Siberian Mathematical Journal, Vol. 41, No. 4, 2000

# FOURIER–RADEMACHER COEFFICIENTS OF FUNCTIONS IN REARRANGEMENT-INVARIANT SPACES

S. V. Astashkin
UDC 517.982.27

## 1. Preliminaries and Definitions

Let $\{w_k\}_{k=1}^{\infty}$ be a uniformly bounded orthonormal system of functions on the interval $[0,1]$. Given a function $x \in L_1[0,1]$, denote its Fourier coefficients with respect to $\{w_k\}_{k=1}^{\infty}$ by $c_k = c_k(x)$.

If $F$ is the operator carrying a function $x$ to the sequence of its Fourier coefficients, $Fx = (c_k(x))_{k=1}^{\infty}$, then $F$ acts boundedly from $L_2$ into $l_2$ and, by Mercer's theorem [1, p. 181], from $L_1$ into $c_0$. Therefore, for every $x \in L_1$ we have

$$\mathscr{K}(t, Fx; c_0, l_2) \le A_1 \mathscr{K}(t, x; L_1, L_2) \quad (t > 0), \tag{1}$$

where

$$\mathscr{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i\}$$

is the Peetre $\mathscr{K}$-functional defined for $x \in X_0 + X_1$ ($(X_0, X_1)$ is an arbitrary Banach pair), $t > 0$, and $A_1 = \max\{1, \sup\{|w_k(t)| : k = 1, 2, \dots,\ t \in [0,1]\}\}$.

By means of (1) the following estimate for the Fourier coefficients of a function $x \in L_1$ was obtained in [2]:

$$\left\{\sum_{k=1}^{n} [c_k^*(x)]^2\right\}^{1/2} \le A_1 \left\{\sqrt{n} \int_0^{1/n} x^*(t)\,dt + \left(\int_{1/n}^{1} [x^*(t)]^2\,dt\right)^{1/2}\right\}. \tag{2}$$

where $n = 1, 2, \dots$, and $x^*(t)$ and $(c_k^*(x))_{k=1}^{\infty}$ are decreasing rearrangements of the function $|x(u)|$ and the sequence $(|c_i(x)|)_{i=1}^{\infty}$.

Relation (2) can be treated as the Bessel inequality for summable functions; moreover, it is exact in a sense for the trigonometric system [2]. At the same time, (2) can be essentially strengthened for lacunary systems, for example, the Rademacher system

$$r_k(t) = \operatorname{sgn} \sin 2^{k-1}\pi t \quad (k = 1, 2, \dots).$$

Recall that a Banach space $X$ of measurable functions $x = x(t)$ on $[0,1]$ is *rearrangement-invariant* if the relations $y^*(t) \le x^*(t)$ and $x \in X$ imply that $y \in X$ and $\|y\|_X \le \|x\|_X$. The theory of rearrangement-invariant spaces (RIS) is presented in detail, for instance, in the monograph [3].

Alongside the $L_p$ spaces ($1 \le p \le \infty$), important examples of RIS are Orlicz, Lorentz, and Marcinkiewicz spaces.

If $S(t) \ge 0$ is a continuous convex function on $[0, \infty)$ such that $S(0) = 0$ then the Orlicz space $L_S$ consists of all $x = x(t)$ such that

$$\|x\|_{L_S} = \inf\left\{u > 0 : \int_0^1 S\left(\frac{|x(t)|}{u}\right) dt \le 1\right\} < \infty.$$

Samara. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 41, No. 4, pp. 729–739, July–August, 2000. Original article submitted December 22, 1998.

If $\varphi(t) \geq 0$ is an increasing concave function on $[0,1]$ then the Marcinkiewicz space $M(\varphi)$ consists of all $x = x(t)$ such that

$$\|x\|_{M(\varphi)} = \sup\left\{\frac{1}{\varphi(t)}\int_0^t x^*(s)\,ds : 0 < t \leq 1\right\} < \infty,$$

and the Lorentz space $\Lambda_p(\varphi)$ $(1 \leq p < \infty)$ consists of all $x = x(t)$ for which

$$\|x\|_{\Lambda_p(\varphi)} = \left\{\int_0^1 [x^*(t)]^p\,d\varphi(t)\right\}^{1/p} < \infty.$$

In particular, we denote $\Lambda_1(\varphi)$ by $\Lambda(\varphi)$.

Suppose that $N(u) = e^{u^2} - 1$, $L_N$ is the Orlicz space constructed for this function, and $G$ is the closure of $L_\infty$ in $L_N$. V. A. Rodin and E. M. Semënov [4] demonstrated that the system $\{r_k\}_{k=1}^\infty$ in a RIS $X$ is equivalent to the standard basis in $l_2$ if and only if $X \supset G$. Equivalence of the system $\{r_k\}_{k=1}^\infty$ in $X$ to the standard basis in $l_2$ implies existence of a constant $C > 0$ such that

$$C^{-1}\|(a_k)\|_2 \leq \left\|\sum_{k=1}^\infty a_k r_k\right\|_X \leq C\|(a_k)\|_2, \tag{3}$$

where $\|(a_k)\|_2 = (\sum_{k=1}^\infty a_k^2)^{1/2}$. S. Montgomery-Smith [5] later strengthened this result by finding the distribution of the Rademacher sums. To this end, he used the $\mathscr{K}$-functional for the Banach pair $(l_1, l_2)$.

Moreover, if the norm of the RIS $X$ is order semicontinuous (i.e., the relations $x_n = x_n(t) \geq 0$ and $x_n(t) \uparrow x(t)$ almost everywhere on $[0,1]$ as $n \to \infty$ imply that $\|x_n\|_X \to \|x\|_X$) then the subspace generated by the Rademacher system is complemented in $X$ if and only if $G \subset X \subset G'$ (see [6] or [7, 2.b.4]). Here and in the sequel, $X'$ is the dual of a RIS $X$ and consists of all $y = y(t)$ such that

$$\|y\|_{X'} = \sup\left\{\int_0^1 y(t)x(t)\,dt : \|x\|_X \leq 1\right\} < \infty.$$

Furthermore, note that if $G \subset X \subset G'$ then the orthogonal projection

$$Px(t) = \sum_{k=1}^\infty c_k(x) r_k(t)$$

is bounded in $X$; here $c_k(x) = \int_0^1 x(t) r_k(t)\,dt$ are the Fourier–Rademacher coefficients of a function $x$.

The above results imply that the operator $Fx = (c_k(x))_{k=1}^\infty$ acts boundedly from $G'$ into $l_2$. Indeed, owing to (3) and boundedness of $P$ in $G'$, we have

$$\|Fx\|_2 = \left\|(c_k(x))_{k=1}^\infty\right\|_2 \leq C\|Px\|_{G'} \leq C\|P\|_{G' \to G'}\|x\|_{G'}.$$

By the definition of $\mathscr{K}$-functional, in the case of the Rademacher system instead of (1) we thus obtain the following inequality:

$$\mathscr{K}(t, Fx; c_0, l_2) \leq A_2 \mathscr{K}(t, x; L_1, G') \quad (t > 0). \tag{4}$$

Let us replace the $\mathscr{K}$-functionals in this relation with equivalent expressions.

The Orlicz space $L_N$ is well known [8] to coincide with the Marcinkiewicz space $M(\psi_1)$ with $\psi_1(u) = u\log^{1/2} e/u$. Therefore, $G' = \Lambda(\psi_1)$ [3, p. 158]. Since $L_1$ is also the Lorentz space constructed for the function $\psi_0(u) = u$, we have

$$\mathscr{K}(t, x; L_1, G') \asymp \int_0^1 x^*(s)\, d(\min(s, ts\log^{1/2} e/s))$$

[3, p. 164] (this means validity of inequalities like (3) with some constant independent of $x \in L_1$ and $t > 0$). After simple transformations we conclude that

$$\mathscr{K}(t, x; L_1, G') \asymp \int_0^{\nu(t)} x^*(s)\, ds + t \int_{\nu(t)}^1 x^*(s)\log^{1/2} e/s\, ds, \tag{5}$$

where $\nu(t) = \exp(1 - 1/t^2)$.

Putting $t = 1/\sqrt{n}$ ($n = 1, 2, \dots$), from (4), (5), and the relation

$$\mathscr{K}\left(\frac{1}{\sqrt{n}}, (a_k); c_0, l_2\right) \asymp \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n [a_k^*]^2\right)^{1/2},$$

resulting from Holmstedt's formula [9], we arrive at the inequality

$$\frac{1}{\sqrt{n}} \left\{\sum_{k=1}^n [c_k^*(x)]^2\right\}^{1/2} \le A_2 \left\{\int_0^{e^{1-n}} x^*(s)\, ds + \frac{1}{\sqrt{n}} \int_{e^{1-n}}^1 x^*(s)\log^{1/2} e/s\, ds\right\}$$

or

$$\left\{\sum_{k=1}^n [c_k^*(x)]^2\right\}^{1/2} \le A_2 \left\{\sqrt{n} \int_0^{e^{1-n}} x^*(s)\, ds + \int_{e^{1-n}}^1 x^*(s)\log^{1/2} e/s\, ds\right\}, \tag{6}$$

where the constant $A_2 > 0$ is independent of $x \in L_1$ and $n = 1, 2, \dots$.

Using

$$x^{**}(t) = \frac{1}{t} \int_0^t x^*(s)\, ds,$$

we can derive an inequality similar to (6) but with a sole expression on the right-hand side.

First of all, for every $0 < t \le 1$ we have

$$\begin{aligned}\int_t^1 x^{**}(s)\, ds &= \int_0^t x^*(u) \int_t^1 \frac{ds}{s}\, du + \int_t^1 x^*(u) \int_u^1 \frac{ds}{s}\, du \\ &= \log\frac{1}{t} \int_0^t x^*(u)\, du + \int_t^1 x^*(u) \log\frac{1}{u}\, du.\end{aligned} \tag{7}$$

Note that for $0 < u \le e^{-2}$ we have $\log^{1/2} e/u \le \log 1/u$. Therefore,

$$\int_t^1 x^*(u) \log 1/u\, du \ge \int_t^{e^{-2}} x^*(u) \log 1/u\, du \ge \int_t^{e^{-2}} x^*(u)\log^{1/2} e/u\, du;$$

moreover, if $0 < t \le e^{-3}$ then

$$\int_t^1 x^*(u)\log 1/u\,du \ge \int_{e^{-3}}^{e^{-2}} x^*(u)\log 1/u\,du \ge \frac{2(e-1)}{e^3}x^*(e^{-2}) \ge \frac{1}{e(e+1)}\int_{e^{-2}}^1 x^*(u)\log^{1/2} e/u\,du.$$

Thus, for $0 < t \le e^{-3}$ we have

$$\int_t^1 x^*(u)\log 1/u\,du \ge \frac{1}{e(e+2)}\int_t^1 x^*(u)\log^{1/2} e/u\,du;$$

whence, by (7), for all $n \ge 4$

$$\int_{e^{1-n}}^1 x^{**}(u)\,du \ge (n-1)\int_0^{e^{1-n}} x^*(u)\,du + \frac{1}{e(e+2)}\int_{e^{1-n}}^1 x^*(u)\log^{1/2} e/u\,du$$
$$\ge \frac{1}{e(e+2)}\left(\sqrt{n}\int_0^{e^{1-n}} x^*(u)\,du + \int_{e^{1-n}}^1 x^*(u)\log^{1/2} e/u\,du\right).$$

Finally, from (6) we obtain

$$\left\{\sum_{k=1}^n \left[c_k^*(x)\right]^2\right\}^{1/2} \le A_3\int_{e^{1-n}}^1 x^{**}(u)\,du, \tag{8}$$

where the constant $A_3 > 0$ is independent of $x \in L_1$ and $n \ge 4$.

However, it is possible to obtain (and this is the main goal of the article) much more exact information on the Fourier–Rademacher coefficients of the functions in a RIS than (6) and (8).

Given measurable functions $x = x(t)$ and $y = y(t)$ on $[0,1]$, put

$$(x,y) = \int_0^1 x(t)y(t)\,dt \quad \text{(if the integral exists)}.$$

Similarly, given sequences $a = (a_k)_{k=1}^\infty$ and $b = (b_k)_{k=1}^\infty$, put

$$(a,b) = \sum_{k=1}^\infty a_k b_k \quad \text{(if the series converges)}.$$

If

$$Ta(t) = \sum_{k=1}^\infty a_k r_k(t) \tag{9}$$

then the following relation holds for every function $x = x(t)$ and every sequence $a = (a_k)_{k=1}^\infty$ with finitely many nonzero terms:

$$(Fx,a) = \sum_{k=1}^\infty c_k(x)a_k = \int_0^1 x(t)\sum_{k=1}^\infty a_k r_k(t)\,dt = (x,Ta). \tag{10}$$

By duality, we easily obtain a description for the set of the Fourier–Rademacher coefficients of a RIS $X$ if $X \subset G'$. First of all, due to the Khinchin's inequality [10] for the $L_1$ space, we have

$$\frac{1}{\sqrt{2}}\|Ta\|_{L_2} \le \|Ta\|_{L_1} \le \|Ta\|_{L_2}.$$

Consequently, it follows from $\|Ta\|_{L_2} = \|a\|_2$ that $T$ is an injective bounded operator from $l_2$ into $L_1$. But then the adjoint $T^*$ is a bounded operator from $L_\infty$ onto $l_2$ [11, p. 20]. By (10), $T^* = F$ and hence $F(L_\infty) = l_2$. Since the embedding $X \supset L_\infty$ holds for every RIS $X$ on $[0,1]$ [3, p. 123], we certainly have $F(X) \supset l_2$. At the same time, as shown above, $F(G') \subset l_2$. We thus arrive at

**Proposition 1.** *If a rearrangement-invariant space $X$ is contained in $G'$ then the corresponding space of the Fourier–Rademacher coefficients is $F(X) = l_2$.*

We now turn to studying the more interesting case of $X \supset G'$.

Since the Rademacher system possesses the Sidon property, $F$ is a surjective bounded operator from $L_1$ onto $c_0$ [1, p. 295]; i.e.,

$$F(L_1) = c_0. \tag{11}$$

At the same time, by the above proposition

$$F(G') = l_2. \tag{12}$$

Let us show that the surjectivity relations (11) and (12) can be "interpolated" for spaces intermediate between $G'$ and $L_1$, the spaces of the Fourier–Rademacher coefficients still quite definite rearrangement-invariant spaces of sequences.

Suppose that $(X_0, X_1)$ and $(Y_0, Y_1)$ are Banach pairs. A triple $(X_0, X_1, X)$ of spaces such that $X_0 \cap X_1 \subset X \subset X_0 + X_1$ is an *interpolation triple* with respect to a triple $(Y_0, Y_1, Y)$, $Y_0 \cap Y_1 \subset Y \subset Y_0 + Y_1$, if every linear operator $A$, acting boundedly from $X_0$ into $Y_0$ and from $X_1$ into $Y_1$, is a bounded operator from $X$ into $Y$. Moreover, if

$$\|A\|_{X \to Y} \le \max\{\|A\|_{X_i \to Y_i} : i = 0, 1\}$$

then $(X_0, X_1, X)$ is said to be an *exact interpolation triple* with respect to the triple $(Y_0, Y_1, Y)$. If $X_i = Y_i$ $(i = 0, 1)$ and $X = Y$ then we say that $X$ is an (*exact*) *interpolation space* with respect to the pair $(X_0, X_1)$.

In what follows, an important role is played by the real interpolation method. We recall its definition.

Suppose that $E$ is an ideal Banach space (IBS) of two-sided numeric sequences $a = (a_j)_{j=-\infty}^{\infty}$. If $(X_0, X_1)$ is an arbitrary Banach pair then the space $(X_0, X_1)_E^{\mathscr{K}}$ of the $\mathscr{K}$-method consists of all $x \in X_0 + X_1$ such that

$$\|x\| = \|(\mathscr{K}(2^j, x; X_0, X_1))_j\|_E < \infty.$$

The space $(X_0, X_1)_E^{\mathscr{J}}$ of the $\mathscr{J}$-method contains all $x \in X_0 + X_1$ that admit the representation

$$x = \sum_{j=-\infty}^{\infty} u_j \quad \text{(in the sense of convergence in } X_0 + X_1\text{)}, \tag{13}$$

where $u_j \in X_0 \cap X_1$. The norm in $(X_0, X_1)_E^{\mathscr{J}}$ is defined by

$$\inf_{\{u_j\}} \|(\mathscr{J}(2^j, u_j; X_0, X_1))_j\|_E,$$

where the greatest lower bound is calculated over all sequences $\{u_j\}_{j=-\infty}^{\infty}$ for which (13) holds.

If $E$ is an IBS of two-sided sequences and $\alpha = (\alpha_k)_{k=-\infty}^{\infty}$ is a sequence of nonnegative numbers then the space $E(\alpha_k)$ consists of all $a = (a_k)_{k=-\infty}^{\infty}$ such that $(a_k\alpha_k)_{k=-\infty}^{\infty} \in E$, and $\|a\|_{E(\alpha_k)} = \|(a_k\alpha_k)\|_E$. Suppose that $E \supset \Delta(\vec{l}_\infty) = l_\infty(\max(1.2^{-k}))$ ($\{0\} \neq E \subset \Sigma(\vec{l}_1) = l_1(\min(1, 2^{-k}))$). Then the mapping $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathscr{K}}$ ($(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathscr{J}}$) determines an exact interpolation functor. This means that, for arbitrary Banach pairs $(X_0, X_1)$ and $(Y_0, Y_1)$, the triple $(X_0, X_1, (X_0, X_1)_E^{\mathscr{K}})$ is an exact interpolation triple with respect to the triple $(Y_0, Y_1, (Y_0, Y_1)_E^{\mathscr{K}})$ (a similar result is valid for the $\mathscr{J}$-method). The collection of all such functors is called the *real $\mathscr{K}$-interpolation method* (*$\mathscr{J}$-interpolation method*).

In particular, for $0 < \theta < 1$ and $1 \leq p \leq \infty$ we obtain the classical interpolation spaces

$$(X_0, X_1)_{\theta,p} = (X_0, X_1)_{l_p(2^{-k\theta})}^{\mathscr{K}} = (X_0, X_1)_{l_p(2^{-k\theta})}^{\mathscr{J}}.$$

## 2. The Main Results

As mentioned, the space $G'$ coincides with the Lorentz space $\Lambda(u\log^{1/2} e/u)$. Therefore, it is $q$-concave for every $q > 1$ [12] and elastic in the sense of [13]. But then, by Corollary 5.10 of [13], interpolation in the Banach pair $(G', L_1)$ is described by the $\mathscr{K}$-method. This means that for every interpolation space $X$ with respect to this pair we have

$$X = (G', L_1)_E^{\mathscr{K}} \tag{14}$$

for some IBS $E$ of two-sided sequences which can be assumed to be an exact interpolation space with respect to the pair $\vec{l}_\infty = (l_\infty, l_\infty(2^{-k}))$ [14].

**Theorem 1.** *Suppose that $X$ is a separable interpolation* RIS *with respect to the pair $(G', L_1)$ such that* (14) *holds. Then the corresponding space $F(X)$ of the Fourier–Rademacher coefficients coincides with the separable space*

$$R = (l_2, c_0)_E^{\mathscr{K}}. \tag{15}$$

PROOF. By duality [11, p. 20], it suffices to verify that the adjoint operator $F^*$ is injective from $R^*$ into $X^*$.

First of all, we can take the BIS $E$ in (14) so as to fulfill the conditions of the duality theorem for the real interpolation method [14, p. 128]:

$$\Delta(\vec{l}_\infty) \quad \text{is everywhere dense in } E \tag{16}$$

and

$$E \not\subset l_\infty \quad \text{and} \quad E \not\subset l_\infty(2^{-k}). \tag{17}$$

To prove (16), denote by $E_0$ the closure of $\Delta(\vec{l}_\infty)$ in $E$ and show that

$$X = (G', L_1)_{E_0}^{\mathscr{K}}. \tag{18}$$

Since $X$ is separable, the space $L_\infty$ is everywhere dense in $X$ [3, p. 138]. Therefore, by the definition of the spaces of the $\mathscr{K}$-method, for every $x \in X$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset L_\infty$ such that

$$\|x_n - x\|_X = \|(\mathscr{K}(2^k, x_n - x; G', L_1))_k\|_E \to 0$$

as $n \to \infty$. For each $t > 0$ the $\mathscr{K}$-functional $\mathscr{K}(t, x; X_0, X_1)$ is a norm on the space $X_0 + X_1$. Consequently, the preceding relation implies that as $n \to \infty$

$$\|(\mathscr{K}(2^k, x_n; G', L_1))_k - (\mathscr{K}(2^k, x; G', L_1))_k\|_E \to 0. \tag{19}$$

By the definition of the $\mathscr{K}$-functional, we have $\mathscr{K}(2^k, x_n; G', L_1) \leq 2^k\|x_n\|_{L_1}$ for $k \leq 0$ and $\mathscr{K}(2^k, x_n; G', L_1) \leq \|x_n\|_{G'}$ for $k > 0$. Thereby for each $n = 1, 2, \ldots$ the sequence $(\mathscr{K}(2^k, x_n; G', L_1))_{k=-\infty}^{\infty}$ lies in $\Delta(\vec{l}_\infty)$ and (19) implies that $(\mathscr{K}(2^k, x; G', L_1))_{k=-\infty}^{\infty} \in E_0$; i.e., $x \in (G', L_1)_{E_0}^{\mathscr{K}}$. Relation (18) is proven: hence, we may assume that (16) is satisfied.

We now turn to (17). Since $\|y\|_{L_1} \leq 2\|y\|_{G'}$, for $y \in L_1$ and $0 < t \leq 1$ we have

$$\frac{1}{2}t\|y\|_{L_1} \leq \mathscr{K}(t, y; G', L_1) \leq t\|y\|_{L_1}.$$

In view of the embedding $E \supset \Delta(\vec{l}_\infty) = l_\infty(\max(1, 2^{-k}))$, we may therefore assume that $E \supset c_0$ (otherwise we take $E + c_0$ instead of $E$, observing that $\Delta(\vec{l}_\infty)$ is also everywhere dense in $E + c_0$). We derive in particular that $E \not\subset l_\infty(2^{-k})$.

If $E \subset l_\infty$ then from completeness of $G'$ with respect to $L_1$ we obtain

$$G' \subset X = (G', L_1)_E^{\mathscr{K}} \subset (G', L_1)_{l_\infty}^{\mathscr{K}} = G'$$

[15, p. 384]; i.e., $X = G'$. Similarly, $(l_2, c_0)_E^{\mathscr{K}} = l_2$ and the claim of the theorem is contained in (12). Thus, we can suppose that $E \not\subset l_\infty$ and conditions (17) are satisfied too.

Since $(G')^* = L_N$, with $N(u) = e^{u^2} - 1$ [16, p. 97]; applying the duality theorem for the real method [14, p. 128], we obtain

$$X^* = (L_N, L_\infty)_{E^+}^{\mathscr{J}} \quad \text{and} \quad R^* = (l_2, l_1)_{E^+}^{\mathscr{J}}, \tag{20}$$

where $E^+$ is the BIS of two-sided sequences that is dual to $E$ with respect to the bilinear functional

$$\langle a, b\rangle = \sum_{k=-\infty}^{\infty} a_k b_k 2^{-k}, \quad a = (a_k)_{k=-\infty}^{\infty}, \quad b = (b_k)_{k=-\infty}^{\infty};$$

i.e.,

$$\|a\|_{E^+} = \sup\{\langle a, b\rangle : \|b\|_E \leq 1\}.$$

It follows from (16) and (17) that $l_2$ is everywhere dense in $R$ [14, p. 106]; hence, $R$ is separable. Therefore, $X^*$ and $R^*$ coincide with $X'$ and $R'$ [17, p. 254], and $F^* = T$ in view of (10). Since $E$ is an exact interpolation space with respect to the pair $\vec{l}_\infty$, it follows that $E^+$ is an exact interpolation space with respect to the pair $\vec{l}_1 = (l_1, l_1(2^{-k}))$. Moreover, (17) implies that $E^+ \not\subset l_1$ and $E^+ \not\subset l_1(2^{-k})$. Applying [14, p. 68], we can therefore replace the $\mathscr{J}$-spaces in (20) with the $\mathscr{K}$-spaces

$$X' = (L_N, L_\infty)_{\widetilde{E}}^{\mathscr{K}} \quad \text{and} \quad R' = (l_2, l_1)_{\widetilde{E}}^{\mathscr{K}},$$

where $\widetilde{E} = (\vec{l}_\infty)_{E^+}^{\mathscr{J}}$. Owing to the equality $\mathscr{K}(t, x; X_0, X_1) = t\mathscr{K}(1/t, x; X_1, X_0)$, we hence obtain

$$X' = (L_\infty, L_N)_{\widetilde{E}_1}^{\mathscr{K}} \quad \text{and} \quad R' = (l_1, l_2)_{\widetilde{E}_1}^{\mathscr{K}},$$

where $\|(a_k)\|_{\widetilde{E}_1} = \|(2^k a_{-k})_k\|_{\widetilde{E}}$.

Since $L_\infty \subset G$ and $G$ is a subspace of $L_N$, for $x \in G$ we have

$$\mathscr{K}(t, x; L_\infty, G) = \mathscr{K}(t, x; L_\infty, L_N). \tag{21}$$

Therefore, the space $Y = (L_\infty, G)_{\widetilde{E}_1}^{\mathscr{K}}$ is isometrically embedded in $X'$. By Theorem 1 of [18] (see also [19]), $T$ is an injective operator from $R'$ into $Y$ and hence into $X'$. As was observed in the beginning of the proof, this implies that the operator $F$ itself is surjective from $X$ into $R$; i.e., $F(X) = R$. The theorem is proven.

Since, by Sparr's theorem [20], interpolation in the pair $(l_2, c_0)$ is also described by the $\mathscr{K}$-method, we can similarly prove the following assertion:

**Theorem 2.** *Suppose that $R$ is a separable Banach space of sequences which is an interpolation space with respect to the pair $(l_2, c_0)$ and satisfies* (15). *Then the equality $F(X) = R$ holds for every separable* RIS *$X$ satisfying* (14).

The last assertion sharply contrasts with the case of $X \subset G'$ considered in Proposition 1.

**Theorem 3.** *Suppose that $X_0$ and $X_1$ are separable interpolation* RIS's *on $[0,1]$ with respect to the pair $(G', L_1)$. If the corresponding spaces $F(X_0)$ and $F(X_1)$ of the Fourier–Rademacher coefficients coincide then $X_0 = X_1$ and their norms are equivalent.*

PROOF. As was observed before Theorem 1, interpolation in the pair $(G', L_1)$ is described by the $\mathscr{K}$-method. Therefore, there exist BIS's $E_0$ and $E_1$ of two-sided sequences which are exact interpolation spaces with respect to the pair $\vec{l}_\infty$ and such that

$$X_i = (G', L_1)^{\mathscr{K}}_{E_i} \quad (i = 0, 1).$$

Then, by the hypothesis and Theorem 1,

$$F(X_0) = F(X_1) = (l_2, c_0)^{\mathscr{K}}_{E_0} = (l_2, c_0)^{\mathscr{K}}_{E_1}.$$

Denote this space by $R$.

As in the proof of Theorem 1, we may assume that $E_i$ $(i = 0, 1)$ satisfy (16) and (17). Consequently, $T$ (see (9)) is an injective bounded operator from $R'$ into $X_0'$ and $X_1'$. In other words, for every sequence $a = (a_k)_{k=1}^\infty \in R'$ we have

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_0'} \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_1'} \asymp \|a\|_{R'}.$$

If $Y_i$ is the closure of $L_\infty$ in $X_i'$ $(i = 0, 1)$ then

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{Y_0} \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{Y_1}.$$

Since $X_0'$ and $X_1'$ are $\mathscr{K}$-interpolation spaces with respect to the pair $(L_N, L_\infty)$ (see the proof of Theorem 1), in view of (21) $Y_0$ and $Y_1$ are interpolation spaces with respect to the pair $(G, L_\infty)$. Consequently, they satisfy the conditions of Theorem 3 of [18]; hence, $Y_0 = Y_1$. From the definition of the dual space we obtain $Y_i' = X_i''$ $(i = 0, 1)$ and thus $X_0'' = X_1''$. In view of separability of $X_0$ and $X_1$, we hence conclude that $X_0 = X_1$ and their norms are equivalent. The theorem is proven.

## 3. Concluding Remarks and Examples

REMARK 1. Suppose that $Z_0 = \{x \in L_1 : Fx = 0\}$ and $Z_1 = \{x \in G' : Fx = 0\}$. Then the quotient space $G'/Z_1$ is naturally embedded in the quotient space $L_1/Z_0$, $\tilde{x} = x + Z_1 \mapsto x + Z_0$, and this embedding is continuous with respect to the quotient norm. Define the linear operator $\widetilde{F}$ on $L_1/Z_0$ as follows: $\widetilde{F}\tilde{x} = \widetilde{F}(x + Z_0) = Fx = (c_k(x))_{k=1}^\infty$. In view of (11) and (12), this operator is an isomorphism between the spaces $L_1/Z_0$ and $c_0$, as well as between $G'/Z_1$ and $l_2$. Consequently,

$$\mathscr{K}(t, \tilde{x}; G'/Z_1, L_1/Z_0) \asymp \mathscr{K}(t, \widetilde{F}\tilde{x}; l_2, c_0), \tag{22}$$

with some constant independent of $\tilde{x} \in L_1/Z_0$ and $t > 0$.

Suppose that a BIS $E$ of two-sided sequences satisfies (16) and (17). Then, by Theorem 1, the operator $\widetilde{F}$ is an isomorphism between the spaces $(G', L_1)^{\mathscr{K}}_E / Z_E$ and $(l_2, c_0)^{\mathscr{K}}_E$, where $Z_E = \{x \in (G', L_1)^{\mathscr{K}}_E : Fx = 0\}$. Therefore, by (22)

$$(G'/Z_1, L_1/Z_0)^{\mathscr{K}}_E = (G', L_1)^{\mathscr{K}}_E / Z_E$$

(the norms are equivalent).

REMARK 2. By Khinchin's inequality [1, pp. 153–154], the equality $F(L_p) = l_2$ is valid for $1 < p \leq \infty$. Nevertheless, the assertions of Theorems 1 and 2 become false if we replace $G'$ in their statements with $L_p$ for some $p$ in this interval. Indeed, $(L_p, L_1)_{\theta,q} = L_q$ if $\theta \in (0,1)$ is such that $q = p/(p\theta - \theta + 1)$ [21, p. 157]. At the same time, the equality $(l_2, c_0)_{\theta,q} = l_{r,q}$ holds for the same $\theta$ and $q$, where $r = 2q(p-1)/p(q-1)$ [21, p. 146]. Since $q < p$, we have $r > 2$ and therefore $F(L_q) = l_2 \neq l_{r,q}$.

The above results enable us to find rearrangement-invariant spaces of functions with given spaces of the Fourier–Rademacher coefficients. We exhibit two examples.

EXAMPLE 1. Suppose that $R = l_p$, $2 \leq p < \infty$. Since $l_p = (l_2, c_0)_{\theta,p}$, where $\theta = (p-2)/p$ [21, p. 146], the space $X_p$ with $F(X_p) = l_p$ is determined by the formula

$$X_p = (G', L_1)_{\theta,p}.$$

The spaces $G'$ and $L_1$ coincide with the respective Lorentz spaces $\Lambda(u\log^{1/2} e/u)$ and $\Lambda(u)$. Therefore,

$$G' = (L_1, L_\infty)^{\mathscr{J}}_{l_1(2^{-k}(1-k)^{-1/2})}, \quad L_1 = (L_1, L_\infty)^{\mathscr{J}}_{l_1(2^{-k})};$$

moreover, since we consider RIS of functions defined on $[0,1]$, as the parameters of the $\mathscr{J}$-method we take BIS's of sequences $a = (a_j)_{j=0}^{-\infty}$ [22]. Applying the reiteration theorem for the $\mathscr{J}$-method, we now infer that $X_p$ coincides with the Lorentz space $\Lambda_p(\varphi_p)$, where $\varphi_p(u) = u\log^{1/p} e/u$.

EXAMPLE 2. Suppose that $R = \lambda(1/k)$ is the Lorentz space of all sequences $a = (a_k)_{k=1}^{\infty}$ for which the norm

$$\|a\|_{\lambda(1/k)} = \sum_{k=1}^{\infty} \frac{a_k^*}{k}$$

is finite. This space is separable. Choosing a suitable BIS $E$ of sequences, we can demonstrate that $(l_2, c_0)^{\mathscr{K}}_E = \lambda(1/k)$. However, we proceed alternatively, using Example 2 of [18] (see [19] for details).

Observe that

$$\lambda(1/k)^* = \lambda(1/k)' = l_1(\log) \quad \text{and} \quad l_1(\log)' = \lambda(1/k), \tag{23}$$

where the Marcinkiewicz space $l_1(\log)$ consists of all sequences $a = (a_k)_{k=1}^{\infty}$ for which the norm

$$\|a\|_{l_1(\log)} = \sup_{k=1,2,\dots} \log_2^{-1}(2k) \sum_{i=1}^{k} a_i^*$$

is finite [23, p. 190]. Since $l_1(\log)$ is an interpolation space with respect to the pair $(l_1, l_2)$ (see Example 2 of [18]), $\lambda(1/k)$ is an interpolation space with respect to the pair $(l_2, c_0)$ by (23). Thus, by Theorem 2, $F$ is a surjective operator from some separable RIS $X$ to $\lambda(1/k)$. Moreover, its adjoint $T$ is an injection from $\lambda(1/k)' = l_1(\log)$ to $X'$. From the results of [18] we then obtain $X' = M(\varphi)$, where $\varphi(u) = u\log_2\log_2(16/u)$. In view of separability of $X$, we have $X = \Lambda(\varphi)$ [3, p. 138]. Thus, $F(\Lambda(\varphi)) = \lambda(1/k)$.

## References


1. *Kaczmarz S. and Steinhaus H.*, Theory of Orthogonal Series [Russian translation], Fizmatgiz, Moscow (1958).
2. *Ovchinnikov V. I., Raspopova V. D., and Rodin V. A.*, "Exact estimates for the Fourier coefficients of integrable functions and $\mathscr{K}$-functionals," Mat. Zametki, **32**, No. 3, 292–302 (1982).
3. *Kreĭn S. G., Petunin Yu. I., and Semënov E. M.*, Interpolation of Linear Operators [in Russian], Nauka, Moscow (1978).
4. *Rodin V. A. and Semyonov E. M.*, "Rademacher series in symmetric spaces," Anal. Math., **1**, No. 3, 207–222 (1975).

5. *Montgomery-Smith S.*, "The distribution of Rademacher sums," Proc. Amer. Math. Soc., **109**, No. 2, 517–522 (1990).
6. *Rodin V. A. and Semënov E. M.*, "Complementability of the subspace generated by the Rademacher system in a rearrangement-invariant space," Funktsional. Anal. i Prilozhen., **13**, No. 2, 91–92 (1979).
7. *Lindenstrauss J. and Tzafriri L.*, Classical Banach Spaces. Vol. 2: Function Spaces, Springer-Verlag, Berlin (1979).
8. *Rutitskiĭ Ya. B.*, "On some classes of measurable functions," Uspekhi Mat. Nauk, **20**, No. 4, 205–208 (1965).
9. *Holmstedt T.*, "Interpolation of quasi-normed spaces," Math. Scand., **26**, No. 1, 177–199 (1970).
10. *Szarek S. J.*, "On the best constants in the Khinchin inequality," Studia Math., **58**, 197–208 (1976).
11. *Pietsch A.*, Operator Ideals [Russian translation], Mir, Moscow (1982).
12. *Novikov S. Ya.*, "Cotype and type of Lorentz function spaces," Mat. Zametki, **32**, No. 3, 213–221 (1982).
13. *Kalton N. J.*, "Calderon couples of rearrangement invariant spaces," Studia Math., **103**, No. 3, 233–277 (1993).
14. *Brudnyĭ Yu. A. and Kruglyak N. Ya.*, "Functors of real interpolation," submitted to VINITI, 1981, No. 2620–81.
15. *Ovchinnikov V. I.*, "The method of orbits in interpolation theory," Math. Rep. Ser. 1, No. 2, 349–515 (1984).
16. *Krasnosel'skiĭ M. A. and Rutitskiĭ Ya. B.*, Convex Functions and Orlicz Spaces [in Russian], Fizmatgiz, Moscow (1958).
17. *Kantorovich L. V. and Akilov G. P.*, Functional Analysis [in Russian], Nauka, Moscow (1977).
18. *Astashkin S. V.*, "On series with respect to the Rademacher system in rearrangement-invariant spaces 'close' to $L_\infty$," Funktsional. Anal. i Prilozhen., **32**, No. 3, 62–65 (1998).
19. *Astashkin S. V.*, "On interpolation of subspaces of rearrangement-invariant spaces generated by Rademacher system," Izv. Ross. Akad. Estestv. Nauk Mat. Mat. Model. Inform. Upravlen., **1**, No. 1, 18–35 (1997).
20. *Sparr G.*, "Interpolation of weighted $L_p$-spaces," Studia Math., **62**, 229–271 (1978).
21. *Triebel H.*, Interpolation Theory: Function Spaces; Differential Operators [Russian translation], Mir, Moscow (1980).
22. *Dmitriev V. I., Kreĭn S. G., and Ovchinnikov V. I.*, "Fundamentals of the interpolation theory of linear operators," in: Mezhvuz. Temat. Sb. Nauch. Trudov [in Russian], Yaroslavsk. Univ., Yaroslavl', 1977, pp. 31–74.
23. *Gokhberg I. Ts. and Kreĭn M. G.*, An Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space [in Russian], Nauka, Moscow (1965).