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ABOUT INTERPOLATION OF SUBSPACES OF REARRANGEMENT INVARIANT SPACES GENERATED BY RADEMACHER SYSTEM

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ABSTRACT. The Rademacher series in rearrangement invariant function spaces "close" to the space L_∞ are considered. In terms of interpolation theory of operators, a correspondence between such spaces and spaces of coefficients generated by them is stated. It is proved that this correspondence is one-to-one. Some examples and applications are presented.

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1. Introduction.

Let

$$r_k(t) = \operatorname{sign}\sin 2^{k-1}\pi t \quad (k = 1,2,\dots) \tag{1.1}$$

be the Rademacher functions on the segment $[0,1]$. Define the linear operator

$$Ta(t) = \sum_{k=1}^{\infty} a_k r_k(t) \quad \text{for } a = (a_k)_{k=1}^{\infty} \in l_2. \tag{1.2}$$

It is well known (cf. [23, pages 340–342]) that Ta is an almost everywhere finite function on $[0,1]$. Moreover, from Khintchine's inequality it follows that

$$\|Ta\|_{L_p} \asymp \|a\|_2 \quad \text{for } 1 \le p < \infty, \tag{1.3}$$

where $\|a\|_p = (\sum_{k=1}^{\infty} |a_k|^p)^{1/p}$. The symbol $\asymp$ means the existence of two-sided estimates with constants depending only on p. Also, it can easily be checked that

$$\|Ta\|_{L_\infty} = \|a\|_1. \tag{1.4}$$

A more detailed information on the behaviour of Rademacher series can be obtained by treating them in the framework of general rearrangement invariant spaces.

Recall that a Banach space X of measurable functions $x = x(t)$ on $[0,1]$ is said to be a rearrangement invariant space (r.i.s.) if the inequality $x^*(t) \le y^*(t)$, for $t \in [0,1]$ and $y \in X$, implies $x \in X$ and $\|x\| \le \|y\|$. Here and in what follows $z^*(t)$ is the nonincreasing rearrangement of a function $|z(t)|$ with respect to the Lebesgue measure denoted by meas [10, page 83].

Important examples of r.i.s.'s are Marcinkiewicz and Orlicz spaces. Let $\mathcal{P}$ denote the cone of nonnegative increasing concave functions on the semiaxis $(0,\infty)$.

If $\varphi \in \mathcal{P}$, then the Marcinkiewicz space $M(\varphi)$ consists of all measurable functions $x = x(t)$ such that

$$\|x\|_{M(\varphi)} = \sup\left\{\frac{1}{\varphi(t)}\int_o^t x^*(s)\,ds : 0 < t \le 1\right\} < \infty. \tag{1.5}$$

If $S(t)$ is a nonnegative convex continuous function on $[0,\infty)$, $S(0)=0$, then the Orlicz space L_S consists of all measurable functions $x=x(t)$ such that

$$\|x\|_S = \inf\left\{u>0:\int_0^1 S\left(\frac{|x(t)|}{u}\right)dt\le 1\right\}<\infty. \tag{1.6}$$

In particular, if $S(t)=t^p$ $(1\le p<\infty)$, then $L_S=L_p$.

For any r.i.s. X on $[0,1]$ we have $L_\infty\subset X\subset L_1$ [10, page 124]. Let X^0 denote the closure of L_∞ in an r.i.s. X.

In problems discussed below, a special role is played by the Orlicz space L_N, where $N(t)=\exp(t^2)-1$ or, more precisely, by the space $G=L_N^0$. In [19], V. A. Rodin and E. M. Semenov proved a theorem about the equivalence of Rademacher system to the standard basis in the space l_2.

Theorem 1.1. *Suppose that X is an r.i.s. Then*

$$\|Ta\|_X=\left\|\sum_{k=1}^\infty a_k r_k\right\|_X \asymp \|a\|_2 \tag{1.7}$$

if and only if $X\supset G$.

By Theorem 1.1, the space G is the minimal space among r.i.s.'s X such that the Rademacher system is equivalent in X to the standard basis of l_2.

In this paper, we consider problems related to the behaviour of Rademacher series in r.i.s.'s intermediate between L_∞ and G. Here a major role is played by concepts and methods of interpolation theory of operators.

For a Banach couple (X_0,X_1), $x\in X_0+X_1$ and $t>0$, we introduce the Peetre $\mathcal{K}$-functional

$$\mathcal{K}(t,x;X_0,X_1)=\inf\left\{\|x_0\|_{X_0}+t\|x_1\|_{X_1}: x=x_0+x_1,\ x_0\in X_0,\ x_1\in X_1\right\}. \tag{1.8}$$

Let Y_0 be a subspace of X_0 and Y_1 a subspace of X_1. A couple (Y_0,Y_1) is called a $\mathcal{K}$-subcouple of a couple (X_0,X_1) if

$$\mathcal{K}(t,y;Y_o,Y_1)\asymp\mathcal{K}(t,y;X_0,X_1), \tag{1.9}$$

with constants independent of $y\in Y_0+Y_1$ and $t>0$.

In particular, if $Y_i=P(X_i)$, where P is a linear projector bounded from X_i into itself for $i=0,1$, then (Y_0,Y_1) is a $\mathcal{K}$-subcouple of (X_0,X_1) (see [3] or [21, page 136]). At the same time, there are many examples of subcouples that are not $\mathcal{K}$-subcouples (see [21, page 589], [22], and Remark 3.2 of this paper).

Let $T(l_1)$ (respectively $T(l_2)$) denote the subspace of L_∞ (of G) consisting of all functions of the form $x=Ta$, where T is given by (1.2) and $a\in l_1(\in l_2)$. From (1.4) and Theorem 1.1 it follows that

$$\mathcal{K}(t,Ta;T(l_1),T(l_2))\asymp\mathcal{K}(t,a;l_1,l_2). \tag{1.10}$$

In spite of the fact that $T(l_1)$ is uncomplemented in L_∞ (see [17] or [11, page 134]) the following assertion holds.

Theorem 1.2. *The couple $(T(l_1),T(l_2))$ is a $\mathcal{K}$-subcouple of the couple (L_∞,G). In other words (see (1.10)),*

$$\mathcal{K}(t,Ta;L_\infty,G) \asymp \mathcal{K}(t,a;l_1,l_2), \tag{1.11}$$

with constants independent of $a=(a_k)_{k=1}^\infty \in l_2$ and $t>0$.

We will use in the proof of Theorem 1.2 an assertion about the distribution of Rademacher sums. It was proved by S. Montgomery-Smith [13].

Theorem 1.3. *There exists a constant $A \geq 1$ such that for all $a=(a_k)_{k=1}^\infty \in l_2$ and $t>0$*

$$\begin{aligned} &\operatorname{meas}\left\{s\in[0,1]:\sum_{k=1}^\infty a_k r_k(s)>\varphi_a(t)\right\}\leq \exp\left(-\frac{t^2}{2}\right),\\ &\operatorname{meas}\left\{s\in[0,1]:\sum_{k=1}^\infty a_k r_k(s)>A^{-1}\varphi_a(t)\right\}\geq A^{-1}\exp\left(-At^2\right), \end{aligned} \tag{1.12}$$

where $\varphi_a(t)=\mathcal{K}(t,a;l_1,l_2)$.

Now we need some definitions from interpolation theory of operators. We say that a linear operator U is bounded from a Banach couple $\vec{X}=(X_0,X_1)$ into a Banach couple $\vec{Y}=(Y_0,Y_1)$ (in short, $U:\vec{X}\to\vec{Y}$) if U is defined on X_0+X_1 and acts as bounded operator from X_i into Y_i for $i=0,1$.

Let $\vec{X}=(X_0,X_1)$ be a Banach couple. A space X such that $X_0\cap X_1\subset X\subset X_0+X_1$ is called an interpolation space between X_0 and X_1 if each linear operator $U:\vec{X}\to\vec{X}$ is bounded from X into itself.

To every r.i.s. X assign the sequence space F_X of Rademacher coefficients of functions of the form (1.2) from X:

$$\|(a_k)\|_{F_X}=\left\|\sum_{k=1}^\infty a_k r_k\right\|_X. \tag{1.13}$$

Well-known properties of Rademacher functions imply that F_X is an r.i. sequence space [19]. Furthermore, Theorem 1.3 and properties of the $\mathcal{K}$-functional show that F_X is an interpolation space between l_1 and l_2 (see the proof of Theorem 1.2 later). For interpolation r.i.s. between L_∞ and G the correspondence $X\mapsto F_X$ can be defined by using the real interpolation method.

For every $p\in[1,\infty]$, we denote by $l_p(u_k)$, $u_k\geq 0$ $(k=0,1,\dots)$ the space of all two-sided sequences of real numbers $a=(a_k)_{k=-\infty}^\infty$ such that the norm $\|a\|_{l_p(u_k)}=\|(a_k u_k)\|_p$ is finite. Let E be a Banach lattice of two-sided sequences, $(\min(1,2^k))_{k=-\infty}^\infty \in E$. If (X_0,X_1) is a Banach couple, then the space of the real $\mathcal{K}$-method of interpolation $(X_0,X_1)_E^{\mathcal{K}}$ consists of all $x\in X_0+X_1$ such that

$$\|x\|=\|(\mathcal{K}(2^k,x;X_0,X_1))_k\|_E<\infty. \tag{1.14}$$

It is readily checked that the space $(X_0,X_1)_E^{\mathcal{K}}$ is an interpolation space between X_0 and X_1 (cf. [15, page 422]). In the special case $E=l_p(2^{-k\theta})$ $(0<\theta<1,\ 1\leq p\leq\infty)$ we obtain the spaces $(X_0,X_1)_{\theta,p}$ (for the detailed exposition of their properties see [4]).

A couple $\vec{X} = (X_0, X_1)$ is said to be a $\mathcal{K}$-monotone couple if for every $x \in X_0 + X_1$ and $y \in X_0 + X_1$ there exists a linear operator $U : \vec{X} \to \vec{X}$ such that $y = Ux$ whenever

$$\mathcal{K}(t, y; X_0, X_1) \leq \mathcal{K}(t, x; X_0, X_1) \quad \forall t > 0. \tag{1.15}$$

As it is well known (cf. [15, page 482]), any interpolation space X with respect to a $\mathcal{K}$-monotone couple (X_0, X_1) is described by the real $\mathcal{K}$-method. It means that for some E

$$X = (X_0, X_1)_E^{\mathcal{K}}. \tag{1.16}$$

In particular, by the Sparr theorem [20] the couple (l_1, l_2) is a $\mathcal{K}$-monotone couple. Therefore, if F is an interpolation space between l_1 and l_2, then there exists E such that

$$F = (l_1, l_2)_E^{\mathcal{K}}. \tag{1.17}$$

Hence Theorem 1.2 allows to find an r.i.s. that contains Rademacher series with coefficients belonging to an arbitrary interpolation space between l_1 and l_2. In [19], the similar result was obtained for sequence spaces satisfying more restrictive conditions (see Remark 3.3).

Theorem 1.4. *Let F be an interpolation sequence space between l_1 and l_2 and $F = (l_1, l_2)_E^{\mathcal{K}}$. Then for the r.i.s. $X = (L_\infty, G)_E^{\mathcal{K}}$ we have*

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \asymp \|a\|_F \tag{1.18}$$

with constants independent of $a = (a_k)_{k=1}^{\infty}$.

Combining Theorem 1.4 with the above remarks, we get the following assertion. If F is a sequence space, then

$$\|(a_k)\|_F \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \quad \text{for some r.i.s. } X \tag{1.19}$$

if and only if F is an interpolation space between l_1 and l_2.

The last result shows that the restriction of the correspondence (1.13) to interpolation r.i.s. between L_∞ and G is bijective.

Theorem 1.5. *Let r.i.s.'s X_0 and X_1 be two interpolation spaces between L_∞ and G. If*

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_0} \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_1}, \tag{1.20}$$

then $X_0 = X_1$ and the norms of X_0 and X_1 are equivalent.

In [16, 19], the similar results were obtained by additional conditions with respect to spaces X_0 and X_1.

2. Proofs

Proof of Theorem 1.2. It is known [10, page 164] that the $\mathcal{K}$-functional of a couple of Marcinkiewicz spaces is given by the formula

$$\mathcal{K}(t,x;M(\varphi_0),M(\varphi_1)) = \sup_{0<u\le 1} \frac{\int_0^u x^*(s)\,ds}{\max(\varphi_0(u),\varphi_1(u)/t)}. \tag{2.1}$$

If $N(t)=\exp(t^2)-1$, then the Orlicz space L_N coincides with the Marcinkiewicz space $M(\varphi_1)$, where $\varphi_1(u)=u\log_2^{1/2}(2/u)$ [12]. In addition, $L_\infty = M(\varphi_0)$, where $\varphi_0(u)=u$. Therefore,

$$\mathcal{K}(t,x;L_\infty,G) = \sup_{0<u\le 1}\left\{\frac{1}{u}\int_0^u x^*(s)\,ds \min\left(1,t\log_2^{-1/2}\left(\frac{2}{u}\right)\right)\right\} \quad \text{for } x\in G. \tag{2.2}$$

Since $x^*(u)\le 1/u\int_0^u x^*(s)\,ds$, then from (2.2) it follows that

$$\mathcal{K}(t,x;L_\infty,G)\ge \sup_{k=0,1,\dots}\{x^*(2^{-k})\min(1,t(k+1)^{-1/2})\}. \tag{2.3}$$

Hence,

$$\mathcal{K}(t,x;L_\infty,G)\ge x^*(2^{-k_t}) \quad \text{for } t\ge 1, \tag{2.4}$$

where $k_t=[t^2]-1$ ($[z]$ is the integral part of a number z).

Now let $a=(a_k)_{k=1}^\infty\in l_2$ and $x(t)=Ta(t)=\sum_{k=1}^\infty a_k r_k(t)$. By the Holmstedt formula [7],

$$\varphi_a(t)\le \sum_{k=1}^{[t^2]} a_k^* + t\left\{\sum_{k=[t^2]+1}^{\infty}(a_k^*)^2\right\}^{1/2} \le B\varphi_a(t), \tag{2.5}$$

where $\varphi_a(t)=\mathcal{K}(t,a;l_1,l_2)$, $(a_k^*)_{k=1}^\infty$ is a nonincreasing rearrangement of the sequence $(|a_k|)_{k=1}^\infty$, and $B>0$ is a constant independent of $a=(a_k)_{k=1}^\infty$ and $t>0$.

Assume, at first, that $a\notin l_1$. Then inequality (2.5) shows that

$$\lim_{t\to 0+}\varphi_a(t)=0, \qquad \lim_{t\to\infty}\varphi_a(t)=\infty. \tag{2.6}$$

The function φ_a belongs to the class $\mathcal{P}$ [4, page 55]. Therefore it maps the semiaxis $(0,\infty)$ onto $(0,\infty)$ one-to-one, and there exists the inverse function φ_a^{-1}. By Theorem 1.3, we have

$$n_{|x|}(\tau)=\operatorname{meas}\{s\in[0,1]:|x(s)|>\tau\}\ge\psi(\tau) \quad \text{for } \tau>0, \tag{2.7}$$

where $\psi(\tau)=A^{-1}\exp\{-A[\varphi_a^{-1}(\tau A)]^2\}$. Passing to rearrangements we obtain

$$x^*(s)\ge\psi^{-1}(s) \quad \text{for } 0<s<A^{-1}. \tag{2.8}$$

Obviously, by condition $t\ge C_1=C_1(A)=\sqrt{2\log_2(2A)}$, it holds

$$2^{-k_t/2}<A^{-1} \quad \text{for } k_t=[t^2]-1. \tag{2.9}$$

Hence (2.4) and (2.8) imply

$$\mathcal{K}(t,x;L_\infty,G)\ge\psi^{-1}(2^{-k_t}). \tag{2.10}$$

Combining the definition of the function ψ with (2.9), we obtain

$$\begin{aligned}\psi^{-1}(2^{-k_t}) &= A^{-1}\varphi_a(A^{-1/2}\ln^{1/2}(A^{-1}2^{k_t})) \ge A^{-1}\varphi_a\left(\sqrt{\frac{k_t\ln 2}{2A}}\right)\\ &\ge A^{-3/2}\sqrt{\frac{\ln 2}{2}}\varphi_a\left(\sqrt{k_t}\right) \ge A^{-3/2}\sqrt{\frac{\ln 2}{2}}t^{-1}\sqrt{k_t}\varphi_a(t).\end{aligned} \tag{2.11}$$

From the inequality $t \ge C_1 \ge \sqrt{2}$ it follows that

$$\frac{\sqrt{k_t}}{t} \ge \frac{\sqrt{[t^2]-1}}{\sqrt{[t^2]+1}} \ge 3^{-1/2}. \tag{2.12}$$

Therefore, by (2.10), we have

$$\mathcal{K}(t,x;L_\infty,G) \ge C_2\varphi_a(t) \quad \text{for } t \ge C_1, \tag{2.13}$$

where $C_2 = C_2(A) = \sqrt{\ln 2/6}A^{-3/2}$.

If now $t \ge 1$, then the concavity of the $\mathcal{K}$-functional and the previous inequality yield

$$\mathcal{K}(t,x;L_\infty,G) \ge C_1^{-1}\mathcal{K}(tC_1,x;L_\infty,G) \ge \frac{C_2}{C_1}\varphi_a(C_1t) \ge \frac{C_2}{C_1}\varphi_a(t). \tag{2.14}$$

Using the inequalities $\|a\|_2 \le \|a\|_1$ $(a \in l_1)$ and $\|x\|_G \le \|x\|_\infty$ $(x \in L_\infty)$, the definition of the $\mathcal{K}$-functional, and Theorem 1.1, we obtain

$$\mathcal{K}(t,x;L_\infty,G) = t\|x\|_G \ge C_3t\|a\|_2 = C_3\varphi_a(t) \quad \text{for } 0 < t \le 1. \tag{2.15}$$

Thus,

$$\mathcal{K}(t,a;l_1,l_2) \le C\mathcal{K}(t,Ta;L_\infty,G), \tag{2.16}$$

if $C = \max(C_3^{-1}, C_1/C_2)$.

Suppose now $a \in l_1$. By (2.5), without loss of generality, we can assume that the function φ_a maps the semiaxis $(0,\infty)$ injectively onto the interval $(0,\|a\|_1)$. Hence we can define the mappings $\varphi_a^{-1} : (0,\|a\|_1) \to (0,\infty)$, $\psi : (0,A^{-1}\|a\|_1) \to (0,A^{-1})$, and $\psi^{-1} : (0,A^{-1}) \to (0,A^{-1}\|a\|_1)$. Arguing as above, we get inequality (2.16).

The opposite inequality follows from Theorem 1.1 and relation (1.4). Indeed,

$$\begin{aligned}\mathcal{K}(t,Ta;L_\infty,G) &\le \inf\{||Ta^0||_\infty + t||Ta^1||_G : a = a^0 + a^1,\ a^0 \in l_1,\ a^1 \in l_2\}\\ &\le D\mathcal{K}(t,a;l_1,l_2).\end{aligned} \tag{2.17}$$

□

Proof of Theorem 1.4. It is sufficient to use Theorem 1.2 and the definition of the real $\mathcal{K}$-method of interpolation. □

For the proof of Theorem 1.5 we need some definitions and auxiliary assertions. These results are also of some independent interest.

Let $f(t)$ be a function defined on the interval $(0,l)$, where $l = 1$ or $l = \infty$. Then the dilation function of f is defined as follows:

$$\mathcal{M}_f(t) = \sup\left\{\frac{f(st)}{f(s)} : s, st \in (0,l)\right\}, \quad \text{if } t \in (0,l). \tag{2.18}$$

Since this function is semimultiplicative, then there exist numbers

$$\gamma_f = \lim_{t\to 0+} \frac{\ln \mathcal{M}_f(t)}{\ln t}, \qquad \delta_f = \lim_{t\to\infty} \frac{\ln \mathcal{M}_f(t)}{\ln t}. \tag{2.19}$$

A Banach couple $\vec{X} = (X_0, X_1)$ is called a partial retract of a couple $\vec{Y} = (Y_0, Y_1)$ if each element $x \in X_0 + X_1$ is orbitally equivalent to some element $y \in Y_0 + Y_1$. The last means that there exist linear operators $U : \vec{X} \to \vec{Y}$ and $V : \vec{Y} \to \vec{X}$ such that $Ux = y$ and $Vy = x$.

Proposition 2.1. *Suppose that $M(\varphi)$ is a Marcinkiewicz space on $[0,1]$. If $\gamma_\varphi > 0$, then $\vec{X} = (L_\infty, M(\varphi))$ is a $\mathcal{K}$-monotone couple.*

Proof. It is sufficient to show that the couple $\vec{X}$ is a partial retract of the couple $\vec{Y} = (L_\infty, L_\infty(\tilde{\varphi}))$, where

$$\|x\|_{L_\infty(\tilde{\varphi})} = \sup_{0<t\le 1} \tilde{\varphi}(t)\,|x(t)|, \quad \tilde{\varphi}(t) = \frac{t}{\varphi}(t). \tag{2.20}$$

Indeed, a partial retract of a $\mathcal{K}$-monotone couple is a $\mathcal{K}$-monotone couple [15, page 420], and by the Sparr theorem [20] $\vec{Y}$ is a $\mathcal{K}$-monotone couple.

First note that the inclusion $L_\infty \subset M(\varphi)$ implies $L_\infty + M(\varphi) = M(\varphi)$. So, let $x \in M(\varphi)$. Without loss of generality [10, page 87], assume that $x(t) = x^*(t)$. Define the operator

$$U_1 y(t) = \sum_{k=1}^{\infty} 2^k \int_0^{2^{-k}} y(s)\,ds\,\chi_{(2^{-k},2^{-k+1}]}(t) \quad \text{for } y \in M(\varphi). \tag{2.21}$$

Clearly, U_1 maps L_∞ into itself. In addition, the concavity of the function φ and properties of the nonincreasing rearrangement imply

$$||U_1 y||_{L_\infty(\tilde{\varphi})} \le 2 \sup_{k=1,2,\ldots} \left(\varphi(2^{-k+1})\right)^{-1} \int_0^{2^{-k}} y^*(s)\,ds \le 2\|y\|_{M(\varphi)}. \tag{2.22}$$

Hence $U_1 : \vec{X} \to \vec{Y}$. Since $x(t)$ is nonincreasing, then $U_1 x(t) \ge x(t)$. Therefore the linear operator

$$Uy(t) = \frac{x(t)}{U_1 x(t)} U_1 y(t) \tag{2.23}$$

is bounded from the couple $\vec{X}$ into the couple $\vec{Y}$. In addition, $Ux(t) = x(t)$.

Take for V the identity mapping, that is, $Vy(t) = y(t)$. Since $\gamma_f > 0$, then, by [10, page 156], we have

$$\|Vy\|_{M(\varphi)} \le C \sup_{0<t\le 1} \tilde{\varphi}(t) y^*(t) \le C \sup_{0<t\le 1} \tilde{\varphi}(t)\,|y(t)| = C\|y\|_{L_\infty(\tilde{\varphi})}. \tag{2.24}$$

Therefore $V : \vec{Y} \to \vec{X}$ and $Vx = x$.

Thus an arbitrary element $x \in M(\varphi)$ is orbitally equivalent to itself as to element of the space $L_\infty + L_\infty(\tilde{\varphi})$. This completes the proof. □

Corollary 2.2. *If $\gamma_\varphi > 0$, then $(L_\infty, M(\varphi)^0)$ is a $\mathcal{K}$-monotone couple.*

Proof. Assume that x and y belong to the space $M(\varphi)^0$ and

$$\mathcal{K}(t,y;L_\infty,M(\varphi)^0) \le \mathcal{K}(t,x;L_\infty,M(\varphi)^0) \quad \text{for } t>0. \tag{2.25}$$

If $z \in M(\varphi)^0$, then

$$\mathcal{K}(t,z;L_\infty,M(\varphi)^0) = \mathcal{K}(t,z;L_\infty,M(\varphi)). \tag{2.26}$$

Therefore,

$$\mathcal{K}(t,y;L_\infty,M(\varphi)) \le \mathcal{K}(t,x;L_\infty,M(\varphi)) \quad \text{for } t>0. \tag{2.27}$$

Hence, by Proposition 2.1, there exists an operator $T:(L_\infty,M(\varphi)) \to (L_\infty,M(\varphi))$ such that $y = Tx$. It is readily seen that $M(\varphi)^0$ is an interpolation space of the couple $(L_\infty,M(\varphi))$. Therefore $T:(L_\infty,M(\varphi)^0) \to (L_\infty,M(\varphi)^0)$. □

We define now two subcones of the cone $\mathcal{P}$. Denote by $\mathcal{P}_0$ the set of all functions $f \in \mathcal{P}$ such that $\lim_{t\to 0+} f(t) = \lim_{t\to\infty} f(t)/t = 0$. If $f \in \mathcal{P}$, then $0 \le \gamma_f \le \delta_f \le 1$ [10, page 76]. Let $\mathcal{P}^{+-}$ be the set of all $f \in \mathcal{P}$ such that $0 < \gamma_f \le \delta_f < 1$. It is obvious that $\mathcal{P}^{+-} \subset \mathcal{P}_0$.

A couple (X_0,X_1) is called a $\mathcal{K}_0$-complete couple if for any function $f \in \mathcal{P}_0$ there exists an element $x \in X_0+X_1$ such that

$$\mathcal{K}(t,x;X_0,X_1) \asymp f(t). \tag{2.28}$$

In other words, the set $\mathcal{K}(X_0+X_1)$ of all $\mathcal{K}$-functionals of a $\mathcal{K}_0$-complete couple (X_0,X_1) contains, up to equivalence, the whole of the subcone $\mathcal{P}_0$.

Proposition 2.3. *The Banach couple $(L_1(0,\infty),L_2(0,\infty))$ is a $\mathcal{K}_0$-complete couple.*

Proof. By the Holmstedt formula for functional spaces [7],

$$\mathcal{K}(t,x,L_1,L_2) \asymp \max\left\{\int_0^{t^2} x^*(s)\,ds,\, t\left[\int_{t^2}^\infty (x^*(s))^2\,ds\right]^{1/2}\right\}. \tag{2.29}$$

If $f \in \mathcal{P}_0$, then $g(t) = f(t^{1/2})$ belongs to $\mathcal{P}_0$. We denote $x(t) = g'(t)$. Then $x(t) = x^*(t)$ and

$$\int_0^t x(s)\,ds = g(t). \tag{2.30}$$

Assume that $f \in \mathcal{P}^{+-}$. If $\delta_f < 1$, then there exists $\varepsilon > 0$ such that for some $C > 0$

$$G(s) = f(s^{1/2}) \le C\left(\sqrt{\frac{s}{t}}\right)^{1-\varepsilon} f(t^{1/2}), \quad \text{if } s \ge t. \tag{2.31}$$

Since $g \in \mathcal{P}_0$, then $g'(t) \le g(t)/t$. Therefore for $t > 0$

$$\int_t^\infty (x(s))^2\,ds \le \int_t^\infty \frac{g^2(s)}{s^2}\,ds \le C^2 t^{\varepsilon-1}(f(t^{1/2}))^2 \int_t^\infty s^{-1-\varepsilon}\,ds = C^2 \varepsilon t^{-1}(g(t))^2. \tag{2.32}$$

Combining this with (2.29) and (2.30), we obtain

$$\mathcal{K}(t,x;L_1,L_2) \asymp g(t^2) = f(t). \tag{2.33}$$

Thus $\mathcal{K}(L_1+L_2)\supset\mathcal{P}^{+-}$. Hence, in particular, the intersection $\mathcal{K}(X_0+X_1)\cap\mathcal{P}^{+-}$ is not empty. Therefore, by [6, Theorem 4.5.7], (L_1,L_2) is a $\mathcal{K}_0$-complete Banach couple. This completes the proof. □

Let $\mathcal{K}(l_1+l_2)$ be the set of all $\mathcal{K}$-functionals corresponding to the couple (l_1,l_2). By $\mathcal{F}$ we denote the set of all functions $f\in\mathcal{P}$ such that

$$f(t)=f(1)t \quad \text{for } 0<t\le 1, \qquad \lim_{t\to\infty}\frac{f(t)}{t}=0. \tag{2.34}$$

Corollary 2.4. *Up to equivalence,*

$$\mathcal{K}(l_1+l_2)\supset\mathcal{F}. \tag{2.35}$$

Proof. It is well known (cf. [4, page 142]) that for $x\in L_1(0,\infty)+L_\infty(0,\infty)$ and $u>0$

$$\mathcal{K}(u,x;L_1,L_\infty)=\int_0^u x^*(s)\,ds. \tag{2.36}$$

In addition,

$$L_1=(L_1,L_\infty)^{\mathcal{K}}_{l_\infty}, \qquad L_2=(L_1,L_\infty)^{\mathcal{K}}_{l_2(2^{-k/2})}. \tag{2.37}$$

The spaces l_∞ and $l_2(2^{-k/2})$ are interpolation spaces with respect to the couple $(l_\infty,l_\infty(2^{-k}))$ [4]. Therefore, by the reiteration theorem (see [5] or [14]),

$$\mathcal{K}(t,x;L_1,L_2)\asymp\mathcal{K}(t,\mathcal{K}(\cdot,x;L_1,L_\infty);l_\infty,l_2(2^{-k/2})) \quad \text{for } x\in L_1+L_2. \tag{2.38}$$

Introduce the average operator:

$$Qx(t)=\sum_{k=1}^{\infty}\int_{k-1}^{k}x(s)\,ds\,\chi_{(k-1,k]}(t), \quad \text{if } t>0. \tag{2.39}$$

From (2.36) it follows that

$$\mathcal{K}(t,Qx^*;L_1,L_\infty)=\mathcal{K}(t,x;L_1,L_\infty) \tag{2.40}$$

for all positive integers t. Both functions in (2.40) are concave. Therefore,

$$\mathcal{K}(t,Qx^*;L_1,L_\infty)\asymp\mathcal{K}(t,x;L_1\cdot L_\infty) \quad \forall t\ge 1. \tag{2.41}$$

Hence (2.38) yields

$$\mathcal{K}(t,Qx^*;L_1,L_2)\asymp\mathcal{K}(t,x;L_1,L_2), \quad \text{if } t\ge 1. \tag{2.42}$$

Now let $f\in\mathcal{F}$. Since $\mathcal{F}\subset\mathcal{P}_0$, then, by Proposition 2.3, there exists a function $x\in L_1(0,\infty)+L_2(0,\infty)$ such that

$$\mathcal{K}(t,x;L_1,L_2)\asymp f(t). \tag{2.43}$$

Clearly, the operator Q is a projector in the spaces L_1 and L_2 with norm 1. Moreover, $Q(L_1)=l_1$ and $Q(L_2)=l_2$. Hence, by the theorem about complemented subcouples mentioned in Section 1 (see [3] or [21, page 136]),

$$\mathcal{K}(t,Qx^*;L_1,L_2) \asymp \mathcal{K}(t,a;l_1,l_2) \quad \text{for } t>0, \tag{2.44}$$

where $a = (\int_{k-1}^{k} x^*(s)\,ds)_{k=1}^{\infty}$.

Thus (2.42) and (2.43) imply

$$\mathcal{K}(t,a;l_1,l_2) \asymp f(t) \quad \text{for } t \geq 1. \tag{2.45}$$

The last relation also holds if $0<t\leq 1$. Indeed, in this case

$$\mathcal{K}(t,a;l_1,l_2) = t\|a\|_2 = t\mathcal{K}(1,a;l_1,l_2) \asymp tf(1) = f(t). \tag{2.46}$$

This completes the proof. □

Proof of Theorem 1.5. As it was already mentioned in the proof of Theorem 1.2, the Orlicz space L_N, $N(t)=\exp(t^2)-1$, coincides with the Marcinkiewicz space $M(\varphi_1)$, for $\varphi_1(u) = u\log_2^{1/2}(2/u)$. Since $\gamma_{\varphi_1}=1$, then Corollary 2.2 implies that the couple (L_∞,G) is a $\mathcal{K}$-monotone couple. Hence,

$$X_0 = (l_\infty,G)_{E_0}^{\mathcal{K}}, \qquad X_1 = (l_\infty,G)_{E_1}^{\mathcal{K}}, \tag{2.47}$$

for some parameters of the real $\mathcal{K}$-method of interpolation E_0 and E_1. By Theorem 1.4,

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_i} \asymp \|(a_k)\|_{F_i}, \tag{2.48}$$

where $F_i = (l_1,l_2)_{E_i}^{\mathcal{K}} (i=0,1)$. So

$$(l_1,l_2)_{E_0}^{\mathcal{K}} = (l_1,l_2)_{E_1}^{\mathcal{K}}. \tag{2.49}$$

Equation (2.49) means that the norms of spaces E_0 and E_1 are equivalent on the set $\mathcal{K}(l_1+l_2)$. It is readily to check that this set coincides, up to the equivalence, with the set $\mathcal{K}(L_\infty+G)$ of all $\mathcal{K}$-functionals corresponding to the couple (L_∞,G). More precisely,

$$\mathcal{K}(l_1+l_2) = \mathcal{K}(L_\infty+G) = \mathcal{F}. \tag{2.50}$$

In fact, by Theorem 1.2 and Corollary 2.2, $\mathcal{F}\subset\mathcal{K}(l_1+l_2)\subset\mathcal{K}(L_\infty+G)$. On the other hand, since $L_\infty\subset G$ with the constant 1 and L_∞ is dense in G, then $\mathcal{K}(L_\infty+G)\subset\mathcal{F}$ [15, page 386].

Now let $x\in X_0$. By (2.47), we have $(\mathcal{K}(2^k,x;L_\infty,G))_k\in X_0$. Using (2.50), we can find $a\in l_2$ such that

$$\mathcal{K}(2^k,a;l_1,l_2) \asymp \mathcal{K}(2^k,x;L_\infty,G) \tag{2.51}$$

for all positive integers k. Since a parameter of $\mathcal{K}$-method is a Banach lattice, then this implies $(\mathcal{K}(2^k,a;l_1,l_2))_k\in E_0$. Therefore, by (2.49), $(\mathcal{K}(2^k,a;l_1,l_2))_k\in E_1$, that is, $(\mathcal{K}(2^k,x;L_\infty,G))_k\in E_1$ or $x\in X_1$. Thus $X_0\subset X_1$. Arguing as above, we obtain the converse inclusion, and $X_0=X_1$ as sets. Since X_0 and X_1 are Banach lattices, then their norms are equivalent. This completes the proof. □

3. Final remarks and examples

Remark 3.1. Combining Theorems 1.2, 1.4, and 1.5 with results obtained in [8], we can prove similar assertions for lacunary trigonometric series. Moreover, taking into account the main result of [1], we can extend Theorems 1.2, 1.4, and 1.5 to Sidon systems of characters of a compact abelian group.

Remark 3.2. In Theorem 1.2, we cannot replace the space G by L_q with some $q < \infty$. Indeed, suppose that the couple $(T(l_1), T(l_2))$ is a $\mathcal{K}$-subcouple of the couple (L_∞, L_q), that is,

$$\mathcal{K}(t, a; l_1, l_2) \asymp \mathcal{K}(t, Ta; L_\infty, L_q). \tag{3.1}$$

Let $E = l_p(2^{-\theta k})$, where $0 < \theta < 1$ and $p = q/\theta$. Applying the $\mathcal{K}$-method of interpolation $(\cdot,\cdot)_E^{\mathcal{K}}$ to the couples (l_1, l_2) and (L_∞, L_q), we obtain

$$\|Ta\|_p \asymp \|a\|_{r,p} = \left\{ \sum_{k=1}^{\infty} (a_k^*)^p k^{p/r-1} \right\}^{1/p}. \tag{3.2}$$

Since $r = 2/(2-\theta) < 2$ [4, page 142], then this contradicts with (1.3).

Remark 3.3. Clearly, a partial retract of a couple $\vec{Y} = (Y_0, Y_1)$ is a $\mathcal{K}$-subcouple of $\vec{Y}$. The opposite assertion is not true, in general (nevertheless, some interesting examples of $\mathcal{K}$-subcouples and partial retracts simultaneously are given in [9]). Indeed, by Theorem 1.2, the subcouple (l_1, l_2) is a $\mathcal{K}$-subcouple of the couple (L_∞, G). Assume that (l_1, l_2) is a partial retract of this couple. Then (see the proof of Proposition 2.1) (l_1, l_2) is a partial retract of the couple $(L_\infty, L_\infty(\log_2^{-1/2}(2/t)))$, as well. Therefore, by Lemma 1 from [2] and [4, page 142] it follows that

$$[l_1, l_2]_\theta = (l_1, l_2)_{\theta,\infty} = l_{p,\infty}, \tag{3.3}$$

where $[l_1, l_2]_\theta$ is the space of the complex method of interpolation [4], $0 < \theta < 1$, and $p = 2/(2-\theta)$. On the other hand, it is well known [4, page 139] that

$$[l_1, l_2]_\theta = l_p \quad \text{for } p = \frac{2}{2-\theta}. \tag{3.4}$$

This contradiction shows that the couple (l_1, l_2) is not a partial retract of the couple (L_∞, G).

Using Theorem 1.4, we can find coordinate sequence spaces of coefficients of Rademacher series belonging to certain r.i.s.'s.

Example 3.4. Let X be the Marcinkiewicz space $M(\varphi)$, where $\varphi(t) = t\log_2\log_2(16/t)$, $0 < t \le 1$. Show that

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{M(\varphi)} \asymp \|a\|_{l_1(\log)}, \tag{3.5}$$

where $l_1(\log)$ is the space of all sequences $a = (a_k)_{k=1}^\infty$ such that the norm

$$\|a\|_{l_1(\log)} = \sup_{k=1,2,\dots} \log_2^{-1}(2k) \sum_{i=1}^{k} a_i^* \tag{3.6}$$

is finite. Taking into account Theorem 1.4, it is sufficient to check that

$$(l_1,l_2)_F^{\mathcal{K}} = l_1(\log), \tag{3.7}$$

$$(l_\infty,G)_F^{\mathcal{K}} = M(\varphi), \tag{3.8}$$

for some parameter F of the $\mathcal{K}$-method of interpolation. More precisely, we will prove that (3.7) and (3.8) are true for $F = l_\infty(u_k)$, where $u_k = 1/(k+1)$ $(k \ge 0)$ and $u_k = 1$ $(k < 0)$.

By the Holmstedt formula (2.5),

$$\varphi_a(2^k) \le \sum_{i=1}^{2^{2k}} a_i^* + 2^k\left[\sum_{i=2^{2k}+1}^{\infty} (a_i^*)^2\right]^{1/2} \le B\varphi_a(2^k) \quad \text{for } k = 0,1,2,\dots, \tag{3.9}$$

where, as before, $\varphi_a(t) = \mathcal{K}(t,a;l_1,l_2)$. Without loss of generality, assume that $a_i = a_i^*$. If $\|a\|_{l_1(\log)} = R < \infty$, then by (3.6),

$$\sum_{i=1}^{2^{2k}} a_i^* \le 2R(k+1). \tag{3.10}$$

In particular, this implies $a_{2^{2k}} \le 2^{-2k+1}R(k+1)$, for nonnegative integer k. Using (3.10), we obtain

$$\begin{aligned}\sum_{i=2^{2k}+1}^{\infty} a_i^2 &= \sum_{j=k}^{\infty}\sum_{i=2^{2j}+1}^{2^{2(j+1)}} a_i^2 \le 3\sum_{j=k}^{\infty} 2^{2j}a_{2^{2j}}^2 \le 12R^2\sum_{j=k}^{\infty} 2^{-2j}(j+1)^2 \\ &\le 192R^2\int_{k+1}^{\infty} x^2 2^{-2x}\,dx \le 144R^2(k+1)^2 2^{-2k}.\end{aligned} \tag{3.11}$$

Hence the second term in (3.9) does not exceed $12R(k+1)$. Therefore, if $E = (l_1,l_2)_F^{\mathcal{K}}$, then (3.10) implies

$$\|a\|_E = \sup_{k=0,1,\dots} \frac{\varphi_a(2^k)}{k+1} \le 14\|a\|_{l_1(\log)}. \tag{3.12}$$

Conversely, if $2^{2j}+1 \le k \le 2^{2(j+1)}$ for some $j = 0,1,2,\dots$, then from (3.9) it follows that

$$\sum_{i=1}^{k} a_i \le B\varphi_a(2^{j+1}) \le \sum_{i=1}^{2^{2(j+1)}} a_i \le B\|a\|_E(j+2) \le 2B\log_2(2k)\|a\|_E. \tag{3.13}$$

Therefore, $\|a\|_{l_1(\log)} \le 2B\|a\|_E$ and (3.7) is proved.

We pass now to function spaces. At first, we introduce one more interpolation method which is, actually, a special case of the real method of interpolation. For a function $\varphi \in \mathcal{P}$ and an arbitrary Banach couple (X_0,X_1) define generalized Marcinkiewicz space as follows:

$$M_\varphi(X_0,X_1) = \left\{x \in X_0+X_1 : \sup_{t>0} \frac{\mathcal{K}(t,x;X_0,X_1)}{\varphi(t)} < \infty\right\}. \tag{3.14}$$

Let $\varphi_0(t) = \min(1,t)$, $\varphi_1(t) = \min(1, t\log_2^{1/2}[\max(2,2/t)])$, and $N(t) = \exp(t^2) - 1$, as before. By equation (2.36), we have

$$L_\infty = M_{\varphi_0}(L_1, L_\infty), \qquad L_N = M_{\varphi_1}(L_1, L_\infty), \tag{3.15}$$

(here L_∞ and L_N are functional spaces on the segment $[0,1]$). In addition, using similar notation, it is easy to check that

$$(X_0, X_1)_F^{\mathcal{K}} = M_\rho(X_0, X_1), \tag{3.16}$$

for an arbitrary Banach couple (X_0, X_1) and $\rho(t) = \log_2(4+t)$. Hence, by the reiteration theorem for generalized Marcinkiewicz spaces [15, page 428], we obtain

$$(L_\infty, L_N)_F^{\mathcal{K}} = M_\rho(M_{\varphi_0}(L_1, L_\infty), M_{\varphi_1}(L_1, L_\infty)) = M_{\varphi_\rho}(L_1, L_\infty) = M(\varphi_\rho), \tag{3.17}$$

where $\varphi_\rho(t) = \varphi_0(t)\rho(\varphi_1(t)/\varphi_0(t))$. A simple calculation gives $\varphi_\rho(t) \asymp \varphi(t)$, if $t > 0$. Thus,

$$(L_\infty, L_N)_F^{\mathcal{K}} = M(\varphi). \tag{3.18}$$

It is readily seen that $\mathcal{K}(t, x; L_\infty, G) = \mathcal{K}(t, x; L_\infty, L_N)$, for all $x \in G$. Therefore, for such x the norm $\|x\|_{M(\varphi)}$ is equal to the norm $\|x\|_Y$, where $Y = (L_\infty, G)_F^{\mathcal{K}}$. On the other hand, for $x \in M(\varphi)$

$$\frac{1}{t\log_2^{1/2}(2/t)} \int_0^t x^*(s)\,ds \le \|x\|_{M(\varphi)} \frac{\log_2\log_2(16/t)}{\log_2^{1/2}(2/t)} \longrightarrow 0 \quad \text{as } t \longrightarrow 0+. \tag{3.19}$$

This implies that $M(\varphi) \subset G$ [10, page 156]. Thus $Y = M(\varphi)$, and (3.8) is proved. Equivalence (3.5) follows now, as already stated, from (3.7) and (3.8).

Remark 3.5. Theorems 1.4 and 1.5 strengthen results of [18, 19], where similar assertions are obtained for sequence spaces F satisfying more restrictive conditions. For instance, we can readily show that the norm of the dilation operator

$$\sigma_n a = \left(\underbrace{a_1, \cdot, a_1}_{n}, \underbrace{a_2, \cdot, a_2}_{n}, \dots \right) \tag{3.20}$$

in the space $l_1(\ln)$ (see Example 3.6) is equal to n. Therefore, condition (11) from [19] fails for this space and the theorems obtained in [18, 19] cannot be applied to it. Similarly, the Marcinkiewicz space $M(\varphi)$ from Example 3.4 does not satisfy the conditions of Theorem 8 of [19].

Using Theorems 1.4 and 1.5, we can derive certain interpolation relations.

Example 3.6. Let $\varphi \in \mathcal{P}$ and $1 \le p < \infty$. Recall that the Lorentz space $\Lambda_p(\varphi)$ consists of all measurable functions $x = x(s)$ such that

$$\|x\|_{\varphi,p} = \left\{ \int_0^1 (x^*(s))^p \, d\varphi(s) \right\}^{1/p} < \infty. \tag{3.21}$$

In [19], V. A. Rodin and E. M. Semenov proved that

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{\varphi,p} \asymp \|(a_k)\|_p, \tag{3.22}$$

where $\varphi(s) = \log_2^{1-p}(2/s)$ and $1 < p < 2$. Moreover, the space $\Lambda_p(\varphi)$ is the unique r.i.s. having this property. Note that $l_p = (l_1, l_2)_{\theta,p}$, where $\theta = 2(p-1)/p$ [4, page 142]. Therefore, by Theorem 1.4, we obtain

$$(L_\infty, G)_{\theta,p} = \Lambda_p(\varphi) \tag{3.23}$$

for the same p and θ.

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