Functional Analysis and Its Applications, Vol. 36, No. 3, pp. 217–219, 2002
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 36, No. 3, pp. 60–63, 2002

On the Exact $\mathscr{K}$-Monotonicity of Banach Couples

S. V. Astashkin

Received March 19, 2001

Abstract. We present necessary and sufficient conditions for a Banach couple formed by the space L_∞ and an arbitrary Lorentz space $\Lambda(\varphi)$ to be exact $\mathscr{K}$-monotone. The proof relies on the description of the set of extreme points of a $\mathscr{K}$-orbit for appropriate finite-dimensional couples. As a consequence of this description, we obtain a generalization of a well-known Markus theorem.

Key words: Peetre $\mathscr{K}$-functional, exact $\mathscr{K}$-monotone Banach couple, Lorentz space, rearrangement, extreme point, convex hull.

1. Introduction. Let (X_0, X_1) be a Banach couple, let $x \in X_0 + X_1$, and let $t > 0$. The Peetre $\mathscr{K}$-functional is defined by

$$\mathscr{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\,\|x_1\|_{X_1} : x = x_0 + x_1,\ x_i \in X_i\}.$$

A couple (X_0, X_1) is said to be $\mathscr{K}$-monotone (or a Calderon–Mityagin couple) if there is a constant C, $C > 0$, such that, for any $x, y \in X_0 + X_1$ with

$$\mathscr{K}(t, y; X_0, X_1) \leqslant \mathscr{K}(t, x; X_0, X_1) \qquad (t > 0), \tag{1}$$

there is a linear operator $T\colon X_0 + X_1 \to X_0 + X_1$ such that $y = Tx$ and $\max_{i=0,1} \|T\|_{X_i \to X_i} \leqslant C$. If this condition holds for $C = 1$, then we say that (X_0, X_1) is an exact $\mathscr{K}$-monotone couple. If a couple is $\mathscr{K}$-monotone, then all interpolation spaces for this couple are described as spaces of the real $\mathscr{K}$-method [1].

In the middle of the sixties, Calderon [2] and Mityagin [3] independently obtained the first important example of an exact $\mathscr{K}$-monotone couple, namely, $(L_1(0,\infty), L_\infty(0,\infty))$. Later on, Sedaev and Semenov showed that any couple $(L_1(w_0), L_1(w_1))$ of "weighted" L_1-spaces also has this property [4]. Afterwards, Sedaev generalized this result to the case of couples $(L_p(w_0), L_p(w_1))$ $(1 \leqslant p \leqslant \infty)$ [5]. (The case $p_0 = p_1 = \infty$ was earlier considered by Peetre [6]). Finally, in 1978, Sparr [7] obtained the most general result that an arbitrary couple of spaces of the form $(L_{p_0}(w_0), L_{p_1}(w_1))$ $(1 \leqslant p_0, p_1 \leqslant \infty)$ is exact $\mathscr{K}$-monotone.

At the same time, an example of a Banach couple of three-dimensional spaces that is not exact $\mathscr{K}$-monotone was given in [4]. To describe this example, and also to state the problems treated here, we present necessary definitions.

Let $v = (v_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative reals ($n \in \mathbb{N}$ or $n = \infty$). The related Lorentz sequence space $\lambda^n(v)$ consists of all sequences $x = (x_i)_{i=1}^n$ for which $\|x\|_v = \sum_{i=1}^n v_i x_i^* < \infty$, where (x_i^*) is the nonincreasing rearrangement of $(|x_i|)_{i=1}^n$. The corresponding Lorentz function space $\Lambda(\varphi)$ (φ is a nonnegative increasing concave function on $[0,\infty)$, $\varphi(0) = 0$) is formed by the measurable functions $x = x(t)$ on $(0,\infty)$ such that $\|x\|_\varphi = \int_0^\infty x^*(t)\,d\varphi(t) < \infty$ ($x^*(t)$ stands for the nonincreasing rearrangement of $|x(t)|$). As usual, denote by l_∞^n (L_∞) the space of bounded sequences $x = (x_i)_{i=1}^n$ (of essentially bounded functions $x = x(t)$) with the norm $\|x\|_\infty = \sup_{i=1,\dots,n} |x_i|$ ($\|x\|_\infty = \operatorname{ess\,sup}_{t>0} |x(t)|$).

The above example of a couple that is not exact $\mathscr{K}$-monotone is $(\lambda^3(v), l_\infty^3)$, where $v_1 = v_2 = 1$ and $v_3 = 0$ [4]. In this connection, Sedaev posed the question of whether the couples $(\lambda^n(v), l_\infty^n)$ and $(\Lambda(\varphi), L_\infty)$ are (exact) $\mathscr{K}$-monotone. In 1981, Cwikel proved that every such couple is $\mathscr{K}$-monotone with the constant equal to 4 [8], thus giving a partial affirmative answer to the question. In this note, the problem of exact $\mathscr{K}$-monotonicity of the above couples is treated. We obtain necessary and sufficient conditions on the weight sequence $v=(v_i)_{i=1}^n$ (the function φ) for the couple $(\lambda^n(v), l_\infty^n)$ $((\Lambda(\varphi), L_\infty))$ to have the desired property.

2. Extreme points of the $\mathscr{K}$-orbit in the couple $(\lambda^n(v), l_\infty^n)$. Let $n\in\mathbb{N}$ and $v_1\geqslant\cdots\geqslant v_n\geqslant 0$. Denote by $\mathscr{V}_x$ the $\mathscr{K}$-orbit of the vector $x=(x_i)_{i=1}^n\in\mathbb{R}^n$ $(x_1\geqslant\cdots\geqslant x_n\geqslant 0)$ in the couple $(\lambda^n(v), l_\infty^n)$. This is the set of all $y=(y_i)_{i=1}^n\in\mathbb{R}^n$ such that relation (1) holds for $X_0=\lambda^n(v)$ and $X_1=l_\infty^n$.

For any $z=(z_i)_{i=1}^n$, the function $\mathscr{K}(t,z)=\mathscr{K}(t,z;\lambda^n(v),l_\infty^n)$ is piecewise linear and has break points at $t=t_k=\sum_{i=1}^k v_i$ $(k=1,\dots,n)$. Moreover, $\mathscr{K}(0,z)=0$, $\mathscr{K}(t_k,z)=\sum_{i=1}^k v_i z_i^*$ $(k=1,\dots,n)$, and $\mathscr{K}(t,z)=\mathscr{K}(t_n,z)$ for $t\geqslant t_n$. Therefore, relation (1) is equivalent to the system

$$\sum_{i=1}^k v_i y_i^* \leqslant \sum_{i=1}^k v_i x_i, \qquad k=1,\dots,n. \tag{2}$$

If $1\leqslant a\leqslant b\leqslant n$ $(a,b\in\mathbb{N})$, then we set $[a,b]=\{k\in\mathbb{N}: a\leqslant k\leqslant b\}$. An interval $\delta=[a,b]$ of this kind is said to be *admissible* with respect to a sequence $v=(v_i)_{i=1}^n$ if either $a=b$ and $v_a\neq 0$ or there exists a $k\in\delta$ such that $k+1\in\delta$ and $v_k>v_{k+1}$.

Let $\Delta=\{\delta_k\}_{k=1}^m$ be a system of admissible intervals partitioning the entire interval $[1,n]$, i.e., $[1,n]=\bigcup_{k=1}^m\delta_k$ and $\delta_i\cap\delta_j=\varnothing$ $(i\neq j)$. Denote by E_Δ the set of all $y=(y_i)_{i=1}^n\in\mathbb{R}^n$ such that

$$y_i^*=\frac{\sum_{j\in\delta_k} v_j x_j}{\sum_{j\in\delta_k} v_j} \quad \text{for } i\in\delta_k, \quad k=1,\dots,m.$$

Theorem 1. *The set of extreme points of the $\mathscr{K}$-orbit $\mathscr{V}_x$ defined by the system of inequalities* (2) *coincides with $\bigcup_\Delta E_\Delta$, where Δ ranges over all partitions of the interval $[1,n]$ into admissible intervals.*

Let $\operatorname{conv} F$ be the convex hull of a set F. Since $\mathscr{V}_x$ is a compact convex subset of $\mathbb{R}^n$, we have

Corollary 2. $\mathscr{V}_x=\operatorname{conv}(\bigcup_\Delta E_\Delta)$.

If $v_1=\cdots=v_n>0$, then the only admissible partition of the interval $[1,n]$ consists of singletons. We obtain the well-known Markus theorem: the set of extreme points of a $\mathscr{K}$-orbit $\mathscr{V}_x$ consists of all sequences $y=(y_i)_{i=1}^n\in\mathbb{R}^n$ such that the y_i's are representable in the form $y_i=\varepsilon_i x_{\pi(i)}$, where $\varepsilon_i=\pm1$ and π is a permutation of the interval $[1,n]$ [9–11].

3. Sequence spaces. Let either $n\in\mathbb{N}$ or $n=\infty$.

Theorem 3. *A Banach couple $(\lambda^n(v), l_\infty^n)$ is exact $\mathscr{K}$-monotone if and only if $v_k=v_1q^{k-1}$ $(k=1,\dots,n)$, where $v_1>0$ and $q\in[0,1]$.*

For $q=0$ we have $\lambda^n(v)=v_1 l_\infty^n$ (if X is a Banach space and $a>0$, then the space aX consists of the same elements as X and $\|x\|_{aX}=a\|x\|_X$). If $q=1$, then $\lambda^n(v)=v_1 l_1^n$, and the couple $(\lambda^n(v), l_\infty^n)$ is exact $\mathscr{K}$-monotone by the Calderon–Mityagin theorem [2, 3].

In the proof of the necessity in Theorem 3 for finite n and $v_i>0$ $(i=1,\dots,n)$, we use the operators

$$Q_k(z_i)=\sum_{i\neq k-1,k} z_i e_i+\frac{z_{k-1}v_{k-1}+z_k v_k}{v_{k-1}+v_k}(e_{k-1}+e_k) \qquad (k=2,\dots,n),$$

where $\{e_i\}_{i=1}^n$ are the standard unit vectors in $\mathbb{R}^n$. It is clear that $\|Q_k\|_{l_\infty^n\to l_\infty^n}=1$. For the norm of the operators Q_k in the Lorentz space, we have the following assertion.

Proposition 4. *If $(\lambda^n(v), l_\infty^n)$ is an exact $\mathscr{K}$-monotone couple, then $\|Q_k\|_{\lambda^n(v)\to\lambda^n(v)}=1$ for all $k=2,\dots,n$.*

4. Function spaces.

Theorem 5. *A couple $(\Lambda(\varphi), L_\infty)$ of spaces on the semiaxis $(0,\infty)$ is exact $\mathscr{K}$-monotone if and only if $\varphi(t)=af(t)$ $(a>0)$, where $f(t)=1$, $f(t)=t$, or $f(t)=1-q^t$ $(0<q<1)$.*

If $\varphi(t) = a$, then $\Lambda(\varphi) = aL_\infty$, and if $\varphi(t) = at$, then $\Lambda(\varphi) = aL_1$. In the nontrivial case $\varphi(t) = a(1 - q^t)$ it follows from the relations

$$\lim_{t\to 0+} \varphi(t)/t = a\log(1/q), \qquad \lim_{t\to+\infty} \varphi(t) = a$$

that $\Lambda(\varphi) = L_1 + L_\infty$ with the norm

$$\|x\|_{\Lambda(\varphi)} = a\log(1/q)\int_0^\infty x^*(t)q^t\,dt.$$

References

1. Yu. A. Brudnyi and N. Ya. Kruglyak, Dokl. Akad. Nauk SSSR, **256**, No. 1, 14–17 (1981); English transl. Soviet Math. Dokl., **23**, 5–8 (1981).
2. A. P. Calderón, Studia Math., **26**, No. 3, 273–299 (1966).
3. B. S. Mityagin, Mat. Sb., **66**, No. 4. 473–482 (1965); English transl. Proc. Conf. Interpolation Spaces and Allied Topics in Anal., Lund, 1983, Lect. Notes in Math., Vol. 1070, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1984, pp. 10-23.
4. A. A. Sedaev and E. M. Semenov, Optimizatsiya, No. 4 (21), 98–114 (1971).
5. A. A. Sedaev, Dokl. Akad. Nauk SSSR, **209**, No. 4, 798–800 (1973); English transl. Soviet Math. Dokl., Vol. 14, 538–541 (1973).
6. J. Peetre, Banach Couples, Technical report, Lund, 1971.
7. G. Sparr, Studia Math., **62**, 229–271 (1978).
8. M. Cwikel, Ark. Mat., **19**, No. 1, 123–136 (1981).
9. A. S. Markus, Dokl. Akad. Nauk SSSR, **146**, 11–14 (1962).
10. A. S. Markus, Usp. Mat. Nauk, **19**, No. 4, 93–123 (1964).
11. B. S. Mityagin, Izv. Akad. Nauk SSSR, **28**, 819–832 (1964).

Samara State University
e-mail: astashkn@ssu.samara.ru

Translated by S. V. Astashkin