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EXACT $\mathscr{K}$-MONOTONICITY OF ONE CLASS OF BANACH PAIRS

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1. Preliminaries and Definitions

Given a Banach pair (X_0, X_1), $x \in X_0 + X_1$, and $t > 0$, define the Peetre $\mathscr{K}$-functional

$$\mathscr{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_i \in X_i\}.$$

The pair (X_0, X_1) is called *$\mathscr{K}$-monotone* (or a *Calderón–Mityagin pair*) if for some $C > 0$ and all $x, y \in X_0 + X_1$ the inequality

$$\mathscr{K}(t, y; X_0, X_1) \leq \mathscr{K}(t, x; X_0, X_1) \quad (t > 0) \tag{1}$$

implies the existence of a linear operator $T : X_0 + X_1 \to X_0 + X_1$ such that $\max_{i=0,1} \|T\|_{X_i \to X_i} \leq C$ and $y = Tx$. If this inequality holds for $C = 1$ then (X_0, X_1) is called an *exact $\mathscr{K}$-monotone pair*. If a pair is $\mathscr{K}$-monotone then all interpolation spaces with respect to this pair are described as the spaces of the real $\mathscr{K}$-method [1].

In the mid 1960s, A. P. Calderón [2] and B. S. Mityagin [3] independently demonstrated that the pair (L_1, L_∞) of function spaces on an arbitrary σ-finite measure space is an exact $\mathscr{K}$-monotone pair. A. A. Sedaev and E. M. Semënov proved afterwards that the same property is enjoyed by every pair $(L_1(w_0), L_1(w_1))$ of "weighted" L_1-spaces [4]; A. A. Sedaev later generalized this result to the pairs $(L_p(w_0), L_p(w_1))$ $(1 \leq p \leq \infty)$ [5] (a similar question for $p = \infty$ was discussed earlier by J. Peetre [6]). Finally, in 1978 G. Sparr obtained the most general result on exact $\mathscr{K}$-monotonicity of an arbitrary pair of spaces of the form $(L_{p_0}(w_0), L_{p_1}(w_1))$ $(1 \leq p_0, p_1 \leq \infty)$ [7].

Besides, in [4] A. A. Sedaev and E. M. Semënov exhibited an example of a pair of spaces of dimension 3 without the exact $\mathscr{K}$-monotonicity property. To describe this example and formulate the problems to be considered here, we introduce some necessary definitions.

Let $v = (v_i)_{i=1}^n$ be a nonincreasing collection of nonnegative numbers $(n \in \mathbb{N})$. Then the n-dimensional Lorentz space λ_v^n is the space $\mathbb{R}^n$ with the norm

$$\|x\|_v = \sum_{i=1}^n v_i x_i^*,$$

where (x_i^*) is the nondecreasing rearrangement of the absolute values of the coordinates of the vector $x = (x_i)_{i=1}^n$. If $v = (v_i)_{i=1}^\infty$ is a nonincreasing sequence of nonnegative numbers then the coordinate Lorentz space consists of all $x = (x_i)_{i=1}^\infty$ such that

$$\|x\|_v = \sum_{i=1}^\infty v_i x_i^* < \infty.$$

In particular, if $v = (1, 0, \dots, 0)$ $(v = (1, 0, \dots))$ then we obtain the space l_∞^n (l_∞) with the usual norm.

The function Lorentz space $\Lambda(\varphi)$ (φ is a nonnegative increasing concave function on $[0, \infty)$ such that $\varphi(0) = 0$) is the set of all measurable functions $x = x(t)$ on $(0, \infty)$ such that

$$\|x\|_\varphi = \int_0^\infty x^*(t)\, d\varphi(t) < \infty$$

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($x^*(t)$ is the nonincreasing rearrangement of $|x(t)|$). As usual, we denote by L_∞ the space of all essentially bounded functions $x = x(t)$ with the norm $\|x\|_\infty = \operatorname{ess\,sup}_{t>0} |x(t)|$.

The above-mentioned example of a nonexact $\mathscr{K}$-monotone pair is the pair $\left(\lambda_v^3, l_\infty^3\right)$ with $v_1 = v_2 = 1$ and $v_3 = 0$ [4]. In this connection, A. A. Sedaev posed the question of (exact) $\mathscr{K}$-monotonicity of pairs like $(\lambda_v^n, l_\infty^n)$, (λ_v, l_∞), and $(\Lambda(\varphi), L_\infty)$. In 1981 M. Cwikel [8] proved that every such pair is $\mathscr{K}$-monotone with constant 4 and thereby gave a partial positive answer to the question.

In this article we consider the problem of exact $\mathscr{K}$-monotonicity of pairs of the indicated shape. We obtain necessary and sufficient conditions on the collection $v = (v_i)_{i=1}^n$ (the sequence $v = (v_i)_{i=1}^\infty$ or the function φ) under which the pair $(\lambda_v^n, l_\infty^n)$ ((λ_v, l_∞) or $(\Lambda(\varphi), L_\infty)$) possesses this property. The key role in the proofs is played by a description for the set of extreme points of the $\mathscr{K}$-orbit of an arbitrary vector in $\mathbb{R}^n$ with respect to the pair $(\lambda_v^n, l_\infty^n)$ which generalizes the well-known theorem of A. S. Markus and is of interest in its own right.

2. Extreme Points of the $\mathscr{K}$-Orbit in the Pair $\left(\lambda_v^n, l_\infty^n\right)$

Suppose that $n \in \mathbb{N}$ and $v_1 \ge v_2 \ge \dots \ge v_n \ge 0$. Denote the $\mathscr{K}$-orbit of a vector $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ ($x_1 \ge x_2 \ge \dots \ge x_n \ge 0$) in the pair $\left(\lambda_v^n, l_\infty^n\right)$ by $\mathscr{V}_x$. This is the set of all $y = (y_i)_{i=1}^n \in \mathbb{R}^n$ such that (1) holds for $X_0 = \lambda_v^n$ and $X_1 = l_\infty^n$.

Given $z = (z_i)_{i=1}^n$, the function $\mathscr{K}(t, z) = \mathscr{K}\left(t, z; \lambda_v^n, l_\infty^n\right)$ is piecewise linear and has fracture points at $t = t_k = \sum_{i=1}^k v_i$ ($k = 0, 1, \dots, n$, $t_0 = 0$). Moreover, as is easy to see, $\mathscr{K}(0, z) = 0$, $\mathscr{K}(t_k, z) = \sum_{i=1}^k v_i z_i^*$ ($k = 1, 2, \dots, n$), and $\mathscr{K}(t, z) = \mathscr{K}(t_n, z)$ ($t \ge t_n$). Therefore, in this case (1) amounts to the system of inequalities

$$\sum_{i=1}^k v_i y_i^* \le \sum_{i=1}^k v_i x_i \quad (k = 1, 2, \dots, n). \tag{2}$$

Henceforth we consider intervals of natural numbers. For example, if $a, b \in \mathbb{N}$ and $a \le b$ then $[a, b] = \{k \in \mathbb{N} : a \le k \le b\}$. Denote by $|\Delta|$ the number of elements in the interval Δ and let e_i ($i = 1, 2, \dots, n$) be the standard basis for $\mathbb{R}^n$.

A collection $\mathscr{T} = \{t_k\}_{k=1}^m$, $1 = t_1 < t_2 < \dots < t_m = n + 1$, of natural numbers is *v-admissible with respect to the interval* $[1, n]$ if for every $k = 1, 2, \dots, m - 1$ either $t_{k+1} = t_k + 1$ and $v_{t_k} > 0$ or there is $i \in [t_k, t_{k+1} - 1)$ such that $v_i > v_{i+1}$.

Suppose that $x \in \mathbb{R}^n$, $\mathscr{T} = \{t_k\}_{k=1}^m$ is a v-admissible collection with respect to the interval $[1, n]$, and $\Delta_k = [t_k, t_{k+1} - 1]$. Denote by $\mathscr{E}_{\mathscr{T}} = \mathscr{E}_{\mathscr{T}}(x)$ the set of all vectors $y = (y_i)_{i=1}^n \in \mathbb{R}^n$ such that for every $k = 1, 2, \dots, m$

$$y_i^* = \alpha_k(x) \quad \text{if} \quad i \in \Delta_k,$$

where

$$\alpha_k(x) = \frac{\sum\limits_{j \in \Delta_k} v_j x_j}{\sum\limits_{j \in \Delta_k} v_j}. \tag{3}$$

Let K be a subset of some vector space E. Recall that a point $x_0 \in K$ is an *extreme point* of K if x_0 is not an interior point of any straight line segment whose endpoints belong to K and differ from x_0. In other words, the representation

$$x_0 = t x_1 + (1 - t) x_2, \quad x_1 \in K,\ x_2 \in K,\ 0 < t < 1,$$

implies that $x_0 = x_1 = x_2$. We denote the set of extreme points of K by $\operatorname{ext} K$.

Theorem 1. *The equality* $\operatorname{ext} \mathscr{V}_x = \bigcup_{\mathscr{T}} \mathscr{E}_{\mathscr{T}}$ *holds for every* $x \in \mathbb{R}^n$, *where the union is calculated over all v-admissible collections with respect to the interval* $[1, n]$. *In particular,* $\mathscr{V}_x = \operatorname{conv}(\bigcup_{\mathscr{T}} \mathscr{E}_{\mathscr{T}})$.

Proof. We first suppose that

$$v_1 \ge v_2 \ge \dots \ge v_n > 0. \tag{4}$$

Define the set $\mathscr{V}_x^+ \subset \mathbb{R}^n$ by the system of $2n$ inequalities

$$y_i \geq y_{i+1} \quad (i = 1, 2, \dots, n-1), \quad y_n \geq 0, \tag{5}$$

and

$$\sum_{j=1}^{k} v_j y_j \leq \sum_{j=1}^{k} v_j x_j \quad (k = 1, 2, \dots, n). \tag{6}$$

Since $\mathscr{V}_x^+$ is a closed convex polyhedron in $\mathbb{R}^n$, we have $\bar{y} \in \operatorname{ext} \mathscr{V}_x^+$ if and only if $\bar{y} \in \mathscr{V}_x^+$ and the n inequalities of (5) and (6) become linearly independent equalities for $y = \bar{y}$.

Suppose that $\bar{y} = (\bar{y}_i)_{i=1}^n \in \mathscr{V}_x^+$ and $\Delta^\circ = [a, b)$ is a half-interval such that $a < b$ and $\bar{y}_i = \bar{y}_{i+1} = \alpha$ for all $i \in \Delta^\circ$. Let us show that the equality

$$\sum_{j=1}^{i} v_j \bar{y}_j = \sum_{j=1}^{i} v_j x_j \tag{7}$$

for some $i \in \Delta^\circ$ implies a similar relation for all $j \in \Delta = [a, b]$ or, equivalently,

$$x_j = \alpha \quad (j \in \Delta). \tag{8}$$

Indeed, the inequality $\bar{y}_i < x_i$ for $i = 1$ is impossible in view of (7); moreover, if $i > 1$ then (6) is violated for $k = i - 1$ by (4). If the reverse inequality $\bar{y}_i > x_i$ holds then $\bar{y}_{i+1} = \bar{y}_i = \alpha > x_{i+1}$ and from (7) we obtain

$$\sum_{j=1}^{i+1} v_j \bar{y}_j > \sum_{j=1}^{i+1} v_j x_j \tag{9}$$

which contradicts (6) again. Thus, $x_i = \bar{y}_i = \alpha$. So

$$\sum_{j=1}^{i-1} v_j \bar{y}_j = \sum_{j=1}^{i-1} v_j x_j.$$

Arguing in a similar way, we conclude that (8) is valid for all $j \in [a, i]$.

Furthermore, $x_{i+1} \leq x_i = \alpha$. If we suppose that $x_{i+1} < \alpha$ then (7) implies (9). Therefore, $x_{i+1} = \alpha$,

$$\sum_{j=1}^{i+1} v_j \bar{y}_j = \sum_{j=1}^{i+1} v_j x_j,$$

and we thereby can conclude that (8) holds for all $j \in \Delta$.

Now, assume that $\bar{y} \in \operatorname{ext} \mathscr{V}_x^+$ and $\bar{y}_n > 0$. Consider the system of all half-intervals $\Delta_k^\circ = [a_k, b_k)$ $(k = 1, 2, \dots, m)$ in the interval $[1, n]$ possessing the following properties:

(a) $a_k < b_k$ $(k = 1, 2, \dots, m)$ and $b_k < a_{k+1}$ $(k = 1, 2, \dots, m-1)$;

(b) $\bar{y}_i = \bar{y}_{i+1}$ $(i \in \Delta_k^\circ,\ k = 1, 2, \dots, m)$;

(c) $\bar{y}_{a_k - 1} > \bar{y}_{a_k}$ and $\bar{y}_{b_k} > \bar{y}_{b_k+1}$ $(k = 1, 2, \dots, m)$ (if $a_1 = 1$ then for $k = 1$ we require only the second of these inequalities; if $b_m = n$ then for $k = m$ we require only the first of them);

(d) to each $k = 1, 2, \dots, m$, there is $i \in \Delta_k^\circ$ such that $x_{i+1} < x_i$.

It is easily seen that (a)–(d) determine the system $\{\Delta_k^\circ\}_{k=1}^m$ uniquely. Moreover, by the above, for all $i \in \Delta_k^\circ$, $k = 1, 2, \dots, m$,

$$\sum_{j=1}^{i} v_j \bar{y}_j < \sum_{j=1}^{i} v_j x_j.$$

Therefore, if $s \notin \bigcup_{k=1}^m \Delta_k^\circ$ then

$$\sum_{j=1}^{s} v_j \bar{y}_j = \sum_{j=1}^{s} v_j x_j$$

or, equivalently,

$$\sum_{j=1}^{b_k} v_j \bar{y}_j = \sum_{j=1}^{b_k} v_j x_j \quad (k = 1, 2, \ldots, m) \tag{10}$$

and

$$\bar{y}_s = x_s \tag{11}$$

for $s \in F$, $F = [1, n] \setminus \bigcup_{k=1}^m \Delta_k$. So

$$\sum_{j=a_k}^{b_k} v_j \bar{y}_j = \sum_{j=a_k}^{b_k} v_j x_j \quad (k = 1, 2, \ldots, m);$$

whence by property (b) of the system $\{\Delta_k^\circ\}$

$$\bar{y}_i = \alpha_k(x) \quad (i \in \Delta_k), \tag{12}$$

where $\alpha_k(x)$ are defined by (3).

Enumerating the set $F \cup \{a_k\}_{k=1}^m \cup \{n+1\}$ in increasing order, we obtain the collection $\mathscr{T} = \{t_k\}_{k=1}^p$, $1 = t_1 < t_2 < \cdots < t_p = n + 1$. In view of (12), $\bar{y} \in \mathscr{E}_{\mathscr{T}} = \mathscr{E}_{\mathscr{T}}(x)$. Let us show that, for every $k = 1, 2, \ldots, m$, there is $i \in \Delta_k^\circ$ such that $v_i > v_{i+1}$. By (4) the collection $\mathscr{T}$ will then be v-admissible.

For a contradiction, suppose that for some $k = 1, 2, \ldots, m$

$$v_j = v_{a_k} \quad (j \in \Delta_k). \tag{13}$$

By property (d) of the intervals $\{\Delta_k^\circ\}$, there is $i \in \Delta_k^\circ$ such that

$$x_i > x_{i+1}. \tag{14}$$

Define the vectors $z^j = \left(z_s^j\right)_{s=1}^n$, $j = 1, 2, \ldots, l$, with $l = |\Delta_k| = b_k - a_k + 1$, as follows: $z_s^j = \bar{y}_s$ ($s \notin \Delta_k$) and $z_s^j = x_{s-a_k+j-1 \ (\mathrm{mod}\ l)+a_k}$ ($s \in \Delta_k$). In view of (14), the vectors z^j are pairwise distinct. Moreover, since the sequence $(x_i)_{i=1}^n$ is monotone decreasing, it follows from (10) and (11) that

$$\sum_{s=a_k}^{i} v_s \left(z_s^j\right)^* = \sum_{s=a_k}^{i} v_s x_s$$

for arbitrary $j = 1, 2, \ldots, n$ and $i \in \Delta_k$. Therefore, $z^j \in \mathscr{V}_x^+$ ($j = 1, 2, \ldots, l$).

Finally, we demonstrate that the vector $z = l^{-1} \sum_{j=1}^l z^j$ coincides with $\bar{y}$. Indeed, if $z = (z_s)_{s=1}^n$ then $z_s = \bar{y}_s$ for $s \notin \Delta_k$. If $s \in \Delta_k$ then, by (13) and (12),

$$z_s = l^{-1} \sum_{j=1}^{l} z_s^j = l^{-1} \sum_{j \in \Delta_k} x_j = \Big(\sum_{j \in \Delta_k} v_j\Big)^{-1} \Big(\sum_{j \in \Delta_k} v_j x_j\Big) = \alpha_k(x) = \bar{y}_s.$$

Thus, $\bar{y}$ is a convex combination of different elements of $\mathscr{V}_x^+$, which is impossible, since $\bar{y} \in \operatorname{ext} \mathscr{V}_x^+$.

Now, suppose that $\bar{y} = (\bar{y}_i)_{i=1}^n \in \operatorname{ext} \mathscr{V}_x^+$ and $\bar{y}_1 \geq \bar{y}_2 \geq \cdots \geq \bar{y}_l > \bar{y}_{l+1} = \cdots = \bar{y}_n = 0$ for some $l < n$. We show that in this case

$$x_i = 0 \quad (i > l). \tag{15}$$

Indeed, if $x_{l+1} > 0$ then, by the definition of $\mathscr{V}_x^+$ and the inequality $v_{l+1} > 0$, we have

$$\sum_{i=1}^{k} v_i \bar{y}_i < \sum_{i=1}^{k} v_i x_i \tag{16}$$

for all $k > l$. Put $h = \delta e_{l+1}$, where $0 < \delta < \min(\bar{y}_l, x_{l+1})$. Then the vectors $z' = \bar{y} + h$ and $z'' = \bar{y} - h$ differ and $\bar{y} = \frac{1}{2}(z' + z'')$. Moreover, $z'_s = z''_s = \bar{y}_s$ $(s \neq l+1)$, $z'_{l+1} = \delta$, and $z''_{l+1} = -\delta$. By the choice of δ, it then follows from (16) that $z', z'' \in \mathscr{V}_x^+$. This contradicts the fact that $\bar{y}$ is an extreme point of $\mathscr{V}_x^+$. Thereby (15) is proven.

Thus, in the case under consideration system (5), (6) is equivalent to a similar system in $\mathbb{R}^l$ if we replace x with the vector $x' = (x_i)_{i=1}^l$. Moreover, the l inequalities of the system at the vector $\bar{y}' = (\bar{y}_i)_{i=1}^l$ become linearly independent equalities. Since $\bar{y}_l > 0$, we find as before a v-admissible collection (with respect to the interval $[1, l]$) $\mathscr{T}' = \{t'_k\}_{k=1}^m$, $1 = t'_1 < t'_2 < \dots < t'_m = l+1$, such that $\bar{y}' \in \mathscr{E}_{\mathscr{T}'}$. Then $\mathscr{T} = \{t_k\}_{k=1}^{n+m-l}$, where $t_k = t'_k$ $(k = 1, \dots, m)$ and $t_k = l + 1 + k - m$ $(k = m+1, \dots, n+m-l)$, is a v-admissible collection with respect to the whole interval $[1, n]$ and $\bar{y} \in \mathscr{E}_{\mathscr{T}}$.

Thus, $\operatorname{ext} \mathscr{V}_x^+ \subset \bigcup_{\mathscr{T}} \mathscr{E}_{\mathscr{T}}$, where the union is calculated over all v-admissible collections $\mathscr{T}$ with respect to the interval $[1, n]$.

We return to $\mathscr{V}_x$. This set is symmetric about the coordinate planes $y_i = 0$ $(i = 1, \dots, n)$ as well as the planes $y_i = y_j$ $(1 \le i \neq j \le n)$. Therefore, if $\bar{y} = (\bar{y}_j)_{j=1}^n$ is an extreme point of $\mathscr{V}_x$ then $\bar{y}^* = (\bar{y}_i^*)_{i=1}^n \in \operatorname{ext} \mathscr{V}_x^+$ and hence there is a v-admissible collection $\mathscr{T}$ in the interval $[1, n]$ such that $\bar{y} \in \mathscr{E}_{\mathscr{T}}$.

Prove the converse: $\mathscr{E}_{\mathscr{T}} \subset \operatorname{ext} \mathscr{V}_x$ for every v-admissible collection $\mathscr{T}$. Suppose that $\bar{y} = (\bar{y}_i)_{i=1}^n \in \mathscr{E}_{\mathscr{T}} = \mathscr{E}_{\mathscr{T}}(x)$. Note that system (2), determining $\mathscr{V}_x$, is equivalent to the system of all possible inequalities of the form

$$\sum_{i=1}^{j} \varepsilon_i^j v_i y_{\pi_j(i)} \le \sum_{i=1}^{j} v_i x_i \quad (j = 1, 2, \dots, n), \tag{17}$$

where $\pi_j = \pi_j(i)$ are arbitrary permutations of the interval $[1, n]$ and $\varepsilon^j = \left\{\varepsilon_i^j\right\}$ are arbitrary collections of signs $(j = 1, 2, \dots, n)$. Let us show that $\bar{y} \in \mathscr{V}_x$ and the n inequalities of (17) for $y = \bar{y}$ become linearly independent equalities. We start with one auxiliary assertion.

Let $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k, x, y$ be arbitrary reals. Consider the following determinant whose rows are permutations of these numbers:

$$V = \begin{Vmatrix}
a_1 & a_2 & a_3 & \dots & a_{m-1} & a_m & x & y & b_1 & \dots & b_{k-1} & b_k \\
a_1 & a_2 & a_3 & \dots & a_{m-1} & a_m & y & x & b_1 & \dots & b_{k-1} & b_k \\
a_1 & a_2 & a_3 & \dots & a_{m-1} & y & a_m & x & b_1 & \dots & b_{k-1} & b_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_1 & y & a_2 & \dots & a_{m-2} & a_{m-1} & a_m & x & b_1 & \dots & b_{k-1} & b_k \\
y & a_1 & a_2 & \dots & a_{m-2} & a_{m-1} & a_m & x & b_1 & \dots & b_{k-1} & b_k \\
y & a_1 & a_2 & \dots & a_{m-2} & a_{m-1} & a_m & b_1 & x & \dots & b_{k-1} & b_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
y & a_1 & a_2 & \dots & a_{m-2} & a_{m-1} & a_m & b_1 & b_2 & \dots & x & b_k \\
y & a_1 & a_2 & \dots & a_{m-2} & a_{m-1} & a_m & b_1 & b_2 & \dots & b_k & x
\end{Vmatrix}.$$

Lemma 1. *The following equality holds:*

$$V = (-1)^{(m+1)(m+2)/2} \left(\sum_{i=1}^{m} a_i + \sum_{j=1}^{k} b_j + x + y \right) (y - x) \prod_{i=1}^{m} (y - a_i) \prod_{j=1}^{k} (x - b_j).$$

In particular, if $a_i \geq 0$ $(i = 1, 2, \dots, m)$, $b_j \geq 0$ $(j = 1, 2, \dots, k)$, $x \geq 0$, $y \geq 0$, $x \neq b_j$ $(j = 1, 2, \dots, k)$, $y \neq a_i$ $(i = 1, 2, \dots, m)$, *and* $x \neq y$ *then* $V \neq 0$.

Proof. Summing the first column of the determinant with the others and distinguishing the common factor, we obtain

$$V = \Big(\sum_{i=1}^{m} a_i + \sum_{j=1}^{k} b_j + x + y\Big) V,' \tag{18}$$

where the determinant V' is obtained from V by replacing the first column with the column of units. Since, if we substitute the numbers $a_1, a_2, \dots, a_m$ and x for y in V', then we obtain equal rows; therefore, V' is divisible by the product $(y - x)\prod_{i=1}^{m}(y - a_i)$. The degree of V' as a polynomial in y equals $m + 1$ and hence

$$V' = C(x)(y - x)\prod_{i=1}^{m}(y - a_i).$$

Moreover,

$$C(x) = (-1)^{k_m} V'', \tag{19}$$

where k_m is the number of inversions in the permutation $m + 2, m + 1, \dots, 2, 1$ and

$$V'' = \begin{Vmatrix} 1 & b_1 & b_2 & b_3 & \dots & b_{k-1} & b_k \\ 1 & x & b_2 & b_3 & \dots & b_{k-1} & b_k \\ 1 & b_2 & x & b_3 & \dots & b_{k-1} & b_k \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & b_1 & b_2 & b_3 & \dots & x & b_k \\ 1 & b_1 & b_2 & b_3 & \dots & b_k & x \end{Vmatrix}.$$

Since $b_1, b_2, \dots, b_k$ are roots of the polynomial $V''(x)$ whose degree equals k and the coefficient of x^k equals 1, we have

$$V''(x) = \prod_{j=1}^{k}(x - b_j). \tag{20}$$

Moreover, $k_m = (m + 1)(m + 2)/2$ and the sought equality follows from (18)–(20). The last assertion of the lemma is immediate from the hypotheses and the above-proven equality.

Remark 1. Similar assertions are valid for the determinants

$$V_1 = \begin{Vmatrix} x & y & b_1 & \dots & b_k \\ y & x & b_1 & \dots & b_k \\ y & b_1 & x & \dots & b_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y & b_1 & b_2 & \dots & x \end{Vmatrix}, \quad V_2 = \begin{Vmatrix} a_1 & \dots & a_m & x & y \\ a_1 & \dots & a_m & y & x \\ a_1 & \dots & y & a_m & x \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ y & \dots & a_{m-1} & a_m & x \end{Vmatrix}.$$

More precisely, we can demonstrate that

$$V_1 = \Big(\sum_{j=1}^{k} b_j + x + y\Big)(y - x)\prod_{j=1}^{k}(x - b_j)$$

and

$$V_2 = (-1)^{(m+1)(m+2)/2}\Big(\sum_{i=1}^{m} a_i + x + y\Big)(y - x)\prod_{i=1}^{m}(y - a_i).$$

Therefore, if $b_j \ge 0$, $x \ge 0$, $y \ge 0$, $x \ne b_j$ $(j = 1, 2, \dots, k)$, and $x \ne y$ then $V_1 \ne 0$; and if $a_i \ge 0$, $x \ge 0$, $y \ge 0$, $y \ne a_i$ $(i = 1, 2, \dots, m)$, and $x \ne y$ then $V_2 \ne 0$. In the case when the collection consists only of two numbers x and y we have $V_3 = \left\| \begin{matrix} x & y \\ y & x \end{matrix} \right\| = x^2 - y^2 \ne 0$ if $x \ge 0$, $y \ge 0$, and $x \ne y$.

We continue the proof of the theorem. Given a v-admissible collection $\mathscr{T} = \{t_k\}_{k=1}^p$, put $\Delta_k = [t_k, t_{k+1} - 1]$, $I_1 = \{k = 1, 2, \dots, p-1 : |\Delta_k| \ge 2\}$, and $I_2 = \{1, 2, \dots, p-1\} \setminus I_1$. Also, put $c_k = \max\{i \in \Delta_k : v_i = v_{t_k}\}$. Since $\mathscr{T}$ is a v-admissible collection, $c_k < t_{k+1} - 1$ for $k \in I_1$.

In the case of $k \in I_1$, applying Lemma 1 to the collection of numbers

$$v_{t_k}, v_{t_k+1}, \dots, v_{c_k - 1},\ x = v_{c_k},\ y = v_{c_k+1}, \dots, v_{t_{k+1}-1},$$

we find permutations ${\sigma_j}^k$ $(j \in \Delta_k)$ of Δ_k such that the determinant V_k whose rows are composed of appropriately rearranged numbers of this collection is nonzero. If $k \in I_2$ then Δ_k consists of a single element t_k and we put $\sigma_{t_k}^k(t_k) = t_k$.

Given $j = 1, 2, \dots, n$, define the permutation σ_j as follows. Suppose that k_j is such that $j \in \Delta_{k_j}$. Then $\sigma_j(i) = i$ if $i \notin \Delta_{k_j}$ and $\sigma_j(i) = {\sigma_j}^{k_j}(i)$ if $i \in \Delta_{k_j}$.

Consider the system of equations

$$\sum_{i=1}^{t_{k+1}-1} v_{\sigma_j(i)} y_i = \sum_{i=1}^{t_{k+1}-1} v_i x_i \quad (j \in \Delta_k,\ k = 1, 2, \dots, p-1) \tag{21}$$

in the variables $y_1, y_2, \dots, y_n$. Its determinant $\prod_{k \in I_1} V_k \prod_{k \in I_2} v_k$ differs from zero. Consequently, system (21) has a unique solution.

If $\bar{y} = (\bar{y}_i)_{i=1}^n \in \mathscr{E}_{\mathscr{T}}(x)$ for the v-admissible collection in question then $\bar{y}_i^* = \alpha_k(x)$ $(i \in \Delta_k$, $k = 1, 2, \dots, p-1)$, where $\alpha_k(x)$ is defined by (3). Therefore, since $(x_i)_{i=1}^n$ decreases, for arbitrary $k = 1, 2, \dots, p-1$ and $m \in \Delta_k$ we have

$$\sum_{i=1}^m v_i \bar{y}_i^* = \sum_{j \le k-1} \sum_{i \in \Delta_j} v_i \bar{y}_i^* + \sum_{i=t_k}^m v_i \bar{y}_i^* = \sum_{j \le k-1} \sum_{i \in \Delta_j} v_i x_i + \bar{y}_{t_k}^* \sum_{i=t_k}^m v_i$$
$$= \sum_{i=1}^{t_k - 1} v_i x_i + \Big(\sum_{i=t_k}^m v_i\Big)\Big(\sum_{i \in \Delta_k} v_i\Big)^{-1}\Big(\sum_{i \in \Delta_k} v_i x_i\Big) \le \sum_{i=1}^{t_k-1} v_i x_i + \sum_{i=t_k}^m v_i x_i = \sum_{i=1}^m v_i x_i$$

and hence $\bar{y} \in \mathscr{V}_x$.

Find a permutation ν of $[1, n]$ and $\delta_i = \pm 1$ such that $\bar{y}_i^* = \delta_i \bar{y}_{\nu(i)}$ $(i = 1, 2, \dots, n)$. The system of the equations

$$\sum_{i=1}^{t_{k+1}-1} \delta_i v_{\sigma_j(i)} y_{\nu(i)} = \sum_{i=1}^{t_{k+1}-1} v_i x_i \quad (j \in \Delta_k,\ k = 1, 2, \dots, p-1)$$

or, which is the same,

$$\sum_{i=1}^{t_{k+1}-1} \varepsilon_i^j v_i y_{\pi_j(i)} = \sum_{i=1}^{t_{k+1}-1} v_i x_i \quad (j \in \Delta_k,\ k = 1, 2, \dots, p-1), \tag{22}$$

where $\pi_j = \nu(\tau_j)$, $\varepsilon_i^j = \delta_{\tau_j(i)}$, and $\tau_j = {\sigma_j}^{-1}$, is equivalent to (21). We now show that the unique solution of (22) is the vector $\bar{y}$.

Since $\tau_j(i) \in \Delta_s$ for all $s = 1, 2, \dots, p-1$, $j = 1, 2, \dots, n$, and $i \in \Delta_s$; we have

$$\sum_{i=1}^{t_{k+1}-1} \varepsilon_i^j v_i \bar{y}_{\pi_j(i)} = \sum_{s \le k} \sum_{i \in \Delta_s} v_i \delta_{\tau_j(i)} \bar{y}_{\nu(\tau_j(i))} = \sum_{s \le k} \sum_{i \in \Delta_s} v_i \bar{y}_{\tau_j(i)}^*$$
$$= \sum_{s \le k} \bar{y}_{t_s}^* \sum_{i \in \Delta_s} v_i = \sum_{s \le k} \sum_{i \in \Delta_s} v_i x_i = \sum_{i=1}^{t_{k+1}-1} v_i x_i.$$

System (22) is derived from (17) and comprises n equations. Consequently, $\bar{y} \in \operatorname{ext} \mathscr{V}_x$.

We are left with settling the case in which for some $l < n$

$$v_1 \geq v_2 \geq \cdots \geq v_l > v_{l+1} = \cdots = v_n = 0. \tag{23}$$

Note that, proving the inclusion $\mathscr{E}_{\mathscr{T}}(x) \subset \operatorname{ext} \mathscr{V}_x$ ($\mathscr{T}$ is a v-admissible collection), we did not use the positivity of "weights." Therefore, it suffices to prove the converse: if $\bar{y} \in \operatorname{ext} \mathscr{V}_x$ then $\bar{y} \in \mathscr{E}_{\mathscr{T}}(x)$ for some v-admissible collection $\mathscr{T}$.

If $\bar{y} \in \operatorname{ext} \mathscr{V}_x$ then $\bar{y}^* \in \operatorname{ext} \mathscr{V}_x^+$, where $\mathscr{V}_x^+$ is defined by (5) and (6). Arguing in a standard way and using (23), we conclude that $d = \min\{k : \bar{y}_k^* = \cdots = \bar{y}_n^*\} \leq l$. If $d = 1$ then the collection $\{t_k\}_{k=1}^2$, $t_1 = 1$, $t_2 = n+1$, is v-admissible and $\bar{y} \in \mathscr{E}_{\mathscr{T}}$. Assume that $d > 1$. Since $\bar{y}^* \in \operatorname{ext} \mathscr{V}_x^+$; therefore, $d-1$ inequalities of a system similar to (5), (6) (with $d-1$ substituted for n) must turn at the vector $(\bar{y}_i^*)_{i=1}^{d-1}$ into linearly independent equalities. As in the beginning of the proof, we can consequently construct a v-admissible collection $\mathscr{T}' = \{t'_k\}_{k=1}^m$, $t'_1 = 1 < \cdots < t'_m = d$, in $[1, d-1]$ such that $(\bar{y}_i^*)_{i=1}^{d-1} \in \mathscr{E}_{\mathscr{T}'}$. Then the collection $\mathscr{T} = \{t_k\}_{k=1}^{m+1}$, $t_k = t'_k$ $(k = 1, 2, \ldots, m)$, $t_{m+1} = n+1$, is v-admissible on the whole interval $[1, n]$ and $\bar{y} \in \mathscr{E}_{\mathscr{T}}$.

The theorem is proven.

Remark 2. In the case of $v_1 = v_2 = \cdots = v_n > 0$ the only v-admissible collection with respect to the interval $[1, n]$ is $\mathscr{T} = \{k\}_{k=1}^{n+1}$. As a consequence of Theorem 1 we obtain the well-known theorem of A. S. Markus: if $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ $(x_1 \geq x_2 \geq \cdots \geq x_n \geq 0)$ and $\mathscr{V}_x$ is the set of all $y = (y_i)_{i=1}^n \in \mathbb{R}^n$ satisfying the system of inequalities

$$\sum_{i=1}^k y_i^* \leq \sum_{i=1}^k x_i \quad (k = 1, 2, \ldots, n),$$

then $\operatorname{ext} \mathscr{V}_x$ consists of all $\bar{y} = (\bar{y}_i)_{i=1}^n \in \mathbb{R}^n$ representable as $\bar{y}_i = \varepsilon_i x_{\pi(i)}$, where $\varepsilon_i = \pm 1$ and π is a permutation of the interval $[1, n]$ [9–11].

Remark 3. The difference between the cases of equal and nonequal "weights" can be lucidly demonstrated for $n = 2$. If $v_1 = v_2 > 0$ then the set $\mathscr{V}_x$, $x = (x_1, x_2)$, $x_1 > x_2 > 0$, is a convex octagon whose vertices (extreme points) have coordinates $(\pm x_1, \pm x_2)$ and $(\pm x_2, \pm x_1)$.

If $v_1 > v_2 > 0$ then we obtain a convex dodecagon. Besides the indicated vertices, it has four more vertices on the bissectrices of the coordinate corners:

$$\left(\pm \frac{v_1 x_1 + v_2 x_2}{v_1 + v_2}, \pm \frac{v_1 x_1 + v_2 x_2}{v_1 + v_2} \right).$$

3. Finite-Dimensional Spaces

Theorem 2. *The Banach pair $(\lambda_v^n, l_\infty^n)$ $(n \in \mathbb{N})$ is an exact $\mathscr{K}$-monotone pair if and only if*

$$v_k = v_1 q^{k-1} \quad (k = 2, 3, \ldots, n),\ v_1 > 0,\ q \in [0, 1]. \tag{24}$$

Proof. First, suppose that (24) is valid. If $q = 0$ then $\lambda_v^n = v_1 l_\infty^n$ (which means that $\|x\|_v = v_1 \|x\|_\infty$). Therefore, the inequality

$$\mathscr{K}(t, y; \lambda_v^n, l_\infty^n) \leq \mathscr{K}(t, x; \lambda_v^n, l_\infty^n) \tag{25}$$

is equivalent to $\|y\|_\infty \leq \|x\|_\infty$. Suppose that $f \in (l_\infty^n)^*$ is such that $\|f\| = 1$ and $f(x) = \|x\|_\infty$. Then the operator $Tz = (\|x\|_\infty)^{-1} f(z) y$ satisfies the conditions $\|T\|_{l_\infty^n \to l_\infty^n} \leq 1$ and $Tx = y$. Thus, we can assume that $q \in (0, 1]$.

Suppose that $x, y \in \mathbb{R}^n$ and (25) is valid. As mentioned, this is equivalent to the system (2) of inequalities with the vector x^* substituted for x. Let us show that there is a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$y = Ax \quad \text{and} \quad \max(\|A\|_{l^n_\infty \to l^n_\infty}, \|A\|_{\lambda^n_v \to \lambda^n_v}) \le 1. \tag{26}$$

In view of Theorem 1, we can assume that $y \in \mathscr{E}_{\mathscr{T}} = \mathscr{E}_{\mathscr{T}}(x^*)$, where $\mathscr{T}$ is an arbitrary v-admissible collection with respect to the interval $[1, n]$.

Take $\mathscr{T} = \{t_k\}_{k=1}^m$ and $\Delta_k = [t_k, t_{k+1} - 1]$ $(k = 1, 2, \dots, m-1)$. Define the operator B: $\mathbb{R}^n \to \mathbb{R}^n$,

$$Bz = \sum_{k=1}^{m-1} b_k(z) \sum_{j \in \Delta_k} e_j, \quad \text{where} \quad b_k(z) = \Big(\sum_{j\in\Delta_k} v_j\Big)^{-1} \Big(\sum_{j\in\Delta_k} z_j v_j\Big).$$

It is easy to verify that $\|B\|_{l^n_\infty \to l^n_\infty} = 1$. Moreover, if $b^*_s(z) = |b_{k_s}(z)|$ $(s = 1, 2, \dots, m-1)$ then

$$\|Bz\|_v = \sum_{s=1}^{m-1} |b_{k_s}(z)| \sum_{i=i_{s-1}+1}^{i_s} v_i, \tag{27}$$

where $i_0 = 0$ and $i_s = \sum_{j=1}^{s} |\Delta_{k_j}|$ $(s = 1, 2, \dots, m-1)$. In view of (24), for all $s = 1, 2, \dots, m-1$ we have

$$|b_{k_s}(z)| \le \Big(\sum_{i\in\Delta_{k_s}} q^{i-1}\Big)^{-1} \Big(\sum_{i\in\Delta_{k_s}} |z_i| q^{i-1}\Big)$$
$$= \Big(\sum_{i=i_{s-1}+1}^{i_s} q^{i-1}\Big)^{-1} \Big(\sum_{i=i_{s-1}+1}^{i_s} |z_{i-i_{s-1}+t_{k_s}-1}| q^{i-1}\Big).$$

Therefore, it follows from (27) that

$$\|Bz\|_v \le v_1 \sum_{s=1}^{m-1} \sum_{i=i_{s-1}+1}^{i_s} |z_{i-i_{s-1}+t_{k_s}-1}| q^{i-1} \le v_1 \sum_{i=1}^{n} z^*_i q^{i-1} = \|z\|_v;$$

whence $\|B\|_{\lambda^n_v \to \lambda^n_v} \le 1$. Since $y \in \mathscr{E}_{\mathscr{T}}(x^*)$, we moreover have $Bx^* = y^*$.

Denote by S_1 and S_2 operators that are generated by permutations of the interval $[1, n]$ and coordinatewise multiplication by ± 1 and are such that $S_1 y^* = y$ and $S_2 x = x^*$. Then (26) is valid for the operator $A = S_1 B S_2$.

To prove the converse, we can assume that $v_2 > 0$. It was demonstrated in [12] that $(\lambda^n_v, l^n_\infty)$ is not an exact $\mathscr{K}$-monotone pair if $v_1 \ge v_2 \ge \dots \ge v_l > v_{l+1} = \dots = v_n = 0$ for some $l \in [2, n]$. Therefore, we consider only the case of $v_i > 0$ $(i = 1, 2, \dots, n)$.

Define the operators $Q_k : \mathbb{R}^n \to \mathbb{R}^n$,

$$Q_k((z_i)) = \sum_{i \ne k-1, k} z_i e_i + \frac{z_{k-1} v_{k-1} + z_k v_k}{v_{k-1} + v_k} (e_{k-1} + e_k) \quad (k = 2, 3, \dots, n),$$

and show that the exact $\mathscr{K}$-monotonicity of the pair $(\lambda^n_v, l^n_\infty)$ implies the inequality

$$\|Q_k\|_{\lambda^n_v \to \lambda^n_v} \le 1 \quad (k = 2, 3, \dots, n). \tag{28}$$

Take $x = x^* = \sum_{i=1}^n (n - i + 1) e_i$. It is easy to see that the vector

$$y = y^* = \sum_{i=1}^{k-2} (n - i + 1) e_i + \sum_{i=k+1}^{n} (n - i + 1) e_i + \alpha_k (e_{k-1} + e_k),$$

where

$$\alpha_k = \frac{(n-k+2)v_{k-1} + (n-k+1)v_k}{v_{k-1}+v_k}, \tag{29}$$

belongs to $\mathscr{V}_x$. Therefore, by hypotheses there is a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ for which (26) is valid.

If the operator is given by a matrix $(a_{i,j})_{i,j=1}^n$ then from the definition of x and y and from (26) we obtain the following system of equations:

$$\begin{cases} na_{1,1} + (n-1)a_{1,2} + \cdots + (n-k+1)a_{1,k} + \cdots + a_{1,n} = n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ na_{k-2,1} + (n-1)a_{k-2,2} + \cdots + (n-k+1)a_{k-2,k} + \cdots + a_{k-2,n} = n-k+3 \\ na_{k-1,1} + (n-1)a_{k-1,2} + \cdots + (n-k+1)a_{k-1,k} + \cdots + a_{k-1,n} = \alpha_k \\ na_{k,1} + (n-1)a_{k,2} + \cdots + (n-k+1)a_{k,k} + \cdots + a_{k,n} = \alpha_k \\ na_{k+1,1} + (n-1)a_{k+1,2} + \cdots + (n-k+1)a_{k+1,k} + \cdots + a_{k+1,n} = n-k \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ na_{n,1} + (n-1)a_{n,2} + \cdots + (n-k+1)a_{n,k} + \cdots + a_{n,n} = 1. \end{cases}$$

Since

$$\|A\|_{l_\infty^n \to l_\infty^n} = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{i,j}|,$$

it follows from (26) that

$$\sum_{j=1}^n |a_{i,j}| \le 1 \quad (i = 1, 2, \dots, n). \tag{30}$$

Therefore, from the first equation of the system we derive $a_{1,1} = 1$ and $a_{1,j} = 0$ $(j = 2, \dots, n)$. Now, $Ae_1 = (1, a_{2,1}, \dots, a_{n,1})$ and $\max_{i=2,\dots,n} |a_{i,1}| \le 1$ by (30). Consequently, $(Ae_1)^* = (1, u_2, \dots, u_n)$. Applying (26) once again, we obtain

$$v_1 + v_2u_2 + \cdots + v_nu_n = \|Ae_1\|_v \le \|e_1\|_v = v_1;$$

whence $a_{2,1} = \cdots = a_{n,1} = 0$. Arguing likewise, we arrive at the equalities $a_{i,i} = 1$ $(1 \le i \le k-2)$ and $a_{i,j} = 0$ if $i \ne j$ and $\min(i,j) \le k-2$.

Now, we rewrite the $(k-1)$th equation of the system:

$$(n-k+2)a_{k-1,k-1} + (n-k+1)a_{k-1,k} + \cdots + a_{k-1,n} = \alpha_k. \tag{31}$$

If $a_{k-1,k-1} < v_{k-1}/(v_{k-1}+v_k)$ then, by (30) and (29), the left-hand side of (31) does not exceed

$$\begin{aligned} &(n-k+2)a_{k-1,k-1} + (n-k+1)\left(\sum_{j=k}^n |a_{k-1,j}|\right) \\ &\le (n-k+2)a_{k-1,k-1} + (n-k+1)(1 - |a_{k-1,k-1}|) < \alpha_k, \end{aligned}$$

and (31) fails. Therefore,

$$a_{k-1,k-1} \ge \frac{v_{k-1}}{v_{k-1}+v_k}. \tag{32}$$

Similarly, from the kth equation we deduce the inequality

$$a_{k,k-1} \ge \frac{v_{k-1}}{v_{k-1}+v_k}. \tag{33}$$

Now,

$$A\Big(\sum_{i=1}^{k-1} e_i\Big) = \sum_{i=1}^{k-2} e_i + \sum_{i=k-1}^{n} a_{i,k-1}e_i.$$

Hence, $|a_{k-1,k-1}|v_{k-1} + |a_{k,k-1}|v_k + \cdots + |a_{n,k-1}|v_n \le v_{k-1}$ by (26). From here, (32), and (33) we obtain

$$a_{k-1,k-1} = a_{k,k-1} = \frac{v_{k-1}}{v_{k-1}+v_k}, \quad a_{j,k-1} = 0 \quad (j = k+1, \ldots, n).$$

The $(k-1)$th and kth equations of the system consequently imply that

$$a_{k-1,k} = a_{k,k} = \frac{v_k}{v_{k-1}+v_k}$$

and $a_{k-1,j} = a_{k,j} = 0$ $(j = k+1, \ldots, n)$. Thus,

$$A\Big(\sum_{i=1}^{k} e_i\Big) = \sum_{i=1}^{k} e_i + \sum_{i=k+1}^{n} a_{i,k}e_i;$$

whence, in view of (26), $a_{j,k} = 0$ $(j = k+1, \ldots, n)$.

Repeating the same arguments as for the first $(k-2)$ equations of the system, we see that $a_{i,i} = 1$ $(i = k+1, \ldots, n)$ and $a_{i,j} = 0$ if $i \ne j$ $(i, j = k+1, \ldots, n)$. Thus, the operator A coincides with Q_k, and (28) is proven.

Let us show that the ratio v_{k-1}/v_k is independent of $k = 2, \ldots, n$.

Consider the vectors

$$a = 2e_1 + e_2 + 3\sum_{i=3}^{k} e_i, \quad b = Q_1 a = \frac{2v_1+v_2}{v_1+v_2}(e_1+e_2) + 3\sum_{i=3}^{k} e_i.$$

It follows from (28) that $\|b\|_v \le \|a\|_v$ or

$$\frac{2v_1+v_2}{v_1+v_2}(v_{k-1}+v_k) \le 2v_{k-1} + v_k;$$

whence

$$\frac{v_2}{v_1} \ge \frac{v_k}{v_{k-1}}. \tag{34}$$

Now, take

$$c = \sum_{i=1}^{k-2} e_i + 3e_{k-1} + 2e_k, \quad d = Q_k c = \sum_{i=1}^{k-2} e_i + \frac{3v_{k-1}+2v_k}{v_{k-1}+v_k}(e_{k-1}+e_k).$$

Again by (28), $\|d\|_v \le \|c\|_v$ which amounts to the inequality

$$\frac{v_2}{v_1} \le \frac{v_k}{v_{k-1}}. \tag{35}$$

Relations (34) and (35) mean that the theorem is proven.

4. Sequence Spaces

Let $v = (v_k)_{k=1}^{\infty}$ be a nonincreasing sequence of nonnegative numbers. Recall that the coordinate Lorentz space λ_v consists of all sequences $x = (x_k)_{k=1}^{\infty}$ such that

$$\|x\|_v = \sum_{k=1}^{\infty} v_k x_k^* < \infty.$$

As usual, l_∞ consists of all bounded sequences $x = (x_k)_{k=1}^{\infty}$ with the norm $\|x\|_\infty = \sup_{k=1,2,\dots} |x_k|$.

Theorem 3. *The Banach pair (λ_v, l_∞) is an exact $\mathscr{K}$-monotone pair if and only if*

$$v_k = v_1 q^{k-1} \quad (k = 2, 3, \dots),\ v_1 > 0,\ q \in [0, 1]. \tag{36}$$

Proof. We prove sufficiency of (36). Given $x \in l_\infty$, define the "truncators" $x^{(n)} = \big(x_i^{(n)}\big)$, where $x_i^{(n)} = x_i$ $(i \le n)$ and $x_i^{(n)} = 0$ $(i > n)$. Then $x^{(n)} \to x$ coordinatewise as $n \to +\infty$ and therefore for each $t > 0$

$$\lim_{n\to\infty} \mathscr{K}(t, x^{(n)}) = \mathscr{K}(t, x), \tag{37}$$

where $\mathscr{K}(t, z) = \mathscr{K}(t, z; \lambda_v, l_\infty)$. Suppose that $x \in l_\infty$, $y \in l_\infty$, and

$$\mathscr{K}(t, y) \le \mathscr{K}(t, x) \quad (t > 0). \tag{38}$$

Let us show that, for arbitrary $m \in \mathbb{N}$ and $\varepsilon > 0$, there is $n = n(m) \in \mathbb{N}$ such that

$$\mathscr{K}(t, y^{(m)}) \le (1 + \varepsilon)\mathscr{K}(t, x^{(n)}) \tag{39}$$

for all $t > 0$

First of all, observe that the convergence in (37) is uniform over the interval $[1, m]$. Therefore, in view of (38), inequality (39) is valid for some $n \in \mathbb{N}$ and all $t \in [1, m]$. Since the function $\mathscr{K}(t, y^{(m)})$ is linear and $\mathscr{K}(0, y^{(m)}) = \mathscr{K}(0, x^{(n)}) = 0$ on $[0, 1]$, (39) is also valid for $t \in [0, 1]$. Finally, (39) holds for $t > m$, since here the function $\mathscr{K}(t, y^{(m)})$ is constant while $\mathscr{K}(t, x^{(n)})$ increases.

Assuming $n \ge m$, observe that $\mathscr{K}(t, y^{(m)})$ and $\mathscr{K}(t, x^{(n)})$ coincide with the $\mathscr{K}$-functionals of the n-dimensional vectors $\bar{y} = (y_1, y_2, \dots, y_m, 0, \dots, 0)$ and $\bar{x} = (x_1, x_2, \dots, x_n)$ with respect to the pair $(\lambda^n_{v(n)}, l^n_\infty)$, where $v^{(n)} = (v_1, v_2, \dots, v_n)$. Hence, by Theorem 2 there is a linear operator $A^{(m)} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$A^{(m)}\bar{x} = \frac{1}{1+\varepsilon}\bar{y}, \quad \max(\|A^{(m)}\|_{\lambda^n_{v(n)} \to \lambda^n_{v(n)}}, \|A^{(m)}\|_{l^n_\infty \to l^n_\infty}) \le 1.$$

Let $S^{(m)} : l_\infty \to l^n_\infty$ and $I^{(m)} : l^n_\infty \to l_\infty$ be linear operators such that

$$S^{(m)}\big((z_i)_{i=1}^{\infty}\big) = (z_i)_{i=1}^{n}, \quad I^{(m)}\big((z_i)_{i=1}^{n}\big) = (z_1, z_2, \dots, z_n, 0, \dots).$$

Then by the above the operator $Q^{(m)} = I^{(m)} A^{(m)} S^{(m)}$ satisfies the conditions

$$Q^{(m)} x = \frac{1}{1+\varepsilon} y^{(m)}, \quad \max(\|Q^{(m)}\|_{\lambda_v \to \lambda_v}, \|Q^{(m)}\|_{l_\infty \to l_\infty}) \le 1. \tag{40}$$

Below we use the approach that was pursued by A. P. Calderón in a similar situation in [2].

Suppose that Φ is a Banach limit, i.e., a bounded linear functional on l_∞ such that

$$\liminf_{m\to+\infty} a_m \le \Phi(a) \le \limsup_{m\to+\infty} a_m \quad (a = (a_m)_{m=1}^{\infty} \in l_\infty). \tag{41}$$

Define the linear operator $Q : l_\infty \to l_\infty$ as follows:

$$Qz = ((Qz)_k)_{k=1}^{\infty}, \quad (Qz)_k = \Phi\big(((Q^{(m)}z)_k)_{m=1}^{\infty}\big).$$

In view of (40) and (41), $\|Q\|_{l_\infty\to l_\infty}\le 1$. We prove that

$$\|Q\|_{\lambda_v\to\lambda_v}\le 1+\varepsilon. \tag{42}$$

Find a sequence of pairwise distinct natural numbers $(s_i)_{i=1}^\infty$ such that $(Qz)_i^*\le(1+\varepsilon)|(Qz)_{s_i}|$ $(i=1,2,\dots)$. Then for every $k=1,2,\dots$ it follows from (41) and (40) that

$$\begin{aligned}\sum_{i=1}^k (Qz)_i^* v_i &\le (1+\varepsilon)\sum_{i=1}^k |(Qz)_{s_i}|v_i=(1+\varepsilon)\sum_{i=1}^k \pm\,\Phi((Q^{(m)}z)_{s_i})v_i\\ &=(1+\varepsilon)\Phi\left(\sum_{i=1}^k \pm\,(Q^{(m)}z)_{s_i}v_i\right)\le(1+\varepsilon)\sup_{m=1,2,\dots}\sum_{i=1}^k|(Q^{(m)}z)_{s_i}|v_i\\ &\le(1+\varepsilon)\sup_{m=1,2,\dots}\|Q^{(m)}z\|_v\le(1+\varepsilon)\sup_{m=1,2,\dots}\|Q^{(m)}\|_{\lambda_v\to\lambda_v}\|z\|_v\le(1+\varepsilon)\|z\|_v.\end{aligned}$$

Passing to the limit as $k\to\infty$, we come to (42). Moreover, from (41) and (40) we find that

$$Qx=\lim_{m\to\infty}Q^{(m)}x=\frac{1}{1+\varepsilon}y.$$

Therefore, if $\overline{Q}=(1+\varepsilon)Q$ then

$$\overline{Q}x=y,\quad \max(\|\overline{Q}\|_{\lambda_v\to\lambda_v},\|\overline{Q}\|_{l_\infty\to l_\infty})\le(1+\varepsilon)^2.$$

Thereby (λ_v,l_∞) is an exact $\mathscr{K}$-monotone pair.

Since the exact $\mathscr{K}$-monotonicity of the pair (λ_v,l_∞) guarantees a similar condition for the pair $\left(\lambda_v^n,l_\infty^n\right)$ with an arbitrary $n\in\mathbb{N}$, the necessity of (36) follows from Theorem 2.

5. Function Spaces

Let φ be a nonnegative increasing concave function on $[0,\infty)$ such that $\varphi(0)=0$. The function Lorentz space $\Lambda(\varphi)$ consists of all measurable functions $x=x(t)$ on $(0,\infty)$ such that

$$\|x\|_\varphi=\int_0^\infty x^*(t)\,d\varphi(t)<\infty$$

($x^*(t)$ is the nonincreasing rearrangement of $|x(t)|$).

As usual, we denote by L_∞ the space of all essentially bounded functions $x=x(t)$ with the norm $\|x\|_\infty=\operatorname{ess\,sup}_{t>0}|x(t)|$.

Theorem 4. *The Banach pair $(\Lambda(\varphi),L_\infty)$ is an exact $\mathscr{K}$-monotone pair if and only if $\varphi(t)=af(t)$ $(a>0)$, where $f(t)=1$ or $f(t)=t$ or $f(t)=1-q^t$ $(0<q<1)$.*

Proof. Given arbitrary $n\in\mathbb{N}$ and $h>0$, define the functions

$$f(t)=\sum_{k=1}^n x_k\chi_{[h(k-1),hk)}(t),\quad g(t)=\sum_{k=1}^n y_k\chi_{[h(k-1),hk)}(t), \tag{43}$$

where χ_e is the characteristic function of a set $e\subset[0,\infty)$. If $(\Lambda(\varphi),L_\infty)$ is an exact $\mathscr{K}$-monotone pair then for every $\varepsilon>0$ the inequality

$$\mathscr{K}(t,g;\Lambda(\varphi),L_\infty)\le\mathscr{K}(t,f;\Lambda(\varphi),L_\infty)\quad(t>0) \tag{44}$$

implies the existence of a linear operator A such that

$$Af=g\quad\text{and}\quad\max(\|A\|_{\Lambda(\varphi)\to\Lambda(\varphi)},\|A\|_{L_\infty\to L_\infty})\le1+\varepsilon.$$

Since [13, pp. 164–165]

$$\mathscr{K}(t,h;\Lambda(\varphi),L_\infty)=\int\limits_0^{\varphi^{-1}(t)} h^*(s)\,d\varphi(s), \tag{45}$$

(44) amounts to

$$\mathscr{K}\big(t,(y_k)_{k=1}^n;\lambda_v^n,l_\infty^n\big)\le\mathscr{K}\big(t,(x_k)_{k=1}^n;\lambda_v^n,l_\infty^n\big)\quad(t>0),$$

where $v=(v_k)_{k=1}^n$ and $v_k=\varphi(hk)-\varphi(h(k-1))$. The norm of the projection

$$Pu(t)=\frac{1}{h}\sum_{k=1}^N\int\limits_{h(k-1)}^{hk}u(s)\,ds\chi_{[h(k-1),hk)}(t)$$

in $\Lambda(\varphi)$ and L_∞ equals 1 [13, pp. 111, 148]. Similarly, if

$$I\big((z_k)_{k=1}^n\big)=\sum_{k=1}^n z_k\chi_{[h(k-1),hk)}(t),\quad S\bigg(\sum_{k=1}^n z_k\chi_{[h(k-1),hk)}(t)\bigg)=(z_k)_{k=1}^n \tag{46}$$

then

$$\|I\|_{\lambda_v^n\to\Lambda(\varphi)}=\|I\|_{l_\infty^n\to L_\infty}=\|S\|_{\Lambda(\varphi)\to\lambda_v^n}=\|S\|_{L_\infty\to l_\infty^n}=1$$

(the operator S is considered only on functions of the form (43)). Therefore, the operator $\widetilde{A}=SPAI:\mathbb{R}^n\to\mathbb{R}^n$ satisfies the conditions

$$\widetilde{A}\big((x_k)_{k=1}^n\big)=(y_k)_{k=1}^n,\quad \max(\|\widetilde{A}\|_{\lambda_v^n\to\lambda_v^n},\|\widetilde{A}\|_{l_\infty^n\to l_\infty^n})\le 1+\varepsilon.$$

Thus, $\big(\lambda_v^n,l_\infty^n\big)$ is an exact $\mathscr{K}$-monotone pair for every $n\in\mathbb{N}$. Consequently, by Theorem 2

$$\varphi(hk)-\varphi(h(k-1))=\varphi(h)q_h^{k-1}\quad(k=2,3,\dots), \tag{47}$$

where $\varphi(h)>0$ and $q_h\in[0,1]$. If $q_h=0$ then $\varphi(hk)=\varphi(h)$ for all natural k. Therefore, $\varphi(t)=a$ $(t>0)$ in view of the arbitrariness of $h>0$.

The two remaining cases $q_h=1$ and $q_h\in(0,1)$ are settled in the following lemmas:

Lemma 2. *Suppose that a function $\varphi(t)$ is nonnegative, concave on $[0,\infty)$, and such that $\varphi(0)=0$. If*

$$\varphi(kh)=k\varphi(h) \tag{48}$$

for some $h>0$ and all $k\in\mathbb{N}$ then $\varphi(t)=at$ $(t\ge 0)$ for some $a>0$.

Proof. Since $\varphi(t)$ is concave, $\varphi(t)/t$ decreases. Therefore, for every $k\in\mathbb{N}$

$$\frac{\varphi(kh)}{kh}\le\frac{\varphi(t)}{t}\le\frac{\varphi(h)}{h}\quad\text{if}\quad t\in[h,kh].$$

Hence, in view of (48), $\varphi(t)$ coincides with the linear function $l(t)=\varphi(h)t/h$ for $t\ge h$.

If $0<t<h$ then $\varphi(t)\ge t\varphi(h)/h=l(t)$. Suppose that $\varphi(t_0)>l(t_0)$ for some $t_0\in(0,h)$. Then

$$\frac{\varphi(h)-\varphi(t_0)}{h-t_0}\le\frac{l(h)-l(t_0)}{h-t_0}=\frac{\varphi(2h-t_0)-\varphi(h)}{h-t_0}$$

which contradicts the concavity of $\varphi(t)$. Consequently, $\varphi(t)=l(t)$, and the lemma is proven.

Lemma 3. *Let $\varphi(t)$ be a positive continuous function for $t > 0$ such that $\varphi(0) = 0$. Suppose that, for every $h > 0$, there is $q_h \in (0, 1)$ such that (47) holds for all $k \in \mathbb{N}$. Then $\varphi(t) = a(1 - q^t)$ for some $a > 0$ and $q \in (0, 1)$.*

PROOF. We start with showing that for $h > 0$ and $m \in \mathbb{N}$

$$q_{h/m} = q_h^{1/m}. \tag{49}$$

First of all

$$\varphi(2h) - \varphi(h) = \varphi(h)q_h. \tag{50}$$

Furthermore,

$$\varphi((1 + 1/m)h) - \varphi(h) = \varphi(h/m)q_{h/m}^m,$$

$$\varphi((1 + 2/m)h) - \varphi((1 + 1/m)h) = \varphi(h/m)q_{h/m}^{m+1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\varphi(2h) - \varphi((2 - 1/m)h) = \varphi(h/m)q_{h/m}^{2m-1}.$$

Summing these equalities, we obtain

$$\varphi(2h) - \varphi(h) = \varphi(h/m)\big(1 + q_{h/m} + \dots + q_{h/m}^{m-1}\big)q_{h/m}^m. \tag{51}$$

On the other hand, for every natural k

$$\varphi(2h/m) - \varphi(h/m) = \varphi(h/m)q_{h/m},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\varphi(kh/m) - \varphi((k - 1)h/m) = \varphi(h/m)q_{h/m}^{k-1};$$

whence

$$\varphi(kh/m) = \varphi(h/m)\big(1 + q_{h/m} + \dots + q_{h/m}^{k-1}\big). \tag{52}$$

Consequently, (49) follows from (50)–(52) for $k = m$.

Putting $h = 1$ in (52) and setting $q = q_1$, by (49) we now obtain

$$\varphi(k/m) = \varphi(1/m)\frac{1 - q^{k/m}}{1 - q^{1/m}} \tag{53}$$

for all natural k and m. In particular, if $k = m$ then

$$\varphi(1) = \varphi(1/m)\frac{1 - q}{1 - q^{1/m}},$$

and we can rewrite (53) as $\varphi(k/m) = \frac{\varphi(1)}{1-q}(1 - q^{k/m})$. Thus, $\varphi(t) = \frac{\varphi(1)}{1-q}(1 - q^t)$ for all rational $t > 0$. Since $\varphi(t)$ is continuous and $\varphi(0) = 0$, this equality holds for all $t \geq 0$. The lemma is proven.

We now prove the converse.

Note that the pair $(\Lambda(\varphi), L_\infty)$ satisfies all conditions of Theorem 1 of [14]. Namely, the spaces $\Lambda(\varphi)$ and L_∞ possess the Fatou property [13, p. 64] and for an arbitrary $x \in \Lambda(\varphi) + L_\infty$

$$\frac{t}{\varphi(t)}\mathscr{K}(\varphi(t), x; \Lambda(\varphi), L_\infty) = \frac{t}{\varphi(t)}\int_0^t x^*(s)\,d\varphi(s) \to 0 \quad \text{as} \quad t \to 0+$$

(the last condition is fulfilled in both cases $\lim_{t\to 0+}\varphi(t)=0$ and $\lim_{t\to 0+}\varphi(t)>0$). To prove exact $\mathscr{K}$-monotonicity of the pair $(\Lambda(\varphi),L_\infty)$, it therefore suffices to consider only nonnegative nonincreasing finite-valued functions. More precisely, we have to demonstrate that, for every $\varepsilon>0$, the condition

$$\mathscr{K}(t,\bar g)\le \mathscr{K}(t,\bar f), \tag{54}$$

with $\mathscr{K}(t,z)=\mathscr{K}(t,z;\Lambda(\varphi),L_\infty)$, holding for the functions

$$\bar f(t)=\sum_{k=1}^{m}\bar x_k\chi_{0,a_k]}(t),\quad \bar g(t)=\sum_{k=1}^{m}\bar y_k\chi_{0,b_k]}(t)$$

($a_1>a_2>\dots>a_m>0$, $b_1>b_2>\dots>b_m>0$, $\bar x_k>0$, and $\bar y_k>0$) implies the existence of a linear operator $T:\Lambda(\varphi)+L_\infty\to\Lambda(\varphi)+L_\infty$ such that

$$T\bar f=\bar g,\quad \max(\|T\|_{\Lambda(\varphi)\to\Lambda(\varphi)},\|T\|_{L_\infty\to L_\infty})\le 1+\varepsilon. \tag{55}$$

Construct a sequence of nonnegative decreasing functions $f_s(t)$ and $g_s(t)$ of the form (43) (corresponding to some sequence $h_s\downarrow 0$) such that $f_s\uparrow\bar f$ and $g_s\downarrow\bar g$. Then $\mathscr{K}(t,f_s)\uparrow\mathscr{K}(t,\bar f)$ and $\mathscr{K}(t,g_s)\downarrow\mathscr{K}(t,\bar g)$.

Since the function $\mathscr{K}(t,\bar f)$ is linear on the intervals $[a_i,a_{i-1}]$ ($i=2,3,\dots,m+1$, $a_{m+1}=0$) and is constant on $[a_1,\infty)$ while the function $\mathscr{K}(t,\bar g)$ possesses similar properties, by inequality (54)

$$\mathscr{K}(t,g_s)\le(1+\varepsilon)\mathscr{K}(t,f_s)\quad(t>0)$$

for every $\varepsilon>0$ and a sufficiently large $s\in\mathbb{N}$. Suppose that $f_s=f$ and $g_s=g$, where f and g are defined in (43). If φ satisfies the conditions of the theorem then (45) and Theorem 2 imply the existence of a linear operator $B:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$B\big((x_k)_{k=1}^n\big)=\frac{1}{1+\varepsilon}(y_k)_{k=1}^n,\quad \max(\|B\|_{\lambda_v^n\to\lambda_v^n},\|B\|_{l_\infty^n\to l_\infty^n})\le 1,$$

where $v=(v_k)_{k=1}^n$ and $v_k=\varphi(hk)-\varphi(h(k-1))$.

Put

$$M_1z(t)=\frac{f(t)}{\bar f(t)}z(t)\quad\text{if}\quad \bar f(t)\ne 0\quad\text{and}\quad M_1z(t)=0\quad\text{if}\quad \bar f(t)=0$$

and

$$M_2z(t)=\frac{\bar g(t)}{g(t)}z(t)\quad\text{if}\quad g(t)\ne 0\quad\text{and}\quad M_2z(t)=0\quad\text{if}\quad g(t)=0.$$

Since the norms of these operators in $\Lambda(\varphi)$ and L_∞ do not exceed 1, for the operator $T=(1+\varepsilon)M_2IBSM_1$ we obtain (55) (the operators I and S are defined by (46)). The theorem is proven.

Remark 4. If $\varphi(t)=a$ then $\Lambda(\varphi)=aL_\infty$; if $\varphi(t)=at$ then $\Lambda(\varphi)=aL_1$. In the most interesting case of $\varphi(t)=a(1-q^t)$, from the equalities

$$\lim_{t\to 0+}\varphi(t)/t=a\log(1/q),\quad \lim_{t\to+\infty}\varphi(t)=a$$

we obtain $\Lambda(\varphi)=L_1+L_\infty$ with the norm

$$\|x\|_{\Lambda(\varphi)}=a\log(1/q)\int_0^\infty x^*(t)q^t\,dt.$$

Remark 5. It is interesting to compare the conclusion of Theorem 4 with the fact that the pair $(\Lambda(\varphi),L_\infty)$ possesses the exact $\mathscr{K}$-monotonicity property with respect to the pair (L_1,L_∞) for every continuous increasing concave function $\varphi(t)$ such that $\varphi(0) = 0$ and $\varphi(\infty) = \infty$ [15]. More precisely, the following holds: If $x \in \Lambda(\varphi) + L_\infty$ and $y \in L_1 + L_\infty$ then the inequality

$$\mathscr{K}(t, y; L_1, L_\infty) \le \mathscr{K}(t, x; \Lambda(\varphi), L_\infty) \tag{56}$$

implies the existence of a linear operator $T : \Lambda(\varphi) + L_\infty \to L_1 + L_\infty$ such that

$$y = Tx \quad \text{and} \quad \max(\|T\|_{\Lambda(\varphi)\to L_1}, \|T\|_{L_\infty \to L_\infty}) \le 1. \tag{57}$$

A simple proof of this fact can be given by using one idea of [8, Theorem 2]. Let $\tilde{x}$ be the nonincreasing rearrangement of x^* with respect to the measure $d\nu = d\varphi(t) = \varphi'(t)\,dt$ on $\mathbb{R}_+ = [0, \infty)$. Then by [13, p. 108]

$$\mathscr{K}(t, \tilde{x}; L_1(d\nu), L_\infty(d\nu)) = \int\limits_0^t \tilde{x}(s)\,ds = \int\limits_0^{\varphi^{-1}(t)} x^*(s)\,d\varphi(s) = \mathscr{K}(t, x; \Lambda(\varphi), L_\infty).$$

Hence, by the Calderón–Mityagin theorem [2], there is a linear operator $T_1 : L_1(d\nu) + L_\infty(d\nu) \to L_1 + L_\infty$ such that

$$y = T_1 x^* \quad \text{and} \quad \max(\|T_1\|_{L_1(d\nu)\to L_1}, \|T_1\|_{L_\infty(d\nu)\to L_\infty}) \le 1.$$

Denote the identity operator by T_2; i.e., $T_2 z = z$. Since [13, p. 94]

$$\|T_2 z\|_{L_1(d\nu)} = \int\limits_0^\infty |z(s)|\,d\varphi(s) \le \int\limits_0^\infty z^*(s)\,d\varphi(s) = \|z\|_{\Lambda(\varphi)},$$

we have $\max(\|T_2\|_{\Lambda(\varphi)\to L_1(d\nu)}, \|T_2\|_{L_\infty \to L_\infty(d\nu)}) = 1$.

Finally, by the Calderón–Mityagin theorem [2]

$$x^* = T_3 x \quad \text{and} \quad \max(\|T_3\|_{L_1\to L_1}, \|T_3\|_{L_\infty\to L_\infty}) \le 1$$

for some linear operator $T_3 : L_1 + L_\infty \to L_1 + L_\infty$. Since $\Lambda(\varphi)$ is an exact interpolation space with respect to the pair (L_1, L_∞) [13, p. 148], we have $\|T_3\|_{\Lambda(\varphi)\to\Lambda(\varphi)} \le 1$. It remains to observe that the operator $T = T_1 T_2 T_3$ satisfies (57).

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