IMAGES OF OPERATORS IN REARRANGEMENT INVARIANT SPACES AND INTERPOLATION

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For every closed set $G \subset [0,1]$ there exists an injective linear operator $T = T_G$ bounded on $L_1[0,1]$ and $L_\infty[0,1]$ such that T is normally solvable in an Orlicz space $L_M[0,1]$ if and only if $G \cap [\alpha_M, \beta_M] = \emptyset$ (α_M and β_M are the Boyd indices of the space L_M). Analogous result holds also for isomorphisms of spaces of functions defined on the semiaxis $[0,\infty)$.

1. Introduction

Let (X_0, X_1) be a Banach couple, i.e., X_0 and X_1 are two Banach spaces linearly and continuously embedded in some Hausdorff topological vector space. Suppose that a linear operator A defined on the sum $X_0 + X_1$ is a continuous injection from X_i into itself for $i = 0, 1$. Moreover, let A be normally solvable in the both spaces X_0 and X_1 (this means that the images $A(X_0)$ and $A(X_1)$ are closed subspaces of the spaces X_0 and X_1, respectively).

If F is an interpolation functor [1], then a natural formula to hope for is

$$F(A(X_0), A(X_1)) = A(F(X_0, X_1)) \quad \text{(isomorphism)}. \tag{1}$$

In fact, J. Löfström proved [2,p.57] that formula (1) is true for any functor F of the real method of interpolation [1] if

$$A(X_0 \cap X_1) = A(X_0) \cap A(X_1). \tag{2}$$

Nevertheless, equality (1) does not guarantee that A is a normally solvable operator in the interpolation space $F(X_0, X_1)$. The point is that the space $F(A(X_0), A(X_1))$ is not always a subspace of the last space.

At first, the situation when X_0 and X_1 are Hilbert spaces and $X_1 \subset X_0$ was considered (see, for example, [3,p.131] or [4,p.136,589]). In this case condition (2) is fulfilled, and therefore equality (1) is valid for any functor of the real method of interpolation. Later, in the paper [5], V. Shevchik and I. Shneiberg showed that for arbitrary closed subset F of the interval $(0, 1)$ there exist a Banach couple (H_0, H_1) of Hilbert spaces and a couple (N_0, N_1) of their subspaces such that the norms of lower complex method interpolation spaces $[N_0, N_1]_\theta$ and $[H_0, H_1]_\theta$ (see [1] or [6]) are non-equivalent if and only if $\theta \in F$.

Moreover, we cannot interpolate even the property of two-sided invertibility. If $A : X_0 + X_1 \to X_0 + X_1$ is an isomorphism on both spaces X_0 and X_1, then, clearly,

$$F(A(X_0), A(X_1)) \;=\; F(X_0, X_1) \quad \text{(isomorphism).}$$

At the same time the space $A(F(X_0, X_1))$ is not always a subspace of the space $F(X_0, X_1)$ in view of the failure of isomorphism (1), in general.

In [2,p.59-61], J. Löfström considered the operators $T_\lambda = \lambda I - S$ $(\lambda \in \mathbb{R})$, where I is the identity mapping and $Sa = (a_{n-1})_{n=-\infty}^{\infty}$ $(a = (a_n)_{n=-\infty}^{\infty})$. These operators act injectively and continuously on the sequence spaces E_r $(r \in \mathbb{R})$ with norms:

$$\|a\|_{E_r} = \left(\sum_{n=-\infty}^{\infty} (2^{nr}|a_n|)^2 \right)^{1/2} .$$

In addition, the image $T_\lambda(E_r) = E_r$ if and only if $|\lambda| \neq 2^r$. Let $1 \le p < r < q < \infty$ and F be the functor $\Phi_{\theta,2}$ of the real method of interpolation [1], where $\theta \in (0, 1)$ and $r = (1-\theta)p + \theta q$. Then (1) is not true for the operator $A = T_{2^r}$ and the spaces $X_0 = E_p$, $X_1 = E_q$ [1, p.157].

Another interesting example was treated by N.Krugljak, L.Maligranda, and L.-E. Persson in [7]. For arbitrary $p \in [1, \infty)$ and $\alpha \in \mathbb{R}$, denote by $G_{p,\alpha}$ the Banach space of all functions $x = x(t)$ measurable on $(0, \infty)$ such that

$$\|x\|_{p,\alpha} = \left\{ \int_0^\infty \left(\frac{|x(t)|}{t^\alpha} \right)^p \frac{dt}{t} \right\}^{1/p} < \infty.$$

For all $\alpha > -1$ the operator

$$Ax(t) = \; x(t) - \frac{1}{t} \int_0^t x(s)\, ds$$

acts continuously from $G_{p,\alpha}$ into itself and has a bounded inverse one if and only if $\alpha \in (-1, 0) \cup (0, \infty)$. Therefore equality (1) fails for $X_0 = G_{p,\alpha_0}$, $X_1 = G_{p,\alpha_1}$ $(-1 < \alpha_0 < 0 < \alpha_1 < \infty)$, and an appropriate functor of the real or complex method of interpolation [1, p.154-155].

An investigation of normally solvability and invertibility of functional operators in various spaces plays an important role in studying of Fredholm property and $n(d)$ -normality of singular integral operators, stability of functional-differential equations, in the theory of dynamic systems, etc. (see [8] and references therein).

We will consider here these problems in the setting of general rearrangement invariant spaces and, especially, of Orlicz spaces.

For any linear operator T defined on the space $L_1[0,1]$ we introduce the set of exceptional points

$$\kappa(T) = \{\gamma \in [0,1] : T \text{ isn't normally solvable in } L_{1/\gamma}\}.$$

We will show that for arbitrary closed set $G \subset [0,1]$ there exists an injective linear operator $T = T_G$ such that T acts continuously on every space $L_p[0,1]$ $(1 \le p \le \infty)$ and $\kappa(T) = G$. Moreover, we will prove that the image $T(L_M)$ of any Orlicz space L_M on the segment $[0,1]$ is closed in L_M if and only if $G \cap [\alpha_M, \beta_M] = \emptyset$ (α_M and β_M are the Boyd indices of the space L_M). Analogous results hold also for isomorphisms of spaces of functions measurable on the semiaxis $[0,\infty)$.

We remind now some definitions from the theory of rearrangement invariant spaces (for the detailed exposition of their properties see, for example, [6] or [9]).

For every function $x = x(t)$ measurable on $[0,1]$, $x(t) \ge 0$, its distribution function $n_x(\tau)$ and non-increasing rearrangement $x^*(t)$ are given by formulas

$$n_x(\tau) = m\{t \in [0,1] : x(t) > \tau\}, \quad \text{for } \tau > 0,$$

and

$$x^*(t) = \inf\{\tau \ge 0 : n_x(\tau) < t\}, \quad \text{for } t \in [0,1].$$

Here m is usual Lebesgue measure. Two non-negative measurable functions x and y are called equimeasurable if $n_x(\tau) = n_y(\tau)$ $(\tau > 0)$. In particular, any $x(t) \ge 0$ and its rearrangement $x^*(t)$ are equimeasurable functions. By $x^*(t)$ we will denote, also, the non-increasing rearrangement of $|x(t)|$ for arbitrary measurable function $x(t)$.

A Banach space E of functions measurable on $[0,1]$ is called a rearrangement invariant space (RIS) if

1) the relations $|y(t)| \le |x(t)|$ and $x(t) \in E$ imply that $y(t) \in E$ and $\|y\| \le \|x\|$;

2) if $|y(t)|$ and $|x(t)|$ are equimeasurable functions and $x(t) \in E$ then $y(t) \in E$ and $\|y\| = \|x\|$.

For every $\tau > 0$, the dilation operator σ_τ given by

$$\sigma_\tau x(t) = \ x(t/\tau)\, \chi_{[0,\min(1,\tau)]}(t)$$

s well defined in every RIS E and $\|\sigma_\tau\|_{E\to E} \le \max(1,\tau)$. Here and next, we write $\chi_F(t)$ for the characteristic function of a measurable set F. At last, define the lower and upper Boyd indices of an RIS E

$$\alpha_E = \lim_{\tau\to 0+} \frac{\ln \|\sigma_\tau\|_{E\to E}}{\ln \tau} \ \text{ and } \ \beta_E = \lim_{\tau\to +\infty} \frac{\ln \|\sigma_\tau\|_{E\to E}}{\ln \tau}.$$

We have $0 \le \alpha_E \le \beta_E \le 1$ for every RIS E.

An important example of RIS's is an Orlicz space. If M is a non-negative convex function on $[0,\infty)$ and $M(0) = 0$, then the Orlicz space L_M (cf. [10], [11]) consists of all measurable functions $x(t)$ on $[0,1]$ for which the norm

$$\|x\|_M \ = \ \inf\Big\{\lambda > 0 : \ \int_0^1 M\Big(\frac{|x(t)|}{\lambda}\Big)\, dt \ \le \ 1\Big\}$$

is finite. The Boyd indices of an Orlicz space L_M are calculated as follows [9,p.139]

$$\alpha_M \ = \ \sup\Big\{\gamma > 0 : \ \sup_{u,v\ge 1} \frac{M(uv)}{M(u)v^{1/\gamma}} \ < \infty\Big\}$$

and

$$\beta_M \ = \ \inf\Big\{\gamma > 0 : \ \inf_{u,v\ge 1} \frac{M(uv)}{M(u)v^{1/\gamma}} \ > 0\Big\}.$$

In particular, for $M(t) = t^p$ we obtain L_p -spaces with usual norms, and $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $1 \le p < \infty$.

Similarly, it can be defined RIS's and Orlicz spaces of functions measurable on the semiaxis $[0,\infty)$ and their Boyd indices (see [6], [9]).

2. Auxiliary Results

Lemma 2.1. *Let E be an RIS on $[0,1]$. If $|\lambda| < 2^{-\beta_E}$, then the operator*

$$L_\lambda y(t) \ = \ \sum_{k=0}^{\infty} \lambda^k y(t2^{-k}) \tag{3}$$

acts continuously on E.

Proof. By the definition of the Boyd indices, for arbitrary $\varepsilon > 0$ there exists $C_1 = C_1(\varepsilon) > 0$ such that

$$\|\sigma_\tau\|_{E\to E} \ \le \ C_1 \tau^{\beta_E+\varepsilon}, \quad \text{for all } \ \tau \ge 1. \tag{4}$$

Take $\varepsilon > 0$ such that

$$|\lambda| 2^{\beta_E + \varepsilon} \;<\; 1. \tag{5}$$

By (4) and (5), we have

$$\|L_\lambda y\|_E \;\le\; C_1 \sum_{k=0}^{\infty} |\lambda|^k (2^k)^{\beta_E+\varepsilon} \|y\|_E \;=\; \frac{C_1}{1 - |\lambda| 2^{\beta_E+\varepsilon}} \|y\|_E.$$

Hence the series from (3) converges absolutely a.e., and

$$\|L_\lambda\|_{E\to E} \;\le\; C_1 (1 - |\lambda| 2^{\beta_E+\varepsilon})^{-1}. \qquad \square$$

Let us define the operators

$$T_\lambda x(t) \;=\; x(2t)\chi_{[0,1/2]}(t) \;-\; \lambda x(t) \;\; (\lambda \in \mathbb{R}) \tag{6}$$

for all functions $x = x(t)$ measurable on the segment $[0,1]$. It is easily shown that every T_λ acts continuously on any RIS E on $[0,1]$. Moreover, it is not hard to check that T_λ is an injection from any RIS E into itself.

Let $\Delta_M = [2^{-\beta_M}, 2^{-\alpha_M}]$, where α_M and β_M are the Boyd indices of an Orlicz space L_M. By $\rho(z,F)$ we will denote the distance from a point $z \in \mathbb{R}$ to a set $F \subset \mathbb{R}$, i.e., $\rho(z,F) \;=\; \inf_{a\in F} |z-a|$.

Lemma 2.2. *Let L_M be an Orlicz space on $[0,1]$, $\lambda \in \mathbb{R}$, and $\rho(|\lambda|, \Delta_M) = \delta$. Then for every $l = 1,2,..$ there exists a sequence $\{w_k\}_{k=1}^{\infty} \subset L_M$ such that* $\operatorname{supp} w_k = \{t \in [0,1] : w_k(t) \ne 0\} \subset [0, 2^{-l}]$,

$$\frac{1}{2} \;\le\; \|w_k\|_M \;\le\; 2 \;\; (k = 1,2,..), \tag{7}$$

and

$$\|T_\lambda w_k\|_M \;\le\; 2\delta \,+\, \max(1,|\lambda|)\alpha_k, \quad \textit{where} \;\; \alpha_k \to 0. \tag{8}$$

Proof. At first, suppose that $\delta = |\lambda - \mu|$, for $\mu < 1$ $(\mu \in \Delta_M)$. Then $\mu = 2^{-1/q}$, where q is a finite number from the segment $[\beta_M^{-1}, \alpha_M^{-1}]$.

The Boyd indices α_M and β_M of an Orlicz space L_M can be characterized in the following way [12,4.a.9] (cf. [9,p.141]): for every $p \in [\beta_M^{-1}, \alpha_M^{-1}]$, $\varepsilon > 0$, and positive integer n, L_M contains a $(1+\varepsilon)$ -copy of $l_p(n)$ generated by n disjointly supported equimeasurable functions in L_M. Therefore for every $m = 1,2,..$ there exist functions x_k $(k = 1,2,.,m)$ such that

$$\operatorname{supp} x_k \;\subset\; [(k-1)/m, k/m] \;,\; n_{|x_k|}(\tau) \;=\; n_{|x_1|}(\tau) \;\; (\tau > 0), \tag{9}$$

and for any $a_k \in \mathbb{R}$

$$\frac{1}{2}\left(\sum_{k=1}^{m} |a_k|^q\right)^{1/q} \;\le\; \left\|\sum_{k=1}^{m} a_k x_k\right\|_M \;\le\; 2\left(\sum_{k=1}^{m} |a_k|^q\right)^{1/q}. \tag{10}$$

Let $m = 2^n$, where $n \geq l$. If $\{x_k\}_{k=1}^{2m}$ is a collection satisfying conditions (9) and (10), then put

$$y_0 = x_1 \ , \ \ y_s = \sum_{i=2^{s-1}+1}^{2^s} x_i \quad (s = 1, 2, ., n-l).$$

By (9), y_s are disjointly supported functions, supp $y_s \subset [0, 2^{-l}]$, and

$$n_{|y_s|}(\tau) \ = \ 2^{s-1} n_{|y_0|}(\tau) \ \ (\tau > 0), \quad \text{for every} \ \ s = 1, 2, ., n-l. \tag{11}$$

Then

$$z_0(t) = \ y_0^*(t) \ , \ z_s(t) = \ z_0(2^{1-s}t - 2^{-n}) \ \ (s = 1, 2, ., n-l)$$

are disjointly supported functions, and supp $z_s \subset [0, 2^{-l}]$. In addition, relation (11) implies that $n_{z_s}(\tau) \ = \ n_{|y_s|}(\tau) \ \ (\tau > 0)$, for any $s = 0, 1, 2, ., n-l$.

Since L_M is an RIS, then

$$\Big\|\sum_{s=0}^{n-l} b_s z_s\Big\|_M \ = \ \Big\|\sum_{s=0}^{n-l} b_s y_s\Big\|_M, \quad \text{for all} \ \ b_s \in \mathbb{R}. \tag{12}$$

Hence, if $\bar{z}_0(t) = \ z_0(t)$ and $\bar{z}_s(t) = \ 2^{(1-s)/q} z_s(t)$, for $s = 1, 2, ., n-l$, then from (10) we get

$$\frac{1}{2} \leq \|\bar{z}_s\|_M \leq 2 \ \text{ and } \ \frac{1}{2}(n-l+1)^{1/q} \leq \left\|\sum_{s=0}^{n-l} \bar{z}_s\right\|_M \leq 2(n-l+1)^{1/q}. \tag{13}$$

In particular, for every $n \geq l$ the function

$$v_n(t) \ = \ (n-l+1)^{-1/q} \sum_{s=0}^{n-l} \bar{z}_s(t)$$

satisfies inequality (7), and supp $v_n \subset [0, 2^{-l}]$.

Next, by the definition of $\bar{z}_s$, we have

$$\bar{z}_{s+1}(2t) \ = \ 2^{-1/q} \bar{z}_s(t), \quad \text{for} \quad t \in [0, 1/2] \quad \text{and} \quad s = 1, 2, ., n-l-1.$$

In view of (6), it follows that

$$T_\lambda v_n(t) \ = \ (n-l+1)^{-1/q} \,[\sum_{s=0}^{n-l} \bar{z}_s(2t)\chi_{[0,1/2]}(t) \ - \ \lambda \sum_{s=0}^{n-l} \bar{z}_s(t)] \ =$$

$$= (n-l+1)^{-1/q} \big[\bar{z}_0(2t) + \bar{z}_1(2t) - \lambda(\bar{z}_0(t) + \bar{z}_{n-l}(t)) + (\mu - \lambda) \sum_{s=1}^{n-l-1} \bar{z}_s(t)\big].$$

Hence relations (13) imply

$$\|T_\lambda v_n\|_M \ \leq \ 2\delta \ + \ 8 \max(1, |\lambda|)(n-l+1)^{-1/q},$$

and the functions $w_k = v_{k+l}$ $(k = 1, 2, ..)$ satisfy (7) and (8).

Suppose now $\delta = |\lambda - 1|$ (in this case $\alpha_M = 0$). Again, for any $m = 1, 2, ..$ we can find functions x_k $(k = 1, 2, ., m)$ satisfying relations (9) such that for arbitrary $a_k \in \mathbb{R}$

$$\frac{1}{2} \max_{k=1,2,.,m} |a_k| \ \leq \ \left\| \sum_{k=1}^{m} a_k x_k \right\|_M \ \leq \ 2 \max_{k=1,2,.,m} |a_k|. \tag{14}$$

Let $m = 2^n$, where $n > l + 6$ and $n - l$ is an odd positive integer. In addition, let the functions $z_s(t)$ $(s = 0, 1, 2, ., n-l)$ be such as above. Define the functions $u_n(t) \ = \ \sum_{s=1}^{n-l-1} \tilde{z}_s(t)$, where

$$\tilde{z}_s(t) = \frac{2s}{n-l} z_s(t), \quad \text{if } 0 \leq s < \frac{n-l}{2},$$

and

$$\tilde{z}_s(t) = \frac{2(n-l-s)}{n-l} z_s(t), \quad \text{if } \frac{n-l}{2} < s \leq n-l.$$

Since

$$z_{s+1}(2t) \ = \ z_s(t), \quad \text{for} \quad t \in [0, 1/2] \quad \text{and} \quad s = 1, 2, ., n-l-1,$$

then we obtain

$$T_\lambda u_n(t) \ = \ \sum_{s=1}^{n-l-1} \tilde{z}_s(2t) \chi_{[0,1/2]}(t) \ - \lambda \sum_{s=1}^{n-l-1} \tilde{z}_s(t) \ =$$

$$= \ \tilde{z}_1(2t) \ - \ \tilde{z}_{n-l-1}(t) \ + \ \sum_{s=1}^{n-l-2} [\tilde{z}_{s+1}(2t) \ - \ \tilde{z}_s(t)] \ + \ (1 - \lambda) \sum_{s=1}^{n-l-1} \tilde{z}_s(t) \ =$$

$$= \ \frac{2}{n-l} [z_1(2t) \ + \ (1-\lambda) \sum_{s=1}^{(n-l-3)/2} z_s(t) \ - \ (1-\lambda) \sum_{s=(n-l+1)/2}^{n-l-2} z_s(t) -$$

$$- \, z_{n-l-1}(t)] \ + \ (1-\lambda) \sum_{s=1}^{n-l-1} \tilde{z}_s(t).$$

Therefore, by the definition of functions $\tilde{z}_s$, (12), and (14), we have

$$\|T_\lambda u_n\|_M \ \leq \ 2\delta \ + \ \frac{12 \max(1, |\lambda|)}{n-l},$$

for all natural n such that $n - l$ is an odd positive integer which is more than 6. If $w_k = u_{2k+l+5}$ $(k = 1, 2, ..)$, then the last inequality implies

$$\|T_\lambda w_k\|_M \ \leq \ 2\delta \ + \ \frac{12 \max(1, |\lambda|)}{2k+5},$$

and w_k satisfy relations (7) and (8).

Let us consider now the case $\lambda < 0$. Then $\delta = |\lambda + \mu|$, where $\mu \in \Delta_M$. Arguing as above, we define, instead of $z_s(t)$, the functions

$$z'_0(t) = y^*_0(t) \quad \text{and} \quad z'_s(t) = (-1)^s z_0(2^{1-s}t - 2^{-n}) \;\; (s = 1, 2, ., n-l).$$

At first, suppose $\mu = 2^{-1/q} < 1$. Define the functions

$$\bar{z}'_s(t) = 2^{(1-s)/q} z'_s(t) \quad \text{and} \quad v'_n(t) = (n-l+1)^{-1/q} \sum_{s=0}^{n-l} \bar{z}'_s(t) \quad \text{for} \;\; n \geq l.$$

The equalities

$$\bar{z}'_{s+1}(2t) \;=\; -2^{-1/q} \bar{z}'_s(t), \quad \text{for} \quad t \in [0, 1/2] \quad \text{and} \quad s = 1, 2, ., n-l-1,$$

yield

$$T_\lambda v'_n(t) \;=\; (n-l+1)^{-1/q} [\bar{z}'_0(2t) + \bar{z}'_1(2t) -$$

$$-\lambda(\bar{z}'_0(t) + \bar{z}'_{n-l}(t)) - (\mu + \lambda) \sum_{s=1}^{n-l-1} \bar{z}'_s(t)].$$

Since the relations similar to (13) are fulfilled for $\bar{z}'_s(t)$ also, then we have, hence, that the functions $w_k = v'_{k+l}$ $(k = 1, 2, ..)$ satisfy conditions (7) and (8).

In the case $\mu = 1$ for every $n > l + 6$ such that $n - l$ is an odd positive integer we put $u'_n(t) = \sum_{s=1}^{n-l-1} \tilde{z}'_s(t)$, where

$$\tilde{z}'_s(t) = \frac{2s}{n-l} z'_s(t), \quad \text{if} \;\; 0 \leq s < \frac{n-l}{2},$$

and

$$\tilde{z}'_s(t) = \frac{2(n-l-s)}{n-l} z'_s(t), \quad \text{if} \;\; \frac{n-l}{2} < s \leq n-l.$$

Since

$$z'_{s+1}(2t) \;=\; -z'_s(t), \quad \text{for} \quad t \in [0, 1/2] \quad \text{and} \quad s = 1, 2, ., n-l-1,$$

then, arguing as above, we get that the functions $w_k = u'_{2k+l+5}$ $(k = 1, 2, ..)$ satisfy conditions (7) and (8). □

Corollary 2.1. *If* $|\lambda| \in \Delta_M$, *then the operator* T_λ *is not normally solvable in an Orlicz space* L_M.

Proof. In this case $\delta = 0$. Therefore, by Lemma 2.2, there exists a sequence of functions $\{w_k\} \subset L_M$ such that

$$\frac{1}{2} \ \leq \ \|w_k\|_M \ \leq \ 2 \ \ (k = 1, 2, ..) \quad \text{and} \quad \|T_\lambda w_k\|_M \to 0 \quad \text{as} \ \ k \to \infty. \tag{15}$$

Suppose that $T_\lambda(L_M)$ is a closed subspace of L_M. As we have already remarked, T_λ is an injective operator in every RIS. Hence, $\|T_\lambda x\|_M \geq c\|x\|_M$, for some $c > 0$. Since the last inequality contradicts (15), then the result follows. □

3. Main Results

It is clear that the projector $P_1 x(t) = x(t)\chi_{[1/2,1]}(t)$ has norm 1 on any RIS.

Theorem 3.1. *1) T_λ is a normally solvable operator in an RIS E on* $[0, 1]$ *whenever $|\lambda| \notin [2^{-\beta_E}, 2^{-\alpha_E}]$. Moreover, if $|\lambda| > 2^{-\alpha_E}$, then $T_\lambda(E) = E$; if $0 < |\lambda| < 2^{-\beta_E}$, then $T_\lambda(E) = \operatorname{Ker}(P_1 L_\lambda)$; and $T_0(E) = \operatorname{Ker} P_1 = \{y \in E :$* supp $y \subset [0, 1/2]\}$;

2) T_λ is a normally solvable operator in an Orlicz space L_M on $[0, 1]$ *if and only if $|\lambda| \notin \Delta_M$, where $\Delta_M = [2^{-\beta_M}, 2^{-\alpha_M}]$.*

Proof. At first, suppose $|\lambda| > 2^{-\alpha_E}$. Consider the following equation for $x(t)$:

$$T_\lambda x(t) \ = \ y(t),$$

or

$$x(2t)\chi_{[0,1/2]}(t) \ - \ \lambda x(t) \ = \ y(t), \tag{16}$$

where $y = y(t) \in E$. Hence, $x(t) = -\lambda^{-1} y(t)$ $(1/2 < t \leq 1)$. Besides, from (16) it follows

$$\lambda^{-1} x(2^2 t)\chi_{[0,2^{-2}]}(t) \ - \ x(2t)\chi_{[0,1/2]}(t) \ = \ \lambda^{-1} y(2t)\chi_{[0,1/2]}.$$

Summing the last equality and (16), we obtain

$$\lambda^{-1} x(2^2 t)\chi_{[0,2^{-2}]}(t) \ - \ \lambda x(t) \ = \ y(t) \ + \ \lambda^{-1} y(2t)\chi_{[0,1/2]}.$$

Hence,

$$x(t) \ = \ -\lambda^{-1}(y(t) \ + \ \lambda^{-1} y(2t)) \quad (2^{-2} < t \leq 2^{-1}).$$

Arguing as above, we get

$$x(t) = \ -\lambda^{-1} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \lambda^{1-k} y(2^{k-1} t)\, \chi_{(2^{-n}, 2^{1-n}]}(t) \ =$$

$$= -\lambda^{-1}\sum_{n=1}^{\infty}\lambda^{1-n}y(2^{n-1}t)\,\chi_{(0,2^{1-n}]}(t) \quad (0\le t\le 1). \tag{17}$$

Let $\varepsilon > 0$ be arbitrary. By the definition of the Boyd indices, there exists a constant $C_2 = C_2(\varepsilon) > 0$ such that

$$\|\sigma_\tau\|_{E\to E} \;\le\; C_2\tau^{\alpha_E-\varepsilon}, \quad \text{for } 0<\tau\le 1. \tag{18}$$

Take $\varepsilon > 0$ such that $|\lambda|2^{\alpha_E-\varepsilon} \;>\; 1$. Then relations (17) and (18) imply

$$\|x\|_E \;\le\; |\lambda|^{-1}\sum_{n=1}^{\infty}|\lambda|^{1-n}\|\sigma_{2^{1-n}}y\|_E \;\le$$

$$\le\; C_2|\lambda|^{-1}\sum_{n=1}^{\infty}\left(|\lambda|2^{\alpha_E-\varepsilon}\right)^{1-n}\|y\|_E \;=\; \frac{C_2 2^{\alpha_E-\varepsilon}}{|\lambda|2^{\alpha_E-\varepsilon}-1}\|y\|_E,$$

and so $x \in E$. It is easily shown that (17) implies $y = T_\lambda x$. Therefore, $\operatorname{Im} T_\lambda = E$.

Suppose now $0 < |\lambda| < 2^{-\beta_E}$. For $0 < t \le 1$, from (16) it follows

$$x(t) \,-\, \lambda x(2^{-1}t) \;=\; y(2^{-1}t) \quad\text{and}\quad \lambda x(2^{-1}t) \,-\, \lambda^2 x(2^{-2}t) \;=\; \lambda y(2^{-2}t).$$

Summing these equalities, we get $x(t) - \lambda^2 x(2^{-2}t) \;=\; y(2^{-1}t) + \lambda y(2^{-2}t)$. Arguing as above, we have

$$x(t) \,-\, \lambda^n x(2^{-n}t) \;=\; \sum_{k=1}^{n}\lambda^{k-1}y(2^{-k}t), \quad \text{for arbitrary } n\in\mathbb{N}.$$

If $\varepsilon > 0$ satisfies inequality (5), then from (4) it follows $\lambda^n x(2^{-n}t) \to 0$ as $n\to\infty$ (convergence in E). Therefore the last equality implies

$$x(t) \;=\; \sum_{k=1}^{\infty}\lambda^{k-1}y(2^{-k}t) \quad (\text{convergence in } E). \tag{19}$$

Thus, if a solution of equation (16) exists, then it is defined by (19). The direct substitution shows that (19) is a solution of (16) if and only if $P_1L_\lambda y(t) \;=\; 0$. In other words, in the case $0 < |\lambda| < 2^{-\beta_E}$ equation (16) has a solution $x \in E$ if and only if $y \in \operatorname{Ker}(P_1L_\lambda)$. Therefore, $T_\lambda(E) = \operatorname{Ker}(P_1L_\lambda)$, and, by Lemma 2.1, this subspace is closed in E.

Since the case $\lambda = 0$ is trivial, the first assertion of Theorem 3.1 is proved. The second one follows now from Corollary 2.1. □

Similarly, on the set of all functions $x = x(t)$ measurable on the semiaxis $[0,\infty)$ define the operators

$$\tilde{T}_\lambda x(t) \;=\; x(2t) \,-\, \lambda x(t) \quad (\lambda\in\mathbb{R}).$$

It is not hard to check that $\tilde{T}_\lambda$ is an injection from the space $L_p = L_p[0,\infty)$ into itself for any $\lambda \in \mathbb{R}$ and $p \in [1,\infty)$.

Arguing as in the proof of Theorem 3.1, one can prove the following assertion.

Theorem 3.2. *1) $\tilde{T}_\lambda$ is an isomorphism of an RIS E on $[0,\infty)$ whenever $|\lambda| \notin [2^{-\beta_E}, 2^{-\alpha_E}]$.*

2) $\tilde{T}_\lambda$ is a normally solvable operator (or an isomorphism) on an Orlicz space L_M on $[0,\infty)$ if and only if $|\lambda| \notin \Delta_M$, where $\Delta_M = [2^{-\beta_M}, 2^{-\alpha_M}]$.

Remark 3.1. Theorem 3.1 shows that we cannot interpolate the property of normally solvability of operators even in the case of embedded RIS's. In fact, if $1 \le p_0 < p < p_1 \le \infty$ and $\lambda = 2^{-1/p}$, then the image $T_\lambda(L_{p_i})$ is closed in the space L_{p_i} $(i = 0,1)$, but the image $T_\lambda(L_p)$ is not closed in L_p. The point is that not all interpolation spaces for the couple $(L_{p_0}, T_\lambda(L_{p_1}))$ (by Theorem 1, $T_\lambda(L_{p_0}) = L_{p_0}$) are subspaces of corresponding interpolation spaces for the couple (L_{p_0}, L_{p_1}).

In the case of RIS's on the semiaxis $[0,\infty)$ we cannot interpolate the property of two-sided invertibility. Indeed, by Theorem 3.2, if $1 \le p_0 < p < p_1 \le \infty$ and $\lambda = 2^{-1/p}$, then $\tilde{T}_\lambda$ is an isomorphism in the spaces L_{p_0} and L_{p_1}, but not in L_p. We cannot interpolate the inverse operator $\tilde{T}_\lambda^{-1}$, because $\tilde{T}_\lambda$ is not an injection as operator defined on the sum $L_{p_0}(0,\infty) + L_{p_1}(0,\infty)$. This implies that the inverse operator (on this sum) does not exist. Equivalently, equality (2) fails for $A = \tilde{T}_\lambda$, $X_0 = L_{p_0}(0,\infty)$, and $X_1 = L_{p_1}(0,\infty)$ (cf. [2,p.57]).

Theorem 3.3. *Let $G \subset [0,1]$ be a closed set. Then there exists an injective linear operator $T = T_G$ such that T acts continuously on every Orlicz space and has the following property:*

T is normally solvable in an Orlicz space L_M if and only if

$$G \cap [\alpha_M, \beta_M] \ = \ \emptyset. \tag{20}$$

Proof. Let the set F consist of all $\lambda = 2^{-t}$, where $t \in G$, and $\{\lambda_k\}_{k=1}^\infty$ be a sequence from F such that its closure $\overline{\{\lambda_k\}} = F$. Clearly, (20) is equivalent to the condition

$$F \cap \Delta_M \ = \ \emptyset. \tag{20'}$$

For any $k = 1,2,..$ and $i = 0,1,2,..,$ denote $d_k^i = (2^{-k}, (2^i+1)2^{-k-i}]$. Then d_k^0 are pairwise disjoint intervals, and $(0,1] = \cup_{k=1}^\infty d_k^0$. Next, let h_k be an affine map from d_k^1 onto d_k^0, i.e., $h_k(t) = 2t - 2^{-k}$ for $t \in d_k^1$, $h_k^{-1} : d_k^0 \to d_k^1$

be the inverse map of h_k, and $h_k^0(t) = t$ $(t \in d_k^0)$ $(k = 1, 2, ..)$. In addition, for every $i = 1, 2, ..$ we introduce the maps

$$h_k^i(t) = \underbrace{h_k(h_k(\cdots(h_k}_{i}(t))\cdots)) \text{ and } h_k^{-i}(t) = \underbrace{h_k^{-1}(h_k^{-1}(\cdots(h_k^{-1}}_{i}(t))\cdots)).$$

It is obvious that $h_k^i : d_k^i \to d_k^0$ and $h_k^{-i} : d_k^0 \to d_k^i$. Denote, also, by a_k^i the extension of h_k^i to the whole segment $[0, 1]$ by zero $(k = 1, 2, ..; i = 0, \pm 1, \pm 2, \cdots)$. At last, we define the projectors $P_k x(t) = x(t)\chi_{d_k^0}(t)$ $(k = 1, 2, ..)$ and the operator

$$Tx(t) = \sum_{k=1}^{\infty} T_k x(t), \quad \text{where } T_k x(t) = P_k x(a_k^1(t)) - \lambda_k P_k x(t).$$

Since $T_k x(t)$ are pairwise disjoint functions for every $x \in L_1$, then

$$\|Tx\|_1 = \sum_{k=1}^{\infty} \|T_k x\|_1 \le 2 \sum_{k=1}^{\infty} \|P_k x\|_1 = 2\|x\|_1$$

and

$$\|Tx\|_\infty = \sup_{k=1,2,..} \|T_k x\|_\infty \le 2 \sup_{k=1,2,..} \|P_k x\|_\infty = 2\|x\|_\infty,$$

i.e., T acts continuously on the spaces L_1 and L_∞. By the interpolation theorem [6,p.142], $T : L_M \to L_M$ is a bounded operator on every Orlicz space L_M. Also, it is easily to check that this operator is injective in every RIS.

At first, we prove that the image $T(L_M)$ is closed in an Orlicz space L_M whenever condition (20) (or (20′)) is fulfilled. Since F is a closed set, there exists $\varepsilon > 0$ such that any positive integer belongs to the one of the sets $K_1 = \{k \in \mathbb{N} : \lambda_k \ge 2^{-\alpha_M + 2\varepsilon}\}$ or $K_2 = \{k \in \mathbb{N} : \lambda_k \le 2^{-\beta_M - 2\varepsilon}\}$. For arbitrary $x \in L_1[0, 1]$ we put

$$x = \pi_1 x + \pi_2 x, \quad \text{where } \pi_i x(t) = \sum_{k \in K_i} P_k x(t) \ \ (i = 1, 2).$$

Let us fix a $k = 1, 2, \cdots$ and consider the following equation for $P_k x$:

$$P_k x(a_k^1(t)) - \lambda_k P_k x(t) = P_k y(t) \quad (y \in E). \tag{21}$$

Assume that $k \in K_1$. In the same way as in the proof of Theorem 3.1, we obtain

$$P_k x(t) = -\sum_{n=1}^{\infty} \lambda_k^{-n} P_k y(a_k^{n-1}(t))\chi_{d_k^{n-1}}(t).$$

By the definition of K_1,

$$|\pi_1 x(t)| \ \le \ \sum_{k\in K_1} |P_k x(t)| \ \le \ \sum_{n=1}^{\infty} \big(2^{-\alpha_M+2\varepsilon}\big)^{-n} g_n^1(t),$$

where

$$g_n^1(t) \ = \ \sum_{k\in K_1} |P_k y(a_k^{n-1}(t))| \chi_{d_k^{n-1}}(t).$$

Since $g_n^1(t)$ and $\sigma_{2^{1-n}}|\pi_1 y(t)|$ are equimeasurable functions for every $n\in\mathbb{N}$, then from (18) it follows

$$\|\pi_1 x\|_M \ \le \ C_2 \sum_{n=1}^{\infty} \big(2^{-\alpha_M+2\varepsilon}\big)^{-n} \big(2^{1-n}\big)^{\alpha_M-\varepsilon} \|\pi_1 y\|_M \ \le \ \frac{2C_2}{2^{\varepsilon}-1}\|y\|_M. \tag{22}$$

Next, let $k\in K_2$. Arguing as in the proof of Theorem 3.1 in the case $0<|\lambda|<2^{-\beta_M}$, we conclude that a solution of equation (21) must be of the form

$$P_k x(t) \ = \ \sum_{n=1}^{\infty} \lambda_k^{n-1} P_k y(a_k^{-n}(t)). \tag{23}$$

By the definition of K_2,

$$|\pi_2 x(t)| \ \le \ \sum_{k\in K_2} |P_k x(t)| \ \le \ \sum_{n=1}^{\infty} \big(2^{-\beta_M-2\varepsilon}\big)^{n-1} g_n^2(t),$$

where

$$g_n^2(t) \ = \ \sum_{k\in K_2} |P_k y(a_k^{-n}(t))|.$$

Again, $g_n^2(t)$ and $\sigma_{2^n}|\pi_2 y(t)|$ are equimeasurable functions ($n\in\mathbb{N}$), and from (4) it follows

$$\|\pi_2 x\|_M \ \le \ C_1 \sum_{n=1}^{\infty} \big(2^{-\beta_M-2\varepsilon}\big)^{n-1} \big(2^{\beta_M+\varepsilon}\big)^{n} \|\pi_2 y\|_M \ \le$$

$$\le \ 4C_1 \sum_{n=1}^{\infty} 2^{-n\varepsilon} \|\pi_2 y\|_M \ \le \ \frac{4C_1}{2^{\varepsilon}-1}\|y\|_M. \tag{24}$$

Moreover, it is not hard to check that function (23) is a solution of equation (21) as $k\in K_2$ if and only if a function y from L_M satisfies the condition

$$\sum_{n=0}^{\infty} \lambda^n P_k y(a_k^{-n}(t)) \ = \ 0 \quad \text{for} \ \ t\in d_k^0\setminus d_k^1.$$

Hence the image $T(L_M)$ consists of all $y \in L_M$ for which the last equality is fulfilled for every $k \in K_2$. Combining (22) and (24), we get

$$\|x\|_M \;\le\; \|\pi_1 x\|_M \;+\; \|\pi_2 x\|_M \;\le\; \frac{2(2C_1 + C_2)}{2^\varepsilon - 1}\|y\|_M,$$

for an unique solution x of the equation $Tx = y$. Thus the image $T(L_M)$ is closed in the space L_M.

Assume now that relation (20) is not fulfilled. In this case there exists a $\lambda \in \Delta_M$ such that

$$\lambda \;=\; \lim_{i\to\infty} \lambda_{k_i}, \tag{25}$$

for some sequence $\{\lambda_{k_i}\} \subset \{\lambda_k\}$. By Lemma 2.2, for every $i = 1, 2, ..$ there is a sequence $\{w_s^i\}_{s=1}^\infty \subset L_M$ such that for all $s = 1, 2, \cdots$ supp $w_s^i \subset [0, 2^{-k_i}]$,

$$\frac{1}{2} \;\le\; \|w_s^i\|_M \;\le\; 2, \tag{26}$$

and

$$\|T_{\lambda_{k_i}} w_s^i\|_M \;\le\; 2\delta_i \;+\; \alpha_s^i, \tag{27}$$

where $\delta_i = \rho(\lambda_{k_i}, \Delta_M)$ and $\alpha_s^i \to 0$ as $s \to \infty$.

For every $i = 1, 2, ..$ we can choose s_i such that $\alpha_{s_i}^i < 1/i$. By (25), $\lim_{i\to\infty} \delta_i = 0$. Therefore (27) implies

$$\lim_{i\to\infty} \|T_{\lambda_{k_i}} w_{s_i}^i\|_M \;=\; 0. \tag{28}$$

Let $v_i(t) = w_{s_i}^i(t - 2^{-k_i})$. Taking into account (26), we get

$$\frac{1}{2} \;\le\; \|v_i\|_M \;\le\; 2 \quad (i = 1, 2, ..). \tag{29}$$

In addition, supp $v_i \subset d_{k_i}^0 = [2^{-k_i}, 2^{1-k_i}]$, and thus

$$Tv_i(t) \;=\; T_{k_i} v_i(t) \;=\; v_i(a_{k_i}^1(t)) \;-\; \lambda_{k_i} v_i(t).$$

By the definition of $a_k^1(t)$, we see that $Tv_i(t)$ and $T_{\lambda_{k_i}} w_{s_i}^i(t)$ are equimeasurable functions. Therefore, by (28),

$$\lim_{i\to\infty} \|Tv_i\|_M \;=\; 0.$$

Combining the last relation and (29), we conclude that the image $T(L_M)$ is not closed in L_M. This completes the proof. $\square$

For any linear operator T that is defined on the space $L_1[0, 1]$ we introduce the set of exceptional points

$$\kappa(T) = \;\{\gamma \in [0, 1] : T \;\text{ isn't normally solvable in } L_{1/\gamma}\}.$$

By Theorem 3.3, we get the folowing assertion.

Corollary 3.1. *For arbitrary closed set $G \subset [0,1]$ there exists an injective linear operator $T = T_G$ such that T is bounded on every space $L_p[0,1]$ $(1 \le p \le \infty)$ and $\kappa(T) = G$.*

Analogous theorems hold also for spaces on the semiaxis $[0,\infty)$.

Theorem 3.4. *Let $G \subset [0,1]$ be a closed set. Then there exists a linear operator $U = U_G$ defined on the sum $L_1[0,\infty) + L_\infty[0,\infty)$ such that U acts continuously on every Orlicz space and has the following property:*

if L_M is an Orlicz space on $[0,\infty)$, then the operator U is an isomorphism of L_M if and only if

$$G \cap [\alpha_M, \beta_M] \ = \ \emptyset.$$

For arbitrary linear operator U that is defined on the sum $L_1[0,\infty) + L_\infty[0,\infty)$ we introduce the set of exceptional points

$$\tilde{\kappa}(U) = \ \{\gamma \in [0,1] : U \ \text{ is not an isomorphism of } \ L_{1/\gamma}[0,\infty)\}.$$

From Theorem 3.4 we obtain

Corollary 3.2. *For any closed set $G \subset [0,1]$ there exists a linear operator $U = U_G$ defined on the sum $L_1[0,\infty) + L_\infty[0,\infty)$ such that U acts injectively and continuously on every space $L_p[0,\infty)$ $(1 \le p \le \infty)$ and $\tilde{\kappa}(U) = G$.*

Proof. The arguments used here are mostly similar to the ones used in the proof of Theorem 3.3. Therefore we can restrict our consideration to the construction of the operator U. The set of non-negative integers can be representable as an union $\cup_{i=1}^{\infty} A_i$, where A_i are infinite disjoint sets. Then the sets

$$R_i \ = \ \cup_{k \in A_i}(k, k+1] \ \ (i = 1, 2, ..)$$

are disjoint, also, and $\cup_{i=1}^{\infty} R_i = (0,\infty)$. Let $f_k(t) = 2t - k$, for $t \in (k, k+1/2]$, and $f_k(t) = 0$, otherwise, and $b_i(t) = \sum_{k \in R_i} f_k(t)$ $(i = 1, 2, \cdots)$. Finally, if $Q_i x(t) = x(t)\chi_{R_i}(t)$ and a sequence $\{\lambda_k\} \subset F = \{\lambda : \lambda = 2^{-t}, t \in G\}$ is such that its closure $\overline{\{\lambda_k\}} = F$, then we put

$$Ux(t) \ = \ \sum_{i=1}^{\infty} U_i x(t), \quad \text{where } \ U_i x(t) \ = \ Q_i x(b_i(t)) \ - \ \lambda_i Q_i x(t).$$

□

Remark 3.2. We use here an important property of an Orlicz space L_M to contain of a $(1+\varepsilon)$ -copy of $l_p(n)$ generated by n disjointly supported equimeasurable functions, for every $p \in [\beta_M^{-1}, \alpha_M^{-1}]$, $\varepsilon > 0$, and integer n. Note that in the case of arbitrary RIS's the last is true only for some (but not all,in general) $p \in [\beta_M^{-1}, \alpha_M^{-1}]$ (see [13], [9,p.141]).

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