phi(2t) \leq C\Phi(t)$ for some constant $C > 0$ and all $t > 0$.

In any r.i. space E , the operator family

$$\sigma_\tau x(t) = \begin{cases} x(t/\tau), & 0 \leq t \leq \min(1, \tau) \\ 0, & \min(1, \tau) < t \leq 1 \end{cases}$$

acts continuously. The numbers

$$\alpha_E = \lim_{\tau \rightarrow 0} \frac{\ln \|\sigma_\tau\|_E}{\ln \tau}, \quad \beta_E = \lim_{\tau \rightarrow \infty} \frac{\ln \|\sigma_\tau\|_E}{\ln \tau}$$

are called the Boyd indices of the space E . We always have $0 \leq \alpha_E \leq \beta_E \leq 1$. We use the Calderon–Lozanovskii construction [2]. If E_0 and E_1 are r.i. spaces and $0 < \theta < 1$, then $E_0^{1-\theta} E_1^\theta$ denotes the space of functions with the norm

$$\|x\| = \inf_{\|x_0\|_{E_0} = \|x_1\|_{E_1} = 1} \sup_{0 \leq t \leq 1} \frac{|x(t)|}{|x_0(t)|^{1-\theta} |x_1(t)|^\theta}.$$

* Samara State University,
ul. Akademika Pavlova 1, Samara, 443011 Russia
e-mail: astashkn@ssu.samara.ru

** Department of Mathematics, Lulea University,
Lulea, 97187 Sweden
e-mail: lech@sm.luth.se

*** Mathematical Faculty, Voronezh State University,
Universitetskaya pl. 1, Voronezh, 394006 Russia
e-mail: nadezhka@ssm.vsu.ru, root@func.vsu.ru

For any r.i. space E , the continuous embeddings $L_\infty \subset E \subset L_1$ hold. For an r.i. space E , E^0 denotes the closure of L_∞ in E . If E is separable, then $E^0 = E$; if $E \neq L_\infty$, then E^0 is separable. The equality of two r.i. spaces means that they coincide as sets. By the closed graph theorem, their norms are equivalent. By E' we denote the space of functions for which

$$\|x\|_{E'} = \sup_{\|y\|_E \leq 1} \int_0^1 x(t)y(t)dt < \infty.$$

The space E' is isometrically embedded in E^* . If E is separable, then $E' = E^*$ and the norms are equal.

There exists a measure-preserving one-to-one mapping of $[0, 1]$ onto the square $[0, 1] \times [0, 1]$; therefore, for any r.i. space E on $[0, 1]$, the measure-preserving mapping generates a space of functions on the square isomeric to this space. We denote this function space on the square by the same symbol E . This allows us to consider the tensor multiplier $(x \otimes y)(t, s) = x(t)y(s)$ as a bilinear operator in L_p , where $1 \leq p \leq \infty$. Every r.i. space E generates a space $\mathfrak{M}(E)$ of tensor multipliers with the norm

$$\|x\|_{\mathfrak{M}(E)} = \sup_{\|y\|_E \leq 1} \|x \otimes y\|_E.$$

Obviously, the embeddings $L_\infty \subset \mathfrak{M}(E) \subset E$ hold. Tensor products of r.i. spaces are considered in [3, 4] and elsewhere, and the space of tensor multipliers are studied in [5]. In more detail, the r.i. spaces are studied in [1, 6]. Subspaces generated by shifts of one function from an r.i. space on $[0, \infty)$ are examined in [7].

The central result of this paper is the following theorem.

Theorem 1. *Let E be an r.i. space. Then,*

$$\mathfrak{N}(E^0) \subset \mathfrak{M}(E) \subset \mathfrak{N}(E).$$

Theorem 1 solves the problem of finding $\mathfrak{N}(E)$ for separable r.i. spaces. In this case, $\mathfrak{N}(E)$ coincides with $\mathfrak{M}(E)$. Using the results describing $\mathfrak{M}(E)$ [3–5], we obtain the following corollary.

Corollary 1. (i) *If $1 < p < \infty$ and $1 \leq q < \infty$, then $\mathfrak{N}(L_{p,q}) = L_{p, \min(p,q)}$ and $\mathfrak{N}(L_{p,\infty}^0) = L_p$.*

(ii) *If $\varphi \in \Omega$ and $\tilde{\varphi}(t) = \sup_{0 < s \leq 1} \frac{\varphi(ts)}{\varphi(s)}$, then*

$$\Lambda(\tilde{\varphi}) \subset \mathfrak{N}(\Lambda(\varphi)) \subset \Lambda(\varphi).$$

The equality $\mathfrak{N}(\Lambda(\varphi)) = \Lambda(\varphi)$ holds if and only if $\varphi(ts) \leq C\varphi(t)\varphi(s)$ for some $C > 0$ and all $t, s \in [0, 1]$.

(iii) *If $\Phi \in \Delta_2$, then $\mathfrak{N}(L_\Phi) = L_\Phi$ if and only if Φ is semimultiplicative, i.e., there exists a $C > 0$ such that $\Phi(uv) \leq C\Phi(u)\Phi(v)$ for all $u, v \geq 1$.*

(iv) *If E is a separable r.i. space, then $\mathfrak{N}(E) = L_\infty$ if and only if $\alpha_E = 0$.*

For nonseparable r.i. spaces E , the set $\mathfrak{N}(E)$ may differ from $\mathfrak{M}(E)$, and the problem of describing $\mathfrak{N}(E)$ becomes more complicated.

Theorem 2. *Suppose that $\varphi \in \Omega$ and $\lim_{t \rightarrow 0} \frac{\varphi(2t)}{\varphi(t)} = 2$.*

Then, $a \in M(\varphi)$ belongs to $\mathfrak{N}(M(\varphi))$ if and only if $a \in L_\infty$ or $a \notin M^0(\varphi)$, i.e.,

$$\mathfrak{N}(M(\varphi)) = L_\infty \cup (M(\varphi) \setminus M^0(\varphi)).$$

Theorem 2 shows that the set $\mathfrak{N}(E)$ may be nonlinear if E is nonseparable.

Theorem 3. *If $\varphi \in \Omega$ and $\sup_{0 < t \leq 1} \frac{\varphi(t)}{\varphi(t^2)} < \infty$, then*

$$\mathfrak{N}(M(\varphi)) = \mathfrak{M}(M(\varphi)) = M(\varphi).$$

Let $\mathfrak{N}$ denote the set of r.i. spaces E coinciding with $\mathfrak{N}(E)$. Theorem 3 shows that some nonseparable r.i. spaces belong to the class $\mathfrak{N}$. The class $\mathfrak{N}$ is stable with respect to the complex interpolation method.

Theorem 4. *Suppose that E_0 and E_1 are separable r.i. spaces; $E_0, E_1 \in \mathfrak{N}$; and $0 < \theta < 1$. Then, $E_0^{1-\theta} E_1^\theta \in \mathfrak{N}$.*

Corollary 1(i) shows that the class $\mathfrak{N}$ is not stable with respect to the real interpolation method. Below, we give yet another description of the class $\mathfrak{N}$.

Theorem 5. *Let E be an r.i. space. The following conditions are equivalent.*

- (i) $\mathfrak{M}(E) = E$;
- (ii) $\mathfrak{N}(E) = E$;
- (iii) *There exists a constant $C > 0$ such that*

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_E \leq C \|a\|_E \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{n,k} \right\|_E$$

for all $a \in E$ and all $c_{n,k} \in \mathbb{R}^1$, where $k = 1, 2, \dots, 2^n$ and $n = 0, 1, \dots$; here, $\chi_{n,k}$ is the characteristic function of the interval $((k-1) \cdot 2^{-n}, k \cdot 2^{-n})$.

The spaces L_p have the following characteristic property in terms of the class $\mathfrak{N}$.

Theorem 6. *Let E be an r.i. space. The following conditions are equivalent.*

- (i) $\mathfrak{N}(E) = E$ and $\mathfrak{N}(E') = E'$;
- (ii) *The tensor product operators $\otimes$ from $E \times E$ to E and from $E' \times E'$ to E' are bounded;*
- (iii) *There exists a $p \in [1, \infty]$ such that $E = L_p$.*

Thus, if an r.i. space E does not coincide with L_p for $1 \leq p \leq \infty$, then the class $\mathfrak{N}$ cannot contain more than one of the spaces E and E' .

The problem of describing $\mathfrak{N}(L_{p,\infty})$ ($1 < p < \infty$) remains unsolved. It is clear only that $\mathfrak{N}(L_{p,\infty}) \cap L_{p,\infty}^0 = L_p$.

ACKNOWLEDGMENTS

This work was financially supported by the grant of Royal Swedish Academy of Sciences in the framework of the program of collaboration of Sweden with the republics of the former USSR (project no. 35147). L.M. was partially supported by the National Scientific Council of Sweden (NFR) (grant no. M5105-20005228/2000). E.M.S. was partially supported by the Russian Foundation for Basic Research (project no. 02-01-00146) and the program “Universities of Russia” (project no. 04.01.051).

S.V.A. and E.M.S. are grateful to Department of Mathematics, Lulea University (Sweden) for hospitality during their visit to Lulea.

REFERENCES

1. Lindenstrauss, J. and Tzafriri, L., *Classical Banach Spaces. Function Spaces*, Berlin: Springer, 1978.
2. Maligranda, L., *Orlicz Spaces and Interpolation: Seminars in Math.*, Campinas: Univ. Campinas, 1989, vol. 5.
3. O’Neil, R., *J. Anal. Math.*, 1968, vol. 21, pp. 1–276.
4. Milman, M., *Proc. Am. Math. Soc.*, 1981, vol. 83, pp. 743–746.
5. Astashkin, S.V., *Funkts. Anal. Ego Prilozh.*, 1996, vol. 30, no. 4, pp. 58–60.
6. Krein, S.G., Petunin, Yu.I., and Semenov, E.M., *Interpolyatsiya lineinykh operatorov* (Interpolation of Linear Operators), Moscow: Nauka, 1978.
7. Hernandez, F.L. and Semenov, E.M., *J. Funct. Anal.*, 1999, vol. 169, pp. 52–80.