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Systems of random variables equivalent in distribution to the Rademacher system and $\mathcal{K}$-closed representability of Banach couples

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Abstract. Necessary and sufficient conditions ensuring that one can select from a system $\{f_n\}_{n=1}^{\infty}$ of random variables on a probability space $(\Omega, \Sigma, \mathsf{P})$ a subsystem $\{\varphi_i\}_{i=1}^{\infty}$ equivalent in distribution to the Rademacher system on $[0,1]$ are found. In particular, this is always possible if $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded orthonormal sequence. The main role in the proof is played by the connection (discovered in this paper) between the equivalence in distribution of random variables and the behaviour of the L_p-norms of the corresponding polynomials.

An application of the results obtained to the study of the $\mathcal{K}$-closed representability of Banach couples is presented.

Bibliography: 26 titles.

Introduction

We say that systems $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of random variables defined on probability spaces $(\Omega, \Sigma, \mathsf{P})$ and $(\Omega', \Sigma', \mathsf{P}')$, respectively, are *equivalent in distribution* (we write $\{f_n\} \overset{\mathsf{P}}{\sim} \{g_n\}$) if there exists $C > 0$ such that for arbitrary $m \in \mathbb{N}$, $a_n \in \mathbb{R}$ $(n = 1, 2, \dots, m)$, and $z > 0$ we have

$$C^{-1}\,\mathsf{P}\biggl\{\biggl|\sum_{n=1}^{m} a_n f_n(\omega)\biggr| > Cz\biggr\} \leqslant \mathsf{P}'\biggl\{\biggl|\sum_{n=1}^{m} a_n g_n(\omega')\biggr| > z\biggr\} \leqslant C\,\mathsf{P}\biggl\{\biggl|\sum_{n=1}^{m} a_n f_n(\omega)\biggr| > C^{-1}z\biggr\}.$$

If only the left-hand side of this relation holds, then we say that the system $\{f_n\}_{n=1}^{\infty}$ is *majorized in distribution* by the system $\{g_n\}_{n=1}^{\infty}$. In a similar way one can introduce these concepts in the case of finite collections $\{f_n\}_{n=1}^{N}$ and $\{g_n\}_{n=1}^{N}$.

As shown in [1], from a uniformly bounded and orthonormal (in L_2) sequence of random variables $\{f_n\}_{n=1}^{\infty}$ one can always select a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ majorized in distribution by the Rademacher system $\{r_n\}_{n=1}^{\infty}$, where

$$r_n(x) = \operatorname{sign}\sin(2^{n-1}\pi x) \qquad (x \in [0,1]).$$

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The main aim of the present paper is to prove that, under the same assumptions, it is possible to select a subsequence that is not merely majorized, but also equivalent to the system $\{r_n\}_{n=1}^\infty$. Moreover, we find necessary and sufficient conditions ensuring that a system of random variables contains a subsystem equivalent in distribution to the Rademacher system. In the case of finite systems we are interested in the density of their subsystems with a similar property.

Problems related to the selection of 'almost independent' subsystems (that is, subsystems with a certain property characteristic for systems of independent functions) are discussed in detail in Gaposhkin's well-known paper "Lacunary series and independent functions" [2], which we shall repeatedly refer to in what follows. We recall the most important results relating to the integration of the sum of a series in a fixed system of functions and to its absolute convergence.

In view of a classical inequality established by Khintchine [3], the sum of a series $\sum_{n=1}^\infty a_n r_n(x)$ ($x \in [0,1]$) belongs to all L_p-spaces ($p > 2$), provided that the sequence of coefficients $a = (a_n)_{n=1}^\infty$ belongs to l_2. For lacunary trigonometric series a similar result was obtained by Zygmund [4]. In this connection, but slightly later, Banach and Sidon introduced the concept of a lacunary system of order p, or briefly, S_p-system. A sequence of random variables $\{f_n\}_{n=1}^\infty$, $f_n \in L_p$ ($p > 2$), is called an *S_p-system* if

$$\biggl\|\sum_{n=1}^m a_n f_n\biggr\|_p \leqslant K_p \biggl\|\sum_{n=1}^m a_n f_n\biggr\|_2, \tag{0.1}$$

where the constant $K_p > 0$ is independent of $m \in \mathbb{N}$ and $a_n \in \mathbb{R}$ ($n = 1, 2, \dots, m$). If $\{f_n\}$ is an S_p-system for each $p > 2$, then it is called an *S_∞-system*.

Banach proved that one can select an S_p-subsystem from each orthonormal system $\{f_n\}$ of measurable functions on $[0,1]$ such that $\limsup_{n\to\infty} \|f_n\|_p < \infty$ ([5] or [6], § 7.2). This, in particular, means that one can select an S_∞-subsystem from each uniformly bounded system of functions. Here is a more precise result due to Stechkin [2], Theorem 1.3.1.

Theorem A. *Let $\{f_n\}_{n=1}^\infty$ be a system of measurable functions on $[0,1]$ and assume that $p > 2$. Then one can select from $\{f_n\}_{n=1}^\infty$ an S_p-subsystem if and only if there exists a subsequence $\{f_{n_k}\}$ such that*

(1) $\|f_{n_k}\|_p \leqslant D$ ($k = 1, 2, \dots$);

(2) $f_{n_k} \to 0$ *weakly in L_2.*

We now proceed to a brief discussion of another property of lacunary series, their absolute convergence. Well known for trigonometric series is Sidon's theorem on the absolute convergence of a lacunary (in the sense of Hadamard) Fourier series of a bounded function. Zygmund established a local version of this result [7]. Similar results were later obtained for the Rademacher system and certain other special lacunary systems of functions [8]. The following concepts were introduced as a natural generalization of these results. A sequence of random variables $\{f_n\}_{n=1}^\infty$ is called a *Sidon system* if

$$\sum_{n=1}^m |a_n| \leqslant C \biggl\|\sum_{n=1}^m a_n f_n\biggr\|_\infty, \tag{0.2}$$

where the constant $C > 0$ is independent of $m \in \mathbb{N}$ and $a_n \in \mathbb{R}$ ($n = 1, 2, \dots, m$).

A system $\{f_n\}_{n=1}^\infty$ of measurable functions on $[0,1]$ is called a *Sidon–Zygmund system* if there exists a subset E of $[0,1]$ of positive Lebesgue measure $|E|$ such that for each interval I, $|I\cap E|>0$, the condition $\sum_{n=1}^\infty a_nf_n(x)\in L_\infty(I)$ implies that $(a_n)_{n=1}^\infty\in l_1$. If one can set $E=[0,1]$ in this definition, then we say that $\{f_n\}_{n=1}^\infty$ is a Sidon–Zygmund system in the narrow sense.

Broad sufficient conditions ensuring the existence of Sidon and Sidon–Zygmund subsystems have been obtained by Gaposhkin [2], Theorems 1.4.1 and 1.4.2.

Theorem B. *Let $\{f_n\}_{n=1}^\infty$ be a system of measurable functions on $[0,1]$ such that*

(1) $\|f_n\|_2=1$ $(n=1,2,\dots)$;
(2) $|f_n(x)|\leqslant D$ $(n=1,2,\dots;\ x\in[0,1])$;
(3) *there exists a subsequence $\{f_{n_k}\}\subset\{f_n\}$ such that $f_{n_k}\to 0$ weakly in L_2.*

Then one can select from $\{f_n\}_{n=1}^\infty$ a Sidon subsequence.

Theorem B′. (a) *Under the assumptions of Theorem* B *one can select from $\{f_n\}_{n=1}^\infty$ a Sidon–Zygmund subsystem.*

(b) *if the subsequence in assumption* (3) *of Theorem* B *satisfies additionally the condition*

$$\liminf_{k\to\infty}\int_F f_{n_k}^2(x)\,dx>0\quad \textit{for each } F\subset[0,1],\ |F|>0, \tag{0.3}$$

then one can select from $\{f_n\}_{n=1}^\infty$ a Sidon–Zygmund subsystem in the narrow sense.

It follows from the above definitions and the properties of the Rademacher system that if $\{f_n\}\overset{\mathsf{P}}{\sim}\{r_n\}$, then

(1) $\{f_n\}$ is an S_∞-system and the constant K_p in inequality (0.1) has the same growth order in p, as $p\to\infty$, as in the case of the Rademacher system, that is, $K_p\asymp\sqrt{p}$;
(2) $\{f_n\}$ is a Sidon system.

Thus, the results established below in § 3 ensure 'nice' properties of the selected subsystem in both respects, improving and complementing the above-stated results.

Alongside the problems of the selection of infinite subsequences with properties characteristic for systems of independent functions, problems of another kind, treating finite collections of random variables, are also of interest. Let $\{f_n\}_{n=1}^N$ be a collection of random variables. We are looking for a possibly large $s=s(N)$ such that there exists a collection $\{f_{n_i}\}_{i=1}^s\subset\{f_n\}_{n=1}^N$ satisfying a certain condition of lacunarity with constant independent of N. We state here just one result on the density of finite Sidon subsystems, which was proved by Kashin (see [9] or [10], § 8.4, Theorem 9).

Theorem C. *Let $\{f_n\}_{n=1}^N$ be an orthonormal collection of functions on $[0,1]$ such that $|f_n(x)|\leqslant D$ $(n=1,\dots,N;\ x\in[0,1])$. Then there exists a collection $\{f_{n_i}\}_{i=1}^s\subset\{f_n\}_{n=1}^N$ $(1\leqslant n_1<\dots<n_s\leqslant N)$ such that $s\geqslant\max\{[\frac16\log_2 N];1\}$ and for all $a_i\in\mathbb{R}$ $(i=1,\dots,s)$ the following inequality holds*:

$$D^{-1}\biggl\|\sum_{i=1}^s a_if_{n_i}\biggr\|_\infty\leqslant\sum_{i=1}^s|a_i|\leqslant 4D\biggl\|\sum_{i=1}^s a_if_{n_i}\biggr\|_\infty.$$

In the present paper, under the same assumptions, we prove a result on the selection of a subsystem $\{f_{n_i}\}_{i=1}^s$ of the same density that satisfies a stronger condition: it is equivalent to the system $\{r_i\}_{i=1}^s$ with constant dependent only on D.

This paper consists of five sections. In § 1 we collect concepts of operator interpolation theory used in what follows (see [11] for greater detail). For completeness we also present there the proof of a formula from [12] for the $\mathcal{K}$-functional corresponding to the pair (l_1, l_2). In § 2 we prove a criterion for the equivalence in distribution of an arbitrary sequence of random variables and the Rademacher system and derive consequences for some particular cases. The central results of this paper, on the selection of subsystems equivalent in distribution to the Rademacher system, are the subject of § 3. The next section is devoted to similar problems for finite collections of functions. Finally, in § 5 we apply these results to the study of the $\mathcal{K}$-closed representability of Banach couples.

In conclusion let us introduce our notation. An expression of the form $F_1 \asymp F_2$ will mean that there exists $C > 0$ such that $C^{-1}F_1 \leqslant F_2 \leqslant CF_1$; moreover, the constant C will, as a rule, be independent of all or some of the variables of the functions F_1 and F_2. If $1 \leqslant p \leqslant \infty$ and $f = f(\omega)$ is a random variable on a probability space $(\Omega, \Sigma, \mathsf{P})$, then

$$\|f\|_p = (\mathsf{E}|f|^p)^{1/p} = \left\{\int_\Omega |f(\omega)|^p \, d\mathsf{P}(\omega)\right\}^{1/p};$$

while if $a = (a_n)_{n=1}^\infty$ is a number sequence, then

$$\|a\|_p = \left\{\sum_{n=1}^\infty |a_n|^p\right\}^{1/p}$$

(the formulae must be modified in the natural way for $p = \infty$). As usual, the space L_p consists of all random variables f such that $\|f\|_p < \infty$, and l_p consists of all sequences $a = (a_n)_{n=1}^\infty$ such that $\|a\|_p < \infty$.

We denote by $|E|$ the Lebesgue measure of the subset E of $[0, 1]$.

§ 1. Petre $\mathcal{K}$-functional and real interpolation method

Let (X_0, X_1) be a Banach couple, let $x \in X_0 + X_1$, and assume that $t > 0$. We consider the Petre $\mathcal{K}$-functional

$$\mathcal{K}(t, x; X_0, X_1) = \inf\bigl\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\bigr\},$$

which plays an important role in operator interpolation theory.

It is easy to show that for fixed $x \in X_0 + X_1$ the $\mathcal{K}$-functional is an increasing continuous concave function of t (see [11], § 3.1).

In what follows we require only the functional $\mathcal{K}_{1,2}(t, a) = \mathcal{K}(t, a; l_1, l_2)$ constructed for the Banach couple (l_1, l_2), two approximations of which are discussed in the lemmas below. The first lemma, due to Montgomery-Smith (see [12]), will be essential in the proof of Theorem 4. For this reason we present its proof here for completeness.

For arbitrary $a=(a_n)_{n=1}^{\infty}$ and $t\in\mathbb{N}$ we define the norm

$$\|a\|_{Q(t)}=\sup\biggl\{\sum_{j=1}^{t}\biggl(\sum_{n\in A_j}a_n^2\biggr)^{1/2}\biggr\}, \tag{1.1}$$

where the supremum is taken over all partitionings $\{A_j\}_{j=1}^{t}$ of the set of positive integers $\mathbb{N}$.

Lemma 1. *If $a=(a_n)_{n=1}^{\infty}\in l_2$ and $t^2\in\mathbb{N}$, then*

$$\|a\|_{Q(t^2)}\leqslant \mathcal{K}_{1,2}(t,a)\leqslant \sqrt{2}\,\|a\|_{Q(t^2)}. \tag{1.2}$$

Proof. It follows from the definition of $Q(t)$ that

$$\|a\|_{Q(t^2)}\leqslant\|a\|_1 \quad\text{and}\quad \|a\|_{Q(t^2)}\leqslant t\|a\|_2.$$

Hence

$$\begin{aligned}\mathcal{K}_{1,2}(t,a)&=\inf\{\|b\|_1+t\|c\|_2: b+c=a,\ b\in l_1,\ c\in l_2\}\\&\geqslant\inf\{\|b\|_{Q(t^2)}+\|c\|_{Q(t^2)}: b+c=a,\ b\in l_1,\ c\in l_2\}\geqslant\|a\|_{Q(t^2)}\end{aligned}$$

which proves the left-hand side of (1.2).

To prove the right-hand side of (1.2) we note first of all that

$$\mathcal{K}_{1,2}(t,a)=\sup\biggl\{\sum_{n=1}^{\infty}a_nb_n: b=(b_n)_{n=1}^{\infty}\in l_2,\ \mathcal{J}_{\infty,2}(t^{-1},b)\leqslant 1\biggr\},$$

where $\mathcal{J}_{\infty,2}(s,b)=\max\{\|b\|_\infty; s\|b\|_2\}$ (see [11], § 2.7).

Hence, for each $\delta>0$ there exists a sequence $b\in l_2$ such that

$$(1-\delta)\mathcal{K}_{1,2}(t,a)\leqslant\sum_{n=1}^{\infty}a_nb_n \quad\text{and}\quad \mathcal{J}_{\infty,2}(t^{-1},b)=1.$$

We select $n_0,n_1,n_2,\dots,n_{t^2}\in\{0,1,\dots,\infty\}$ by induction as follows: if we have already selected n_0, n_1,..., n_m, where $0=n_0<n_1<\dots<n_m$, then

$$n_{m+1}=1+\sup\biggl\{k:\sum_{n=n_m+1}^{k}b_n^2\leqslant 1\biggr\}.$$

Since $\|b\|_\infty\leqslant 1$, it follows that $\sum_{n=n_m+1}^{n_{m+1}}b_n^2\leqslant 2$. In view of the inequality $\|b\|_2\leqslant t$, we have $n_{t^2}=\infty$.

Hence

$$\begin{aligned}(1-\delta)\mathcal{K}_{1,2}(t,a)&\leqslant\sum_{n=1}^{\infty}a_nb_n\\&\leqslant\sum_{m=1}^{t^2}\biggl(\sum_{n=n_{m-1}+1}^{n_m}b_n^2\biggr)^{1/2}\biggl(\sum_{n=n_{m-1}+1}^{n_m}a_n^2\biggr)^{1/2}\leqslant\sqrt{2}\,\|a\|_{Q(t^2)}.\end{aligned}$$

The last relation holds for all $\delta>0$, which proves the lemma.

Another relation, Holmstedt's formula is well known (see [13] or [11], § 5.7).

Lemma 2. *There exists a constant $\alpha > 0$ independent of $a = (a_n)_{n=1}^\infty \in l_2$ and $t > 0$ such that*

$$\alpha^{-1}\biggl\{\sum_{i=1}^{[t^2]} a_i^* + t\biggl[\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\biggr]^{1/2}\biggr\} \leqslant \mathcal{K}_{1,2}(t,a) \leqslant \biggl\{\sum_{i=1}^{[t^2]} a_i^* + t\biggl[\sum_{i=[t^2]+1}^{\infty} (a_i^*)^2\biggr]^{1/2}\biggr\}, \tag{1.3}$$

where $(a_i^*)_{i=1}^\infty$ is a decreasing rearrangement of the sequence $(|a_n|)_{n=1}^\infty$ and $[z]$ is the integer part of the quantity z.

Corollary 1. *The function $\kappa(t,a) = \mathcal{K}_{1,2}(\sqrt{t},a)$ is continuous and increases for $t \geqslant 0$. For each sequence $a = (a_n)_{n=1}^\infty \in l_2$,*

(1) $\lim_{t\to 0+} \kappa(t,a) = 0$;
(2) $\alpha^{-1}\|a\|_1 \leqslant \lim_{t\to +\infty} \kappa(t,a) \leqslant \|a\|_1$.

To conclude the section we recall the definition of the real interpolation method. For an arbitrary Banach couple (X_0,X_1), $0<\theta<1$, $1\leqslant q\leqslant\infty$, we consider the norm

$$\|x\|_{\theta,q} = \biggl\{\sum_{n=-\infty}^{\infty} \bigl[2^{-n\theta}\mathcal{K}(2^n,x;X_0,X_1)\bigr]^q\biggr\}^{1/q}, \qquad x\in X_0+X_1$$

(and its natural modification for $q=\infty$). The spaces

$$(X_0,X_1)_{\theta,q} = \{x \in X_0+X_1 : \|x\|_{\theta,q} < \infty\}$$

are interpolation spaces for the couple (X_0,X_1) (that is, each linear operator bounded in X_0 and X_1 is bounded also in $(X_0,X_1)_{\theta,q}$); they are called the *spaces of the real interpolation method*.

§ 2. Systems of random variables equivalent by distribution to the Rademacher system

Theorem 1. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of random variables on a probability space $(\Omega,\Sigma,\mathsf{P})$. Then the following conditions are equivalent*:

(1) $\{f_n\} \overset{\mathsf{P}}{\sim} \{r_n\}$ *($\{r_n\}_{n=1}^\infty$ is the Rademacher system on $[0,1]$)*;
(2) *the equivalence*
$$\biggl\|\sum_{n=1}^m a_n f_n\biggr\|_t \asymp \biggl\|\sum_{n=1}^m a_n r_n\biggr\|_t$$
holds with constant independent of $t\in[1,\infty)$, $m\in\mathbb{N}$, and $a_n\in\mathbb{R}$ $(n = 1,\dots,m)$;
(3) *the equivalence*
$$\biggl\|\sum_{n=1}^\infty a_n f_n\biggr\|_t \asymp \mathcal{K}_{1,2}(\sqrt{t},a) \tag{2.1}$$
holds with constant independent of $t\in[1,\infty)$ and $a=(a_n)_{n=1}^\infty \in l_2$.

Proof. As shown in [14], with constant independent of $t\in[1,\infty)$ and $a=(a_n)_{n=1}^{\infty}\in l_2$ we have

$$\biggl\|\sum_{n=1}^{\infty} a_n r_n\biggr\|_t \asymp \mathcal{K}_{1,2}(\sqrt{t},a) \tag{2.2}$$

(this relation is already implicit in [12]). Hence the equivalence (2) $\Leftrightarrow$ (3) is obvious. The implication (1) $\Rightarrow$ (2) is a consequence of the definition of systems equivalent in distribution, therefore it remains to establish the implication (3) $\Rightarrow$ (1).

We fix $a=(a_n)_{n=1}^{m}$, $m\in\mathbb{N}$ (where not all the a_n are equal to zero) and show that the distribution of the absolute value of the random variable

$$f(\omega)=\sum_{n=1}^{m} a_n f_n(\omega) \qquad (\omega\in\Omega)$$

is determined by the behaviour of its moments $\|f\|_t$ and therefore, by assumption, by the function $\kappa(t,a)=\mathcal{K}_{1,2}(\sqrt{t},a)$ $(t\geqslant 1)$.

Let $\beta\geqslant 1$ be the constant involved in the equivalence (2.1). In view of the Paley–Zygmund inequality [15], § 1.6 and the concavity of the $\mathcal{K}$-functional for $t\geqslant 1$, we obtain

$$\begin{aligned}\mathsf{P}\{|f(\omega)|\geqslant(2\beta)^{-1}\kappa(t,a)\}&\geqslant \mathsf{P}\{|f(\omega)|^t\geqslant 2^{-t}\|f\|_t^t\}\\ &\geqslant(1-2^{-t})^2\,\frac{\|f\|_t^{2t}}{\|f\|_{2t}^{2t}}\geqslant\beta^{-4t}(1-2^{-t})^2\biggl\{\frac{\kappa(t,a)}{\kappa(2t,a)}\biggr\}^{2t}\geqslant(2\beta)^{-4t}.\end{aligned}$$

The function $\kappa(t,a)$ increases in t, therefore

$$\mathsf{P}\{|f(\omega)|\geqslant(2\beta)^{-1}\kappa(t,a)\}\geqslant(2\beta)^{-4t-4} \tag{2.3}$$

also for all $t>0$.

Conversely, by Chebyshev's inequality and assumption (2.1) we obtain

$$\mathsf{P}\{|f(\omega)|\geqslant\lambda\kappa(t,a)\}\leqslant\lambda^{-t}\biggl[\frac{\|f\|_t}{\kappa(t,a)}\biggr]^t\leqslant\biggl(\frac{\beta}{\lambda}\biggr)^t$$

for arbitrary $t\geqslant 1$ and $\lambda>0$. In particular, if $\lambda=\beta e$, then

$$\mathsf{P}\{|f(\omega)|\geqslant\beta e\kappa(t,a)\}\leqslant e^{-t} \quad (t\geqslant 1)$$

or

$$\mathsf{P}\{|f(\omega)|\geqslant\beta e\kappa(t,a)\}\leqslant e^{1-t} \quad (t>0). \tag{2.4}$$

Following [14] we now define the functionals

$$F(s)=\sup\{t>0:\kappa(t,a)\leqslant s\},\qquad G(s)=\inf\{t>0:\kappa(t,a)\geqslant s\}.$$

If $0 < z < (4\alpha\beta)^{-1}\|a\|_1$, then, by Corollary 1, there exists $t > 0$ such that $\kappa(t, a) \geqslant 4z\beta$. It then follows from (2.3) that

$$\mathsf{P}\{|f(\omega)| > z\} \geqslant \mathsf{P}\{|f(\omega)| \geqslant (2\beta)^{-1}\kappa(t, a)\} \geqslant (2\beta)^{-4t-4}.$$

Hence, by the definition of the functional G,

$$\mathsf{P}\{|f(\omega)| > z\} \geqslant (2\beta)^{-4G(4\beta z)-4},$$

so that

$$\mathsf{P}\{|f(\omega)| > z\} \geqslant C_1^{-1} e^{-C_2 G(4\beta z)} \qquad (z < (4\alpha\beta)^{-1}\|a\|_1), \tag{2.5}$$

where $C_1 = (2\beta)^4$ and $C_2 = 4\ln(2\beta)$.

Setting $C_3 = 4\sqrt{2}\, C_2\beta$ we claim that

$$C_2 G(4\beta z) \leqslant F(C_3 z) \qquad (z > 0).$$

For, in view of the concavity of the $\mathcal{K}$-functional and the definition of G,

$$\kappa(C_2 G(4\beta z), a) \leqslant \sqrt{2}\, C_2 \kappa(2^{-1} G(4\beta z), a) \leqslant 4\sqrt{2}\, C_2 \beta z = C_3 z,$$

and it remains to use the definition of the functional F. Thus, (2.5) can be written as follows:

$$\mathsf{P}\{|f(\omega)| > z\} \geqslant C_1^{-1} e^{-F(C_3 z)} \qquad (z < (4\alpha\beta)^{-1}\|a\|_1). \tag{2.6}$$

Note that by (2.1) we obtain

$$\|f_n\|_\infty = \lim_{t\to+\infty} \|f_n\|_t \leqslant \beta,$$

so that $\|f\|_\infty \leqslant \beta\|a\|_1$. Hence

$$\mathsf{P}\{|f(\omega)| > z\} = 0 \qquad (z \geqslant \beta\|a\|_1). \tag{2.7}$$

Further, by Corollary 1 we can find $t' > 0$ such that $0 < \kappa(t', a) \leqslant (2\beta e)^{-1} z$. It now follows from (2.4) that

$$\mathsf{P}\{|f(\omega)| > z\} \leqslant \mathsf{P}\{|f(\omega)| \geqslant \beta e \kappa(t', a)\} \leqslant e^{1-t'},$$

so that, by the definition of F,

$$\mathsf{P}\{|f(\omega)| > z\} \leqslant e^{1-F(C_4^{-1} z)} \qquad (z > 0), \tag{2.8}$$

where $C_4 = 2\beta e$.

In view of (2.2), relations similar to (2.6)–(2.8) hold also for the Rademacher system. Namely, if $\beta' \geqslant 1$ is the constant involved in the equivalence (2.2), then the function $r(x) = \sum_{n=1}^m a_n r_n(x)$ satisfies the relations

$$|\{|r(x)| > z\}| \geqslant (C_1')^{-1} e^{-F(C_3' z)} \quad (z < (4\alpha\beta')^{-1}\|a\|_1), \tag{2.6$'$}$$

$$|\{|r(x)| > z\}| = 0 \quad (z \geqslant \|a\|_1), \tag{2.7$'$}$$

$$|\{|r(x)| > z\}| \leqslant e^{1-F((C_4')^{-1} z)} \quad (z > 0). \tag{2.8$'$}$$

Let

$$A = \max(C_1 e; C_3 C_4'; C_1' e; C_3' C_4; 4\alpha\beta\beta'). \tag{2.9}$$

Note that, since the constants α and β' are universal, the dependence of A on the system $\{f_n\}_{n=1}^\infty$ reduces to its dependence on β, the constant involved in (2.1).

If $z < (4\alpha\beta)^{-1}\|a\|_1$, then relations (2.8′) and (2.6) and our choice of A show that

$$\begin{aligned} |\{|r(x)| > Az\}| &\leqslant e^{1-F((C_4')^{-1}Az)} \leqslant e^{1-F(C_3 z)} \\ &\leqslant C_1 e\,\mathsf{P}\{|f(\omega)| > z\} \leqslant A\,\mathsf{P}\{|f(\omega)| > z\}. \end{aligned} \tag{2.10}$$

On the other hand if $z \geqslant (4\alpha\beta)^{-1}\|a\|_1$, then it follows from (2.9) that $Az \geqslant \|a\|_1$, so that $|\{|r(x)| > Az\}| = 0$ in view of (2.7′). Hence inequality (2.10) also holds in that case.

Conversely, if $z/A < (4\alpha\beta')^{-1}\|a\|_1$, then it follows from (2.8), (2.6′), and our choice of A that

$$\begin{aligned} \mathsf{P}\{|f(\omega)| > z\} &\leqslant e^{1-F(C_4^{-1}z)} \leqslant e^{1-F(C_3'(z/A))} \\ &\leqslant C_1' e\left|\left\{|r(x)| > \frac{z}{A}\right\}\right| \leqslant A\left|\left\{|r(x)| > \frac{z}{A}\right\}\right|. \end{aligned} \tag{2.11}$$

If $z/A \geqslant (4\alpha\beta')^{-1}\|a\|_1$, then by (2.9) we obtain the inequality $z \geqslant \beta\|a\|_1$. Hence $\mathsf{P}\{|f(\omega)| > z\} = 0$ in view of (2.7), and (2.11) holds again.

Inequalities (2.10) and (2.11) mean that $\{f_n\} \overset{\mathsf{P}}{\sim} \{r_n\}$, which completes the proof of Theorem 1.

Remark 1. Of course, a similar result holds also for finite collections of random variables. Moreover, it is clear from the proof that the constants involved in the equivalences in conditions (1)–(3) of Theorem 1 depend only on one another. For instance, if (2.1) holds for collections $\{f_n\}_{n=1}^N$ with constant independent of $N = 1, 2, \dots$, then $\{f_n\}_{n=1}^N \overset{\mathsf{P}}{\sim} \{r_n\}_{n=1}^N$ also with constant independent of $N = 1, 2, \dots$.

We now apply the above theorem to two more special cases.

Recall that a system of random variables $\{f_n\}_{n=1}^\infty$ is said to be *multiplicative* if for arbitrary, pairwise distinct $n_1, n_2, \dots, n_k$ ($k \in \mathbb{N}$) we have

$$\mathsf{E}(f_{n_1} f_{n_2} \cdots f_{n_k}) = 0.$$

If, in addition,

$$\mathsf{E}(f_{n_1} f_{n_2} \cdots f_{n_k}\, f_n^2) = 0$$

for arbitrary, pairwise distinct $n_1, n_2, \dots, n_k$ and $n \neq n_s$ ($s = 1, 2, \dots, k$), then we say that the system $\{f_n\}$ is *strongly multiplicative*. In the case of finite collections the definitions are completely similar.

We now discuss the most important examples of such systems [16].

Example 1. Let $f_n(x) = \sin(2\pi k_n x)$ ($x \in [0,1]$). If $k_{n+1}/k_n \geqslant 2$, then this is a multiplicative system and if $k_{n+1}/k_n \geqslant 3$, then it is a strongly multiplicative system.

Example 2. A sequence $\{f_n\}_{n=1}^{\infty}$ of independent random variables such that $f_n \in L_2$ and $\mathsf{E}(f_n) = 0$ $(n = 1, 2, \dots)$ is a strongly multiplicative system.

Kwapien and Jakubowski have established the following result [16].

Theorem D. *Let $\{\varphi_n\}_{n=1}^{N}$ be a multiplicative and $\{\psi_n\}_{n=1}^{N}$ a strongly multiplicative system of random variables. If for each $n = 1, \dots, N$,*

$$\|\psi_n\|_\infty \, \|\varphi_n\|_\infty \leqslant \mathsf{E}(\psi_n^2), \tag{2.12}$$

then there exist a probability space $(\Omega', \Sigma', \mathsf{P}')$, a σ-subalgebra Σ_0 of the σ-algebra Σ', and a random vector $(\psi_1', \psi_2', \dots, \psi_N')$ on $(\Omega', \Sigma', \mathsf{P}')$ such that this vector itself is equidistributed with $(\psi_1, \psi_2, \dots, \psi_N)$, while $(\varphi_1, \varphi_2, \dots, \varphi_N)$ is equidistributed with the conditional expectations $\mathsf{E}\big((\psi_1', \psi_2', \dots, \psi_N') \mid \Sigma_0\big)$.

In particular, for an arbitrary convex function $H\colon \mathbb{R}^n \to \mathbb{R}$,

$$\mathsf{E}\big(H(\varphi_1, \dots, \varphi_N)\big) \leqslant \mathsf{E}\big(H(\psi_1, \dots, \psi_N)\big).$$

From Theorems D and 1 we obtain the following result.

Theorem 2. *Let $\{f_n\}_{n=1}^{\infty}$ be a strongly multiplicative system of random variables such that $|f_n(\omega)| \leqslant D$ $(n = 1, 2, \dots;\ \omega \in \Omega)$ and $d = \inf_{n=1,2,\dots} \mathsf{E}(f_n^2) > 0$. Then $\{f_n\} \overset{\mathsf{P}}{\sim} \{r_n\}$, and the constant involved in the equivalence depends only on D and d.*

Proof. Assumption (2.12) of Theorem D holds for the systems $\varphi_n = f_n/D$ and $\psi_n = r_n$. Hence for all $t \geqslant 1$, $m \in \mathbb{N}$, and $a_n \in \mathbb{R}$ $(n = 1, 2, \dots, m)$ we have

$$\Big\| \sum_{n=1}^{m} a_n f_n \Big\|_t \leqslant D \Big\| \sum_{n=1}^{m} a_n r_n \Big\|_t.$$

Conversely, if $\varphi_n = (d/D) r_n$, $\psi_n = f_n$, then (2.12) holds again, so that

$$\Big\| \sum_{n=1}^{m} a_n r_n \Big\|_t \leqslant \frac{D}{d} \Big\| \sum_{n=1}^{m} a_n f_n \Big\|_t.$$

To complete the proof it remains to use Theorem 1.

Corollary 2. *Each sequence of independent random variables $\{f_n\}_{n=1}^{\infty}$ such that $f_n \in L_2$, $\mathsf{E}(f_n) = 0$, $|f_n(\omega)| \leqslant D$ $(n = 1, 2, \dots;\ \omega \in \Omega)$, and $d = \inf_{n=1,2,\dots} \mathsf{E}(f_n^2) > 0$ is equivalent in distribution to the Rademacher system.*

We consider now another situation. Let G be a compact Abelian group, Γ the group of its characters, and μ the Haar measure on G. In accordance with the above definition a subset F of Γ is called a *Sidon set* if

$$\sum_{\gamma \in \Gamma} |\widehat{f}(\gamma)| \leqslant C \, \|f\|_\infty$$

for some $C = C(F)$ and each $f \in C(G)$ such that

$$\widehat{f}(\gamma) = \int_G f \overline{\gamma} \, d\mu = 0 \qquad (\gamma \notin F).$$

Pisier [17] has proved the following result (we present its formulation only in the scalar case).

Theorem E. *If $F = \{\gamma_n\} \subset \Gamma$ is a Sidon set, then the equivalence*

$$\biggl\|\sum_{n=1}^{m} a_n\gamma_n\biggr\|_t \asymp \biggl\|\sum_{n=1}^{m} a_n r_n\biggr\|_t \qquad (t \geqslant 1)$$

holds with constant depending only on the Sidon constant $C(F)$.

Theorems E and 1 immediately give us the following result, which was also proved (by another method, but also on the basis of Theorem E) in [18].

Theorem 3. *Each infinite Sidon system $F = \{\gamma_n\}_{n=1}^{\infty}$ of characters of a compact Abelian group is equivalent in distribution to the Rademacher system. The corresponding equivalence constant depends only on the Sidon constant $C(F)$.*

Corollary 3. *Sequences*

$$f_n(x) = \sin(2\pi k_n x) \quad \textit{and} \quad g_n(x) = \cos(2\pi k_n x) \qquad (x \in [0,1])$$

in which $k_{n+1}/k_n \geqslant \lambda > 1$ are equivalent in distribution to the Rademacher system.

Remark 2. We point out the fact that it was the similarity between the behaviour of Rademacher series and lacunary trigonometric series (see the introduction) that was the starting point of the study of deeper connections between these series and led eventually to the investigation of systems equivalent in distribution to the 'model' Rademacher system. Among papers considering these subjects, besides [16]–[18], we also mention a note [19] by Rodin and Belov, where the authors used Theorem D to carry over the results of [20] on the behaviour of Rademacher series in symmetric function spaces to lacunary trigonometric series. As mentioned in that note, in 1988 one of its authors reported at the North-Caucasian workshop that in many symmetric spaces the norms of polynomials in the Rademacher system and of lacunary trigonometric polynomials are the same up to equivalence. Corollary 3 shows that this actually holds in all symmetric spaces, not just in many, because the norm of a function in such a space is defined in terms of the distribution of its absolute value (see [21], § 2.4). This allows us, in particular, to improve some results of [19]. For instance, in Theorem 2 from that paper, which establishes necessary and sufficient conditions for the equivalence of the norms of lacunary trigonometric polynomials in a symmetric space to the norms of the sequences of their coefficients in the spaces l_1 and l_2, the assumption that the symmetric space is an interpolation space for the couple (L_1, L_∞) is superfluous.

§ 3. Selection of subsystems equivalent in distribution to the Rademacher system

Theorem 4. *Let $\{f_n\}_{n=1}^{\infty}$ be a system of random variables on a probability space $(\Omega, \Sigma, \mathsf{P})$ containing a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that*

(1) $|f_{n_k}(\omega)| \leqslant D$ $(k = 1, 2, \dots;\ \omega \in \Omega)$;
(2) $f_{n_k} \to 0$ *weakly in* L_2;
(3) $d = \inf_{k=1,2,\dots} \|f_{n_k}\|_2 > 0$.

Then there exists a subsystem $\{\varphi_i\}_{i=1}^\infty \subset \{f_n\}_{n=1}^\infty$ such that

$$\biggl\|\sum_{i=1}^\infty a_i\varphi_i\biggr\|_t \asymp \mathcal{K}_{1,2}(\sqrt{t},a) \qquad (a=(a_i)_{i=1}^\infty \in l_2, \quad t\geqslant 1) \tag{3.1}$$

with constant dependent only on D and d.

Proof. As shown by Gaposhkin (see [2], Theorem 1.3.2, a full proof can be found in [1]), if conditions (1) and (2) are satisfied, then one can select a subsequence $\{g_i\}\subset\{f_{n_k}\}$ such that for some $C_1=C_1(D)$ and all $a=(a_i)_{i=1}^\infty\in l_2$ and $t\geqslant 1$ we have the inequality

$$\biggl\|\sum_{i=1}^\infty a_i g_i\biggr\|_t \leqslant C_1\sqrt{t}\,\|a\|_2. \tag{3.2}$$

Hence, for each expansion $(a_i)=(b_i)+(c_i)$ with $(b_i)\in l_1$ and $(c_i)\in l_2$ we have

$$\begin{aligned}\biggl\|\sum_{i=1}^\infty a_i g_i\biggr\|_t &\leqslant \biggl\|\sum_{i=1}^\infty b_i g_i\biggr\|_\infty + \biggl\|\sum_{i=1}^\infty c_i g_i\biggr\|_t \\ &\leqslant D\sum_{i=1}^\infty |b_i| + C_1\sqrt{t}\biggl(\sum_{i=1}^\infty c_i^2\biggr)^{1/2} \leqslant \max(C_1;D)\bigl(\|b\|_1+\sqrt{t}\,\|c\|_2\bigr).\end{aligned}$$

Consequently, by the definition of the $\mathcal{K}$-functional we obtain

$$\biggl\|\sum_{i=1}^\infty a_i g_i\biggr\|_t \leqslant \max(C_1;D)\,\mathcal{K}_{1,2}(\sqrt{t},a) \qquad (a=(a_i)_{i=1}^\infty\in l_2, \quad t\geqslant 1). \tag{3.3}$$

In the proof of the reverse inequality we use the equivalence of the $\mathcal{K}$-functional $\mathcal{K}_{1,2}(t,a)$ and the norm $Q(t^2)$ (see (1.1) and (1.2)), and also the upper estimate of the L_q-norms ($q>1$) of modified Riesz products (for applications of classical products in similar cases, see [2], § 1.4 or [10], § 8.4).

First, assumptions (1) and (3) of the theorem show that $0<d\leqslant\|g_i\|_2\leqslant D$ ($i=1,2,\dots$), therefore, in view of the positive homogeneity of both sides of (3.1) we can assume that $\|g_i\|_2=1$ ($i=1,2,\dots$).

Let $(\varepsilon_i)_{i=1}^\infty$ be a number sequence such that

$$\varepsilon_i\to 0, \qquad 0<\varepsilon_i<\frac{1}{16}\min(1;D), \qquad \sum_{k=i+1}^\infty \varepsilon_k<\varepsilon_i \quad (i=1,2,\dots). \tag{3.4}$$

By the assumptions of the theorem the sequence g_i^2 is weakly compact in L_2. Hence there exists $\{h_k\}\subset\{g_i\}$ such that $h_k^2\to h$ weakly in this space, where $0\leqslant h(\omega)\leqslant D^2$ and $\mathsf{E}(h)=1$. Together with $\{g_i\}$, the sequence of h_k also converges to 0 weakly in L_2. Hence we can find an index k_1 such that the function $\varphi_1=h_{k_1}$ satisfies the relation

$$|\mathsf{E}(\varphi_1)|+|\mathsf{E}(h\varphi_1)|+|\mathsf{E}(\varphi_1^2-h)|\leqslant\frac{\varepsilon_1}{2D}\,.$$

Assume that we have already selected the indices $k_1 < k_2 < \cdots < k_{i-1}$ and the functions $\varphi_1 = h_{k_1}$, $\varphi_2 = h_{k_2}$, $\dots$, $\varphi_{i-1} = h_{k_{i-1}}$ $(i \geqslant 2)$. We now find $k_i > k_{i-1}$ such that for $\varphi_i = h_{k_i}$ we have the relation

$$\sum\Big\{|\mathsf{E}(\varphi_{j_1}\cdots\varphi_{j_s}\varphi_i)| + |\mathsf{E}(h\varphi_{j_1}\cdots\varphi_{j_s}\varphi_i)| + |\mathsf{E}[\varphi_{j_1}\cdots\varphi_{j_s}(\varphi_i^2 - h)]| \\ + \sum_{l=1}^{s}|\mathsf{E}(\varphi_{j_1}\cdots\varphi_{j_{l-1}}\varphi_{j_l}^2\varphi_{j_{l+1}}\cdots\varphi_{j_s}\varphi_i)|\Big\} \leqslant 2^{-i}D^{-1}\varepsilon_i, \tag{3.5}$$

where $\varphi_{j_0} = \varphi_{j_{s+1}} = 1$ and the summation proceeds over all collections of indices such that $1 \leqslant j_1 < j_2 < \cdots < j_l < \cdots < j_s \leqslant i-1$ $(s = 1, 2, \dots, i-1)$.

We claim that the sequence $\{\varphi_i\}_{i=1}^{\infty}$ satisfies (3.1) for arbitrary $t \geqslant 1$.

Let $t \in \mathbb{N}$ first, and let $\{A_j\}_{j=1}^{t}$ be an arbitrary partitioning of the positive integers into t sets. For $N \in \mathbb{N}$ and $j = 1, 2, \dots, t$ let $A_j^N = \{i = 1, \dots, N : i \in A_j\}$ (some of these sets may be empty). We consider now the Riesz products formed by blocks corresponding to the sets A_j^N:

$$R_N(\omega) = \prod_{i=1}^{N}(1 + b_i\varphi_i(\omega)) = \prod_{j=1}^{t}\prod_{i\in A_j^N}(1 + b_i\varphi_i(\omega)),$$

where the b_i are real coefficients such that

$$\sum_{i\in A_j} b_i^2 \leqslant D^{-2}. \tag{3.6}$$

Reasoning along the lines of Sidon's classical method, for an arbitrary function

$$\varphi(\omega) = \sum_{i=1}^{\infty} a_i\varphi_i(\omega), \qquad a = (a_i)_{i=1}^{\infty} \in l_2,$$

we find a lower estimate of the integral

$$I_N = \int_\Omega R_N(\omega)\varphi(\omega)\,d\mathsf{P}(\omega) = \sum_{i=1}^{\infty} a_i\mathsf{E}(R_N\varphi_i) = \sum_{i=1}^{\infty} a_i\beta_{i,N}. \tag{3.7}$$

Here

$$\beta_{i,N} = \mathsf{E}(R_N\varphi_i) = \int_\Omega\Big[1 + \sum_{k=1}^{N} b_k\varphi_k(\omega) + \sum_{1\leqslant k_1<k_2\leqslant N} b_{k_1}b_{k_2}\varphi_{k_1}(\omega)\varphi_{k_2}(\omega) \\ + \cdots + b_1b_2\cdots b_N\varphi_1(\omega)\cdots\varphi_N(\omega)\Big]\varphi_i(\omega)\,d\mathsf{P}(\omega) = b_i\alpha_{i,N} + \gamma_{i,N}, \tag{3.8}$$

where $\alpha_{i,N} = 1$ $(i \leqslant N)$, $\alpha_{i,N} = 0$ $(i > N)$, and

$$\gamma_{i,N} = \mathsf{E}(\varphi_i) + \sum_{k=1,k\neq i}^{N} b_k\mathsf{E}(\varphi_k\varphi_i) + \sum_{1\leqslant k_1<k_2\leqslant N} b_{k_1}b_{k_2}\mathsf{E}(\varphi_{k_1}\varphi_{k_2}\varphi_i) \\ + \cdots + \sum_{1\leqslant k_1<\cdots<k_s\leqslant N} b_{k_1}\cdots b_{k_s}\mathsf{E}(\varphi_{k_1}\cdots\varphi_{k_s}\varphi_i) + \cdots + b_1\cdots b_N\mathsf{E}(\varphi_1\cdots\varphi_N\varphi_i).$$

Then

$$\gamma_{i,N} = S_1^i + S_2^i + S_3^i, \tag{3.9}$$

where the sums S_k^i $(k = 1, 2, 3)$ are defined as follows. Setting

$$\mathsf{E}(\varphi_{k_1} \cdots \varphi_{k_s} \varphi_i^2) = \mathsf{E}(\varphi_{k_1} \cdots \varphi_{k_s} h) + \mathsf{E}[\varphi_{k_1} \cdots \varphi_{k_s}(\varphi_i^2 - h)],$$

we collect in S_1^i all terms containing integrals of the form $\mathsf{E}(\varphi_{k_1} \cdots \varphi_{k_s} \varphi_i)$ or $\mathsf{E}[\varphi_{k_1} \cdots \varphi_{k_s}(\varphi_i^2 - h)]$, where $k_1 < k_2 < \cdots < k_s < i$. The terms containing integrals $\mathsf{E}(\varphi_{k_1} \cdots \varphi_{k_{l-1}} \varphi_i \varphi_{k_l} \cdots \varphi_{k_s})$, where $k_1 < k_2 < \cdots < k_{l-1} \leqslant i < k_l < \cdots < k_s$, $1 \leqslant l \leqslant s$, we collect in S_2^i, and in S_3^i we include the terms containing integrals $\mathsf{E}(\varphi_{k_1} \cdots \varphi_{k_s} h)$.

In view of (3.5), (3.4), and (3.6) we obtain the estimates

$$|S_1^i| \leqslant \frac{\varepsilon_i}{D}, \qquad |S_2^i| \leqslant \frac{1}{D} \sum_{k=i+1}^{\infty} \varepsilon_k < \frac{\varepsilon_i}{D}. \tag{3.10}$$

Since each term in S_3^i contains a factor b_i, it follows by (3.4) that

$$|S_3^i| \leqslant \frac{|b_i|}{D} \sum_{i=1}^{\infty} \varepsilon_i \leqslant \frac{|b_i|}{8} \tag{3.11}$$

(we have not used the coefficient 2^{-i} in (3.5) so far). Moreover, for $i > N$ there are no sums S_2^i and S_3^i in (3.9), therefore

$$\sum_{i=N+1}^{\infty} |\gamma_{i,N}| \leqslant \frac{1}{D} \sum_{i=N+1}^{\infty} \varepsilon_i \leqslant \frac{1}{16D}. \tag{3.12}$$

Assume now that $1 \leqslant i \leqslant N$. For each $j = 1, 2, \ldots, t$, in view of (3.9)–(3.11), (3.4), (3.6), we have

$$\begin{aligned} \biggl(\sum_{i \in A_j^N} \gamma_{i,N}^2\biggr)^{1/2} &\leqslant \sum_{k=1}^{3} \biggl(\sum_{i \in A_j^N} (S_k^i)^2\biggr)^{1/2} \\ &\leqslant \frac{2}{D} \sum_{i \in A_j^N} \varepsilon_i + \frac{1}{8} \biggl(\sum_{i \in A_j^N} b_i^2\biggr)^{1/2} \leqslant \frac{3}{8D}. \end{aligned} \tag{3.13}$$

Relations (3.7) and (3.8) show that

$$\begin{aligned} I_N &= \sum_{i=1}^{N} a_i b_i + \sum_{i=1}^{N} a_i \gamma_{i,N} + \sum_{i=N+1}^{\infty} a_i \gamma_{i,N} \\ &= \sum_{j=1}^{t} \biggl(\sum_{i \in A_j^N} a_i b_i\biggr) + \sum_{j=1}^{t} \biggl(\sum_{i \in A_j^N} a_i \gamma_{i,N}\biggr) + \sum_{i=N+1}^{\infty} a_i \gamma_{i,N}. \end{aligned}$$

For each $j = 1, 2, \dots, t$ we now select b_i $(i \in A_j^N)$ such that (3.6) holds and

$$\sum_{i\in A_j^N} a_i b_i = \frac{1}{D}\biggl(\sum_{i\in A_j^N} a_i^2\biggr)^{1/2};$$

we also select $N \in \mathbb{N}$ such that

$$\|a\|_2 \leqslant 2\biggl(\sum_{i=1}^{N} a_i^2\biggr)^{1/2} \leqslant 2\sum_{j=1}^{t}\biggl(\sum_{i\in A_j^N} a_i^2\biggr)^{1/2}.$$

Then it follows by the above equality and (3.12), (3.13), and (3.4) that

$$\begin{aligned}
I_N &\geqslant \frac{1}{D}\sum_{j=1}^{t}\biggl(\sum_{i\in A_j^N} a_i^2\biggr)^{1/2} - \sum_{j=1}^{t}\biggl(\sum_{i\in A_j^N} a_i^2\biggr)^{1/2}\biggl(\sum_{i\in A_j^N}\gamma_{i,N}^2\biggr)^{1/2} - \sum_{i=N+1}^{\infty}|a_i|\,|\gamma_{i,N}| \\
&\geqslant \frac{5}{8D}\sum_{j=1}^{t}\biggl(\sum_{i\in A_j^N} a_i^2\biggr)^{1/2} - \biggl(\sum_{i=N+1}^{\infty}|a_i|^2\biggr)^{1/2}\biggl(\sum_{i=N+1}^{\infty}\gamma_{i,N}^2\biggr)^{1/2} \\
&\geqslant \frac{1}{2D}\sum_{j=1}^{t}\biggl(\sum_{i\in A_j^N} a_i^2\biggr)^{1/2}.
\end{aligned}$$

The last inequality shows that for each $t \in \mathbb{N}$ and an arbitrary partitioning $\{A_j\}_{j=1}^t$ of positive integers there exist sufficiently large N and coefficients b_i $(i = 1, 2, \dots, N)$ satisfying (3.6) such that

$$I_N = \int_\Omega R_N(\omega)\varphi(\omega)\,d\mathsf{P}(\omega) \geqslant \frac{1}{3D}\sum_{j=1}^{t}\biggl(\sum_{i\in A_j} a_i^2\biggr)^{1/2}. \tag{3.14}$$

The next part of the proof is the estimate of the integral I_N by an expression of the form $C\|\varphi\|_t$, with constant $C > 0$. By Hölder's inequality

$$|I_N| \leqslant \|R_N\|_{t'}\|\varphi\|_t, \qquad t' = \frac{t}{t-1}, \tag{3.15}$$

and the non-negativity of Riesz products it is sufficient for this to find an estimate of the quantity

$$L_N = \|R_N\|_{t'}^{t'} = \int_\Omega [R_N(\omega)]^{t'}\,d\mathsf{P}(\omega) \tag{3.16}$$

(for the proof that $\{\varphi_i\}_{i=1}^\infty$ is a Sidon system we require only the estimate of the L_1-norms of Riesz products (see [2], Theorem 1.4.1)).

We consider $t \in \mathbb{N}$, $t \geqslant 3$. As before, let $\{A_j\}_{j=1}^t$ be a partitioning of $\mathbb{N}$, let $A_j^N = \{i = 1, \dots, N : i \in A_j\}$, and let b_i be coefficients satisfying (3.6).

By the elementary inequality $(1+x)^y \leqslant 1+yx$ $(x \geqslant 1,\ 0<y\leqslant 1)$,

$$
\begin{aligned}
[R_N(\omega)]^{t'} &\leqslant \prod_{i=1}^{N}(1+b_i\varphi_i(\omega))[1+(t-1)^{-1}b_i\varphi_i(\omega)] \\
&\leqslant \prod_{i=1}^{N}\biggl[1+\frac{D^2}{t-1}b_i^2+\frac{t}{t-1}b_i\varphi_i(\omega)\biggr] \\
&= \prod_{j=1}^{t}\prod_{i\in A_j^N}\biggl[1+\frac{D^2}{t-1}b_i^2+\frac{t}{t-1}b_i\varphi_i(\omega)\biggr]. \qquad (3.17)
\end{aligned}
$$

Let $m(B)$ be the cardinality of a subset B of $\mathbb{N}$. Then the inner product in the last expression is equal to the sum

$$
\begin{aligned}
&\biggl\{1+\frac{D^2}{t-1}\sum_{i\in A_j^N}b_i^2+\Bigl(\frac{D^2}{t-1}\Bigr)^2\sum_{\substack{i_1<i_2\\ i_1,i_2\in A_j^N}}b_{i_1}^2b_{i_2}^2+\dots+\Bigl(\frac{D^2}{t-1}\Bigr)^{m(A_j^N)}\prod_{i\in A_j^N}b_i^2\biggr\} \\
&\quad+\sum_{C_j\subset A_j^N}\biggl[1+\Bigl(\frac{D^2}{t-1}\Bigr)^{m(A_j^N)-m(C_j)}\prod_{i\in A_j^N\setminus C_j}b_i^2\biggr]\Bigl(\frac{t}{t-1}\Bigr)^{m(C_j)}\prod_{i\in C_j}b_i\varphi_i(\omega),
\end{aligned}
$$

where the last sum is taken over all non-empty subsets C_j of the set A_j^N. In view of (3.6), for $t\geqslant 3$ the expression in the curly brackets has the estimate $(t-1)/(t-2)$. Hence, by (3.16), (3.17), and, again, by (3.6) we obtain

$$
\begin{aligned}
L_N &\leqslant \Bigl(\frac{t-1}{t-2}\Bigr)^{t}+\Bigl(\frac{t-1}{t-2}\Bigr)^{t-1}\sum_{j=1}^{t}\sum_{C_j\subset A_j^N}\Bigl(\frac{t}{t-1}\Bigr)^{m(C_j)+1}\biggl|\mathsf{E}\biggl[\prod_{i\in C_j}\varphi_i(\omega)\biggr]\biggr| \\
&\quad+\Bigl(\frac{t-1}{t-2}\Bigr)^{t-2}\sum_{1\leqslant j_1<j_2\leqslant t}\ \sum_{C_{j_1}\subset A_{j_1}^N}\ \sum_{C_{j_2}\subset A_{j_2}^N}\Bigl(\frac{t}{t-1}\Bigr)^{m(C_{j_1})+m(C_{j_2})+2} \\
&\qquad\times\biggl|\mathsf{E}\biggl[\prod_{k=1}^{2}\prod_{i\in C_{j_k}}\varphi_i(\omega)\biggr]\biggr| \\
&\quad+\dots+\Bigl(\frac{t-1}{t-2}\Bigr)^{t-s}\sum_{1\leqslant j_1<\dots<j_s\leqslant t}\ \sum_{C_{j_1}\subset A_{j_1}^N}\cdots\sum_{C_{j_s}\subset A_{j_s}^N}\Bigl(\frac{t}{t-1}\Bigr)^{\sum_{k=1}^{s}m(C_{j_k})+s} \\
&\qquad\times\biggl|\mathsf{E}\biggl[\prod_{k=1}^{s}\prod_{i\in C_{j_k}}\varphi_i(\omega)\biggr]\biggr| \\
&\quad+\dots+\sum_{C_1\subset A_1^N}\cdots\sum_{C_t\subset A_t^N}\Bigl(\frac{t}{t-1}\Bigr)^{\sum_{j=1}^{t}m(C_j)+t}\biggl|\mathsf{E}\biggl[\prod_{j=1}^{t}\prod_{i\in C_j}\varphi_i(\omega)\biggr]\biggr|. \qquad (3.18)
\end{aligned}
$$

If $t\geqslant 3$, then

$$
\frac{t}{t-1}\leqslant 2 \quad\text{and}\quad \Bigl(\frac{t-1}{t-2}\Bigr)^{t-s}\Bigl(\frac{t}{t-1}\Bigr)^{s}\leqslant 4e \quad (s=0,1,\dots,t). \qquad (3.19)
$$

Moreover, since the sets A_j^N $(j=1,2,\dots,t)$ are pairwise disjoint, the integrands in the last expression are distinct products of pairwise distinct functions φ_i $(i=1,2,\dots,N)$. The number of these functions is $m(C_j)$ $(j=1,2,\dots,t)$ for terms in the first sum, $m(C_{j_1})+m(C_{j_2})$ $(1\leqslant j_1<j_2\leqslant t)$ for terms in the second, $\dots$, and $\sum_{j=1}^t m(C_j)$ for terms in the last sum. Hence the largest index of functions φ_i in each integrand is at least $m(C_j)$, $m(C_{j_1})+m(C_{j_2})$, $\dots$, $\sum_{j=1}^t m(C_j)$, respectively. As a result, from (3.18) and (3.19) we deduce

$$L_N\leqslant 4e\biggl\{\sum_{i=1}^N 2^i\sum_{1\leqslant j_1<\dots<j_s\leqslant i-1}|\mathsf{E}(\varphi_{j_1}\cdots\varphi_{j_s}\varphi_i)|\biggr\}, \tag{3.20}$$

where the inner sum is taken over all collections of indices $1\leqslant j_1<\dots<j_s\leqslant i-1$. Hence it follows by (3.4) and (3.5) that

$$L_N\leqslant\frac{4e}{D}\sum_{i=1}^\infty\varepsilon_i<2,$$

and therefore, by (3.15) and (3.16) for all $N\in\mathbb{N}$ we obtain

$$|I_N|\leqslant 2\|\varphi\|_t.$$

In view of (3.14), the last inequality shows that

$$\sum_{j=1}^t\biggl(\sum_{i\in A_j}a_i^2\biggr)^{1/2}\leqslant 6D\|\varphi\|_t.$$

Hence, by Lemma 1 for positive integers $t\geqslant 3$ and arbitrary $a=(a_i)_{i=1}^\infty\in l_2$ we obtain the inequality

$$\mathcal{K}_{1,2}(\sqrt{t},a)\leqslant 6\sqrt{2}\,D\|\varphi\|_t. \tag{3.21}$$

We shall also verify this inequality for $t=1,2$.

We claim that the above-constructed system $\{\varphi_i\}_{i=1}^\infty$ is a Riesz basis sequence, that is,

$$\biggl\|\sum_{i=1}^\infty a_i\varphi_i\biggr\|_2\asymp\|(a_i)\|_2. \tag{3.22}$$

In fact, we assume that $\|\varphi_i\|_2=1$, therefore for each $m\in\mathbb{N}$ it follows by (3.5) that

$$\begin{aligned}\biggl\|\sum_{i=1}^m a_i\varphi_i\biggr\|_2^2&=\sum_{i=1}^m a_i^2+2\sum_{1\leqslant i<j\leqslant m}a_ia_j\mathsf{E}(\varphi_i\varphi_j)\\&\geqslant\sum_{i=1}^m a_i^2\biggl\{1-2\biggl[\sum_{1\leqslant i<j\leqslant m}(\mathsf{E}(\varphi_i\varphi_j))^2\biggr]^{1/2}\biggr\}\geqslant\frac{1}{2}\|(a_i)\|_2^2.\end{aligned} \tag{3.23}$$

In a similar way,

$$\biggl\|\sum_{i=1}^m a_i\varphi_i\biggr\|_2^2\leqslant\frac{3}{2}\|(a_i)\|_2^2$$

and the proof of (3.22) is complete.

Relation (3.2) shows that $\{g_i\}_{i=1}^{\infty}$, and therefore also $\{\varphi_i\}_{i=1}^{\infty}$, is an S_p-system for each $p < \infty$. Hence it is a Banach system ([2], Corollary 1.3.1), that is, for some $M = M(D)$ we have

$$\|a\|_2 \leqslant M \left\| \sum_{i=1}^{m} a_i \varphi_i \right\|_1 \qquad (a = (a_i)_{i=1}^{\infty} \in l_2). \tag{3.24}$$

Since $\|a\|_2 \leqslant \|a\|_1$, it follows that $\mathcal{K}(1, a; l_1, l_2) = \|a\|_2$. By the concavity of the $\mathcal{K}$-functional we can now deduce from (3.23) and (3.24) that

$$\mathcal{K}_{1,2}(\sqrt{2}, a) \leqslant \sqrt{2}\, \mathcal{K}_{1,2}(1, a) \leqslant 2\|\varphi\|_2 \tag{3.25}$$

and

$$\mathcal{K}_{1,2}(1, a) \leqslant M\|\varphi\|_1. \tag{3.26}$$

For arbitrary $t \geqslant 1$ we can find a positive integer t_0 such that $t_0 \leqslant t < t_0 + 1$. Now, in view of the concavity of the $\mathcal{K}$-functional again and also by (3.21), (3.25), and (3.26) we obtain

$$\mathcal{K}_{1,2}(\sqrt{t}, a) \leqslant \sqrt{t/t_0}\, \mathcal{K}_{1,2}(\sqrt{t_0}, a) \leqslant \sqrt{2}\, \mathcal{K}_{1,2}(\sqrt{t_0}, a) \leqslant C\|\varphi\|_t,$$

where $C = \sqrt{2}\, \max(2; M; 6\sqrt{2}\, D)$ depends only on D.

Thus, relation (3.3) holds also for the system $\{\varphi_i\}_{i=1}^{\infty}$, which completes the proof of the theorem.

The next two results are immediate consequences of Theorems 1 and 4.

Theorem 5. *From a system $\{f_n\}_{n=1}^{\infty}$ of random variables on a probability space $(\Omega, \Sigma, \mathsf{P})$ one can select a subsystem $\{\varphi_i\}_{i=1}^{\infty}$ equivalent in distribution to the Rademacher system on $[0,1]$ if and only if there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that*

(1) $|f_{n_k}(\omega)| \leqslant D$ $(k = 1, 2, \dots; \omega \in \Omega)$;
(2) $f_{n_k} \to 0$ *weakly in* L_2;
(3) $d = \inf_{k=1,2,\dots} \|f_{n_k}\|_2 > 0$.

Moreover, the constant involved in the equivalence will depend only on D and d.

Proof. By Theorem 4 we can select a subsequence $\{\varphi_i\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that the equivalence (3.1) holds. Then, however, $\{\varphi_i\} \overset{\mathsf{P}}{\sim} \{r_n\}$ by Theorem 1 and the proof of the *sufficiency* is complete.

The *necessity* of conditions (1) and (3) is a consequence of the definition of equivalence in distribution and the fact that the Rademacher system has these properties. As pointed out in the introduction, the Rademacher system is an S_p-system for each $p < \infty$. Hence a system equivalent to it in distribution must also have this property. Thus, the necessity of condition (2) is a consequence of Theorem A stated in the introduction.

Remark 3. In connection with Theorem 5 we recall a problem posed by Alexits [22], § 3.2. Is it always possible to select an infinite multiplicative or even strictly multiplicative subsystem from an arbitrary complete orthogonal system of functions? The result of Theorem 5 answers (in the affirmative) a closely related question: from an arbitrary system of random variables with properties (1)–(3) one can select a subsystem that is equivalent in distribution to the strictly multiplicative (more than that: consisting of independent functions) Rademacher system.

Theorem 6. *If $\{f_n\}_{n=1}^{\infty}$ is an orthonormal system of random variables on a probability space $(\Omega, \Sigma, \mathsf{P})$, $|f_n(\omega)| \leqslant D$ $(n = 1, 2, \dots; \omega \in \Omega)$, then one can select from this system a subsequence $\{\varphi_i\}_{i=1}^{\infty}$ equivalent in distribution to the Rademacher system. The constant involved in the equivalence depends only on D.*

It is clear that a sequence $\{\varphi_i\}_{i=1}^{\infty}$ satisfying (3.1) is a Sidon system (and also an S_p-system for all $p < \infty$, with constant $K_p \asymp \sqrt{p}$). It turns out that its 'Sidon property' holds in a certain 'strong' sense.

Corollary 4. *Let $\{f_n\}_{n=1}^{\infty}$ be a system of random variables that has a subsequence $\{f_{n_k}\}$ satisfying conditions (1)–(3) from Theorem 5. Then $\{f_n\}_{n=1}^{\infty}$ contains a subsystem $\{\varphi_i\}_{i=1}^{\infty}$ such that for some constants $\alpha_1, \alpha_2, \alpha_3 > 0$ dependent only on D and d and each polynomial*

$$T(\omega) = \sum_{i=1}^{m} a_i \varphi_i(\omega)$$

there exists a set $E = E(T) \subset \Omega$, $\mathsf{P}(E) > \alpha_1 2^{-m}$, such that

$$|T(\omega)| \geqslant \alpha_2 \|T\|_\infty \geqslant \alpha_3 \sum_{i=1}^{m} |a_i| \qquad (\omega \in E).$$

Proof. By Theorem 5 there exists a subsequence $\{\varphi_i\} \subset \{f_n\}$ such that for some $C > 0$ and all $z > 0$ we have the inequality

$$C^{-1} \bigl|\{|\widetilde{T}(x)| > Cz\}\bigr| \leqslant \mathsf{P}\{|T(\omega)| > z\} \leqslant C \bigl|\{|\widetilde{T}(x)| > C^{-1} z\}\bigr|,$$

where $\widetilde{T}(x) = \sum_{i=1}^{m} a_i r_i(x)$ $(x \in [0,1])$. In particular, this shows that

$$C^{-1} \|T\|_\infty \leqslant \|\widetilde{T}\|_\infty \leqslant C \|T\|_\infty.$$

Hence it follows by the definition of the Rademacher system that

$$\begin{aligned} \mathsf{P}\{|T(\omega)| > (2C^2)^{-1} \|T\|_\infty\} &\geqslant \mathsf{P}\{|T(\omega)| > (2C)^{-1} \|\widetilde{T}\|_\infty\} \\ &\geqslant C^{-1} \bigl|\{|\widetilde{T}(x)| > 2^{-1} \|\widetilde{T}\|_\infty\}\bigr| \geqslant C^{-1} 2^{-m}. \end{aligned}$$

We set $\alpha_1 = C^{-1}$, $\alpha_2 = (2C^2)^{-1}$, and $E = E(T) = \{\omega \in \Omega : |T(\omega)| > \alpha_2 \|T\|_\infty\}$. Then for $\omega \in E$ we have

$$|T(\omega)| > \alpha_2 \|T\|_\infty \geqslant \alpha_2 C^{-1} \|\widetilde{T}\|_\infty = \alpha_3 \sum_{i=1}^{m} |a_i|,$$

where $\alpha_3 = \alpha_2 C^{-1}$.

As already pointed out in the introduction (see Theorem B′), under the same assumptions about a sequence of measurable functions on $[0,1]$ as in Theorem B one can select a subsystem of this sequence that has the Sidon–Zygmund property and, under the additional assumption (0.3), the Sidon–Zygmund property in the narrow sense. The following two theorems show that even somewhat stronger results actually hold.

Theorem 7. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on* $[0,1]$ *that contains a subsequence $\{f_{n_k}\}$ with properties* (1)–(3) *in Theorem* 5. *Then there exists a subsystem $\{\varphi_i\} \subset \{f_n\}$ with the following property.*

For some subset E of $[0,1]$, $|E| > 0$, and each interval $I \subset [0,1]$, $|I \cap E| > 0$, there exists an index $k_0 = k_0(I)$ such that the sequence $\{\varphi_i \chi_I\}_{i=k_0}^{\infty}$ ($\chi_I(x) = 1$, $x \in I$, and $\chi_I(x) = 0$, $x \notin I$) is equivalent in distribution to the Rademacher system on $[0,1]$. The constant involved in this equivalence depends only on D, d, and the interval I.

Proof. By Theorem 1 it is sufficient to find a subsystem $\{\varphi_i\}$ of $\{f_n\}$ and a subset E of $[0,1]$, $|E| > 0$, such that for each interval $I \subset [0,1]$, $|I \cap E| > 0$, there exists an index $k_0 = k_0(I)$ such that

$$\left\| \sum_{i=1}^{\infty} a_i \varphi_{k_0+i-1} \chi_I \right\|_t \asymp \mathcal{K}_{1,2}(\sqrt{t}, a) \qquad (a = (a_i)_{i=1}^{\infty}, t \geqslant 1)$$

(the constant involved in this equivalence depends on D, d, and I).

The arguments bringing us to this relation are mostly similar to the ones used in the proof of Theorem 4. We merely make several observations.

First of all, inequality (3.2) remains valid, of course, if we replace the functions g_i by $g_i \chi_I$ (here I is an arbitrary subinterval of $[0,1]$).

Next, as in the proof of Theorem 1.4.2 in [2], one can select a subsequence $\{\varphi_i\} \subset \{g_i\}$ such that relations of type (3.5) hold for integrals over each interval $I \subset [0,1]$.

Finally, the well-known Marcinkiewicz's lemma ([2], Lemma 1.2.5) allows us to assume that there exists a subset E of $[0,1]$, $|E| > 0$, such that for each $F \subset E$, $|F| > 0$, the inequality

$$\liminf_{i \to \infty} \int_F \varphi_i^2(x)\, dx > 0 \tag{3.27}$$

holds. After that one virtually repeats the proof of Theorem 4. The only difference is that one must replace integrals over $[0,1]$ by integrals over an interval I such that $|I \cap E| > 0$.

Theorem 8. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on* $[0,1]$ *that contains a subsequence $\{f_{n_k}\}$ satisfying conditions* (1) *and* (2) *of Theorem* 5 *and also condition* (0.3).

Then there exists a subsystem $\{\varphi_i\}$ of $\{f_n\}$ such that for each interval $I \subset [0,1]$, $|I| > 0$, there exists an index $k_0 = k_0(I)$ such that the sequence $\{\varphi_i \chi_I\}_{i=k_0}^{\infty}$ is equivalent in distribution to the Rademacher system on $[0,1]$.

The proof of Theorem 8 is perfectly similar to the proof of the preceding theorem because, in view of (0.3), we can assume that the selected subsystem $\{\varphi_i\}_{i=1}^{\infty}$ satisfies (3.27) for each set $F \subset [0,1]$ of positive measure.

§ 4. Density of subsystems equivalent in distribution to the Rademacher system

We consider now the problem of the selection of finite subsystems of uniformly bounded orthonormal systems of functions $\{f_n\}_{n=1}^{N}$ on $[0,1]$. We claim that such a system always contains a subsystem $\{f_{n_i}\}_{i=1}^s$ of 'logarithmic density' ($s \geqslant C\log_2 N$) that is equivalent in distribution to the system of the first s Rademacher functions with constant independent of s. This is an improvement on Theorem C in the introduction established by Kashin [9].

Theorem 9. *Let $\{f_n\}_{n=1}^N$ be an orthonormal system of functions on $[0,1]$ such that $|f_n(x)| \leqslant D$ $(n=1,2,\dots,N)$. Then it contains a collection of functions $\{f_{n_i}\}_{i=1}^s$, where $s \geqslant \max\{[\frac{1}{6}\log_2 N];1\}$ and $1 \leqslant n_1 < n_2 < \dots < n_s \leqslant N$, such that for some $C>0$ dependent only on D and all $a_i \in \mathbb{R}$ $(i=1,2,\dots,s)$ and $z>0$ the following inequality holds*:

$$C^{-1}\biggl|\biggl\{\biggl|\sum_{i=1}^s a_i r_i(x)\biggr| > Cz\biggr\}\biggr| \leqslant \biggl|\biggl\{\biggl|\sum_{i=1}^s a_i f_{n_i}(x)\biggr| > z\biggr\}\biggr| \leqslant C\biggl|\biggl\{\biggl|\sum_{i=1}^s a_i r_i(x)\biggr| > C^{-1}z\biggr\}\biggr|.$$

We require two lemmas for the proof. The first is well known ([10], Lemma 8.4.1).

Lemma 3. *Each system of functions $\{f_n\}_{n=1}^N$ $(\log_2 N \geqslant 6)$ satisfying the assumptions of Theorem 9 contains functions $\{f_{n_i}\}_{i=1}^s$ $(1 \leqslant n_1 < n_2 < \dots < n_s \leqslant N)$, where $s \geqslant [\frac{1}{6}\log_2 N]$, such that*

$$\sum_{\theta\in\mathcal{A}_s}\biggl\{\mathsf{E}\biggl[\prod_{i=1}^s\biggl(\frac{f_{n_i}(x)}{D}\biggr)^{\theta_i}\biggr]\biggr\}^2 \leqslant 10^{-s}. \tag{4.1}$$

Here $\mathcal{A}_s$ is the set of collections $\theta = (\theta_i)_{i=1}^s$ such that

$$\text{(a)}\ \ \theta_i = 0,\ 1\ \textit{or}\ 2; \qquad \text{(b)}\ \sum_{i:\,\theta_i=1} 1 \geqslant 1; \qquad \text{(c)}\ \sum_{i:\,\theta_i=2} 1 \leqslant 1.$$

The second lemma enables one to extend under certain conditions a uniformly bounded system of functions on $[0,1]$ to a larger set so that it becomes a multiplicative system remaining at the same time uniformly bounded. For similar results on the extension to an orthonormal collection, see [10], § 7.1.

Lemma 4. *Let $\{g_i\}_{i=1}^s$ be a collection of functions on $[0,1]$ such that $|g_i(x)| \leqslant D$ $(i=1,2,\dots,s;\ x\in[0,1])$ and assume that*

$$\max_{\theta\in\mathcal{A}_s'}\biggl|\mathsf{E}\biggl[\prod_{i=1}^s\biggl(\frac{g_i(x)}{D}\biggr)^{\theta_i}\biggr]\biggr| < 2^{-s}, \tag{4.2}$$

where $\mathcal{A}_s'$ is the set of all collections $\theta = (\theta_i)_{i=1}^s \in \mathcal{A}_s$ such that $\theta_i = 0$ or 1.

Then there exists a multiplicative system $\{h_i\}_{i=1}^s$ of functions on $[0,2]$ such that $h_i(x)\chi_{[0,1]}(x) = g_i(x)$ and $\|h_i\|_{L_\infty[0,2]} \leqslant D$ $(i=1,2,\dots,s)$.

Proof. We introduce the partitioning

$$[1,2) = \bigcup_{k=1}^{2^s}\Delta_k, \qquad \Delta_k = [a_{k-1},a_k), \quad a_k = 1 + k2^{-s} \quad (k=1,2,\dots,2^s),$$

and consider some (arbitrary) one-to-one correspondence between the set $\mathcal{A}'_s$ and the family of intervals $\{\Delta_k\}_{k=1}^{2^s-1}$. We shall define the functions h_i on each interval Δ_k in this family.

Let $\theta = (\theta_i)_{i=1}^s$ be an element of $\mathcal{A}'_s$ corresponding to Δ_k. We consider the set $\{i_j\}_{j=1}^m = \{i = 1, 2, \ldots, s : \theta_i = 1\}$, $i_1 < i_2 < \cdots < i_m$ $(m = 1, 2, \ldots, s)$.

For $\alpha \in \Delta_k$ let

$$u_\alpha(x) = \chi_{[a_{k-1},\alpha)}(x) - \chi_{[\alpha,a_k]}(x) \qquad (x \in \Delta_k).$$

Then the quantity

$$v(\alpha) = \int_{\Delta_k} u_\alpha(x)\,dx = 2\alpha - a_{k-1} - a_k \qquad (a_{k-1} \leqslant \alpha \leqslant a_k)$$

ranges over the entire interval $[-2^{-s}, 2^{-s}]$. By (4.2) the functions $\bar{g}_i(x) = g_i(x)/D$ satisfy the inequality $|\mathsf{E}(\bar{g}_{i_1}\bar{g}_{i_2}\cdots\bar{g}_{i_m})| < 2^{-s}$, therefore there exists $\alpha_k \in (a_{k-1}, a_k)$ such that

$$v(\alpha_k) = -\mathsf{E}(\bar{g}_{i_1}\bar{g}_{i_2}\cdots\bar{g}_{i_m}). \tag{4.3}$$

Assume first that $m \geqslant 2$ and consider a family of piecewise constant functions $d_j(x)$ $(x \in [a_{k-1}, \alpha_k)$ and $j = 1, 2, \ldots, m-1)$ such that

$$\begin{aligned} &(1) \quad |d_j(x)| \equiv 1;\\ &(2) \quad \int_{a_{k-1}}^{\alpha_k} d_{j_1}d_{j_2}\cdots d_{j_l}\,dx = 0 \quad (1 \leqslant j_1 < j_2 < \cdots < j_l \leqslant m-1) \end{aligned} \tag{4.4}$$

(one can construct such a family, for instance, by carrying over the Rademacher functions to $[a_{k-1}, \alpha_k)$ from $[0, 1]$). Let $c_j(x)$ $(x \in [\alpha_k, a_k]$ and $j = 1, 2, \ldots, m-1)$ be a similar system on the interval $[\alpha_k, a_k]$.

If $x \in \Delta_k$, then

$$\begin{aligned} &h_i(x) \equiv 0 \quad (i \neq i_j, \quad j = 1, 2, \ldots, m),\\ &h_{i_j}(x) = D[d_j(x)\chi_{[a_{k-1},\alpha_k)}(x) + c_j(x)\chi_{[\alpha_k,a_k]}(x)] \quad (j = 1, 2, \ldots, m-1) \end{aligned}$$

and

$$h_{i_m}(x) = D\biggl[\prod_{j=1}^{m-1} d_j(x)\chi_{[a_{k-1},\alpha_k)}(x) - \prod_{j=1}^{m-1} c_j(x)\chi_{[\alpha_k,a_k]}(x)\biggr].$$

By (4.4) and (4.3),

$$\begin{aligned} \int_{\Delta_k} h_{i_1}h_{i_2}\cdots h_{i_m}\,dx &= D^m \int_{\Delta_k} u_{\alpha_k}(x)\,dx\\ &= D^m v(\alpha_k) = -\mathsf{E}(g_{i_1}g_{i_2}\cdots g_{i_m}). \end{aligned} \tag{4.5}$$

At the same time, for any other collection of indices $1 \leqslant i'_1 < \cdots < i'_k \leqslant s$ it follows from the definition of the functions h_i $(i = 1, 2, \ldots, s)$ and from (4.4) that

$$\int_{\Delta_k} h_{i'_1}h_{i'_2}\cdots h_{i'_l}\,dx = 0. \tag{4.6}$$

On the other hand, if $m=1$, then we set

$$h_i(x)\equiv 0 \quad (i\neq i_1), \qquad h_{i_1}(x)=Du_{\alpha_k}(x),$$

and relations (4.5) and (4.6) again hold.

We now define the functions $h_i(x)$ on the entire interval $[0,2]$ by setting $h_i(x)=g_i(x)$ for $x\in[0,1]$ and $h_i(x)=0$ for $x\in\Delta_{2^s}$. In view of (4.5) and (4.6), $\{h_i\}_{i=1}^s$ is a multiplicative system on $[0,2]$ and $|h_i(x)|\leqslant D$.

Proof of Theorem 9. In view of Theorem 1 and Remark 1 after it, it suffices to show that there exists a collection $\{f_{n_i}\}_{i=1}^s$ ($s\geqslant \max\{[\frac{1}{6}\log_2 N];1\}$) such that for some constant dependent only on D,

$$\Bigl\|\sum_{i=1}^s a_i f_{n_i}\Bigr\|_t \asymp \mathcal{K}_{1,2}(\sqrt{t},a) \qquad (t\geqslant 1, \quad a=(a_i)_{i=1}^s), \tag{4.7}$$

where, as before, $\mathcal{K}_{1,2}(\sqrt{t},a)=\mathcal{K}(\sqrt{t},a;l_1,l_2)$.

Using Lemma 3 we select a subsystem $\{f_{n_i}\}_{i=1}^s$ ($s\geqslant \max\{[\frac{1}{6}\log_2 N];1\}$) satisfying (4.1). Since $\mathcal{A}'_s\subset\mathcal{A}_s$, the functions $g_i=f_{n_i}$ satisfy at the same time (4.2). Hence, by Lemma 4, we can extend the functions $\{f_{n_i}\}_{i=1}^s$ to a multiplicative system $\{h_i\}_{i=1}^s$ on $[0,2]$, $|h_i(x)|\leqslant D$. Then, however, the functions $h'_i(x)=h_i(2x)$ make up a multiplicative system on $[0,1]$, $|h'_i(x)|\leqslant D$, and therefore by Corollary 3 in [16], for arbitrary $t\geqslant 1$ and $a=(a_i)_{i=1}^s$ we obtain

$$\Bigl\|\sum_{i=1}^s a_i h'_i\Bigr\|_{L_t[0,1]} \leqslant C_1\sqrt{t}\,\|a\|_2,$$

where $C_1>0$ depends only on D. Hence

$$\Bigl\|\sum_{i=1}^s a_i f_{n_i}\Bigr\|_t \leqslant 2C_1\sqrt{t}\,\|a\|_2, \tag{4.8}$$

which in a similar way to the proof of Theorem 4 (see inequality (3.20)) shows that

$$\Bigl\|\sum_{i=1}^s a_i f_{n_i}\Bigr\|_t \leqslant C\mathcal{K}_{1,2}(\sqrt{t},a) \qquad (t\geqslant 1), \tag{4.9}$$

where $C=\max(D;2C_1)$ depends only on D.

For the proof of the reverse inequality we set

$$a=(a_i)_{i=1}^s \quad \text{and} \quad P(x)=\sum_{i=1}^s a_i f_{n_i}(x).$$

Let $t\in\mathbb{N}$ first, $t\geqslant 3$. If $s\leqslant 16D^2$, then by the definition of the $\mathcal{K}$-functional we have

$$\|P\|_t\geqslant \|P\|_2=\|a\|_2\geqslant \frac{1}{\sqrt{s}}\sum_{i=1}^s|a_i|\geqslant \frac{1}{4D}\mathcal{K}_{1,2}(\sqrt{t},a). \tag{4.10}$$

Hence it suffices to consider the case when

$$s > 16D^2. \tag{4.11}$$

We consider now the Riesz products

$$R_s(x) = \prod_{i=1}^{s}\biggl(1 + \frac{b_i}{D} f_{n_i}(x)\biggr).$$

For an arbitrary partitioning $\{A_j\}_{j=1}^t$ of the set $\{1, 2, \ldots, s\}$ into t disjoint subsets (some of which may be empty) we assume that

$$\sum_{i\in A_j} b_i^2 \leqslant 1 \qquad (j = 1, 2, \ldots, t). \tag{4.12}$$

We now find a lower estimate for the integral

$$\begin{aligned} I_s &= \int_0^1 P(x)R_s(x)\,dx \\ &= \sum_{i=1}^{s} a_i \int_0^1 f_{n_i}(x)\,dx + \sum_{i=1}^{s} a_i \int_0^1 f_{n_i}(x) \sum_{\theta\in\mathcal{A}'_s} \prod_{k=1}^{s} \biggl[\frac{b_k}{D} f_{n_k}(x)\biggr]^{\theta_k} dx. \end{aligned}$$

Rearranging the terms we obtain

$$I_s = D^{-1}\sum_{i=1}^{s} a_i b_i + \sum_{i=1}^{s} a_i\gamma_i, \tag{4.13}$$

where

$$\begin{gathered} \gamma_i = \int_0^1 f_{n_i}(x)\,dx + \int_0^1 f_{n_i}(x) \sum_{\theta\in\mathcal{A}'_s(i)} \prod_{k=1}^{s} \biggl[\frac{b_k}{D} f_{n_k}(x)\biggr]^{\theta_k} dx, \\ \mathcal{A}'_s(i) = \mathcal{A}'_s \setminus \{\theta^i\}, \qquad \theta^i = (\theta^i_j), \quad \theta^i_i = 1, \quad \theta^i_j = 0 \ \ (j \neq i). \end{gathered}$$

In view of (4.1) and the condition $|b_i| \leqslant 1$ $(i = 1, 2, \ldots, s)$, after applying the Cauchy–Schwarz–Bunyakovskiĭ inequality we obtain

$$\sum_{i=1}^{s} |\gamma_i| \leqslant D \sum_{\theta\in\mathcal{A}_s} \biggl| \int_0^1 \prod_{k=1}^{s} \biggl[\frac{f_{n_k}(x)}{D}\biggr]^{\theta_k} dx \biggr| \leqslant D(s2^s)^{1/2} 10^{-s/2} = D(s5^{-s})^{1/2}.$$

In view of (4.11), $(s5^{-s})^{1/2} \leqslant 8/s < 2^{-1}D^{-2}$, therefore

$$\sum_{i=1}^{s} |\gamma_i| \leqslant \frac{1}{2D}\,. \tag{4.14}$$

For each $j = 1, 2, \ldots, t$ we now find b_i $(i \in A_j)$ such that (4.12) holds and

$$\sum_{i\in A_j} a_i b_i = \Bigl(\sum_{i\in A_j} a_i^2\Bigr)^{1/2}.$$

Then by (4.13) and (4.14) we obtain

$$\begin{aligned} I_s &\geqslant \frac{1}{D}\sum_{j=1}^{t}\Bigl(\sum_{i\in A_j} a_i^2\Bigr)^{1/2} - \sum_{j=1}^{t}\Bigl|\sum_{i\in A_j} a_i\gamma_i\Bigr| \\ &\geqslant \frac{1}{D}\sum_{j=1}^{t}\Bigl(\sum_{i\in A_j} a_i^2\Bigr)^{1/2} - \sum_{j=1}^{t}\Bigl(\sum_{i\in A_j} a_i^2\Bigr)^{1/2}\Bigl(\sum_{i\in A_j} |\gamma_i|\Bigr) \\ &\geqslant \frac{1}{2D}\sum_{j=1}^{t}\Bigl(\sum_{i\in A_j} a_i^2\Bigr)^{1/2}. \end{aligned}$$

The partitioning $\{A_j\}_{j=1}^t$ is arbitrary, therefore by Lemma 1 we finally obtain

$$I_s \geqslant \frac{1}{2\sqrt{2}D}\mathcal{K}_{1,2}(\sqrt{t}, a) \qquad (t \in \mathbb{N}, \quad t \geqslant 3). \tag{4.15}$$

Next, we find an upper estimate for I_s. By Hölder's inequality,

$$|I_s| \leqslant \|R_s\|_{t'}\|P\|_t, \qquad t' = \frac{t}{t-1}.$$

As in the proof of Theorem 4, we claim that for $t \geqslant 3$ we have

$$\|R_s\|_{t'} \leqslant 4e2^s \sum_{\theta\in\mathcal{A}'_s}\biggl|\mathsf{E}\biggl\{\prod_{i=1}^{s}\biggl[\frac{f_{n_i}(x)}{D}\biggr]^{\theta_i}\biggr\}\biggr|,$$

therefore by (4.1), the inclusion $\mathcal{A}'_s \subset \mathcal{A}_s$, and the Cauchy–Schwarz–Bunyakovskiĭ inequality we obtain the relation $\|R_s\|_{t'} \leqslant 4e$. Hence $|I_s| \leqslant 4e\|P\|_t$ and it follows from (4.15) that

$$\mathcal{K}_{1,2}(\sqrt{t}, a) \leqslant 8\sqrt{2}\, eD\|P\|_t \qquad (t \in \mathbb{N}, \quad t \geqslant 3). \tag{4.16}$$

Similar relations hold also for $t = 2$ and $t = 1$. In fact, $\{f_n\}_{n=1}^N$ is an orthonormal system and $\mathcal{K}_{1,2}(1, a) = \|a\|_2$, so that

$$\mathcal{K}_{1,2}(\sqrt{2}, a) \leqslant \sqrt{2}\,\mathcal{K}_{1,2}(1, a) = \sqrt{2}\,\|P\|_2. \tag{4.17}$$

On the other hand, if $t = 1$, then, in view of (4.8) and as in the proof of Theorem 4, we can use [2], Corollary 1.3.1, and therefore

$$\mathcal{K}_{1,2}(1, a) = \|a\|_2 \leqslant M\|P\|_1 \tag{4.18}$$

with $M > 0$ dependent only on D.

Finally, passing from positive integers (see (4.16)–(4.18)) to arbitrary $t \geqslant 1$ we obtain in the standard way

$$\mathcal{K}_{1,2}(\sqrt{t}, a) \leqslant C\|P\|_t,$$

where the constant $C = \sqrt{2}\,\max(M; \sqrt{2}; 8\sqrt{2}\,eD)$ depends only on D. Hence, in view of (4.9), the equivalence (4.7) holds with constant dependent only on D, which completes the proof.

Remark 4. The example of a trigonometric system shows that the result of Theorem 9 (similarly to Theorem C in the introduction) is best possible in order. For Stechkin [23] has shown that if a sequence $\{\sqrt{2}\,\cos 2\pi n_k x\}_{k=1}^{\infty}$ $(x \in [0,1])$ is a Sidon system, then

$$\sum_{k\colon n_k<N} 1 \leqslant C \ln N \qquad (N = 2, 3, \dots).$$

Remark 5. The results on the selection of both infinite and finite subsystems equivalent in distribution to the Rademacher system proved in this section are best possible in the following sense: no further 'thinning' can improve the distributions of the corresponding polynomials. It is easy to see that for each subsequence $\{r_{n_i}\}_{i=1}^{\infty}$ of the Rademacher system and arbitrary $m \in \mathbb{N}$, $a_i \in \mathbb{R}$ $(i = 1, 2, \dots, m)$, and $z > 0$ we have

$$\biggl|\biggl\{\biggl|\sum_{i=1}^{m} a_i r_{n_i}(x)\biggr| > z\biggr\}\biggr| = \biggl|\biggl\{\biggl|\sum_{i=1}^{m} a_i r_i(x)\biggr| > z\biggr\}\biggr|.$$

§ 5. $\mathcal{K}$-closed representability of Banach couples

We say that a Banach couple (X_0, X_1) is $\mathcal{K}$-closed representable in a Banach couple (Y_0, Y_1) if there exists a linear operator $T\colon X_0 + X_1 \to Y_0 + Y_1$ such that

(a) T is a bounded injective operator from X_0 into Y_0 and from X_1 into Y_1;
(b) for some constant $C > 0$ independent of $x \in X_0 + X_1$ and $t > 0$,

$$\mathcal{K}(t, Tx; Y_0, Y_1) \asymp \mathcal{K}(t, x; X_0, X_1).$$

As shown in [24] (see also [25]), a couple (l_1, l_2) is $\mathcal{K}$-closed representable in the couple (L_∞, G) of spaces of measurable functions on $[0,1]$. Here we denote by G the closure of the space L_∞ in the Orlicz space L_N with $N(u) = e^{u^2} - 1$ and we can define the operator T as follows:

$$Ta(x) = \sum_{n=1}^{\infty} a_n r_n(x) \qquad (a = (a_n)_{n=1}^{\infty} \in l_2).$$

As before, $\{r_n\}_{n=1}^{\infty}$ is the Rademacher system on $[0,1]$.

Assume that a sequence $\{f_n\}_{n=1}^{\infty}$ of measurable functions on $[0,1]$ has properties (1)–(3) in Theorem 5. Then we can select a subsystem $\{\varphi_i\}$ of it that is equivalent in distribution to the Rademacher system. All the proofs in [24] use only estimates of the distributions of the absolute values of polynomials in the Rademacher system, therefore the corresponding results hold also for $\{\varphi_i\}$. In particular, the operator

$$T_\varphi a(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x) \qquad (a = (a_i)_{i=1}^{\infty} \in l_2)$$

also brings about a $\mathcal{K}$-closed representability of the couple (l_1, l_2) in (L_∞, G).

We now prove a negative result on the $\mathcal{K}$-closed representability of couples, which is related to the above example.

Theorem 10. *The couple (l_1, l_2) is not $\mathcal{K}$-closed representable in the couple of spaces (L_∞, L_2) of random variables on a probability space $(\Omega, \Sigma, \mathsf{P})$.*

Proof. Assume that, on the contrary, there exists a linear operator $T\colon l_2 \to L_2$ such that

$$\|Ta\|_\infty \asymp \|a\|_1 \quad (a \in l_1), \qquad \|Ta\|_2 \asymp \|a\|_2 \quad (a \in l_2) \tag{5.1}$$

and for some $C_1 > 0$,

$$C_1^{-1}\mathcal{K}_{1,2}(t,a) \leqslant \mathcal{K}(t, Ta; L_\infty, L_2) \leqslant C_1 \mathcal{K}_{1,2}(t,a) \qquad (a \in l_2, \quad t > 0), \tag{5.2}$$

where, as before, $\mathcal{K}_{1,2}(t,a) = \mathcal{K}(t, a; l_1, l_2)$.

Let $f_n = Te_n$, where $e_n = (\delta_n^j)$, $\delta_n^n = 1$, $\delta_n^j = 0$ $(j \neq n)$. Then it follows from (5.1) that there exist $D > 0$ and $d > 0$ such that

$$D^{-1}\|a\|_1 \leqslant \left\| \sum_{n=1}^{\infty} a_n f_n \right\|_\infty \leqslant D\|a\|_1 \qquad (a \in l_1)$$

and

$$d\|a\|_2 \leqslant \left\| \sum_{n=1}^{\infty} a_n f_n \right\|_2 \leqslant d^{-1}\|a\|_2 \qquad (a \in l_2). \tag{5.3}$$

Hence, in particular,

$$|f_n(\omega)| \leqslant D \quad (n = 1, 2, \dots; \quad \omega \in \Omega) \quad \text{and} \quad \|f_n\|_2 \geqslant d. \tag{5.4}$$

Moreover, it follows from (5.3) that $f_n \neq 0$ $(n = 1, 2, \dots)$ and for all $k, m \in \mathbb{N}$, $k \leqslant m$, we have

$$\left\| \sum_{n=1}^{k} a_n f_n \right\|_2 \leqslant d^{-2} \left\| \sum_{n=1}^{m} a_n f_n \right\|_2.$$

Hence $\{f_n\}_{n=1}^{\infty}$ is a basis in the closed (in the L_2-norm) linear hull Z of this system ([26], part I, § 1). In other words, $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis sequence, so that for $g \in Z$ we see that $\mathsf{E}(gf_n) \to 0$ as $n \to \infty$ [2], § 1.1. If $g \in L_2$ is arbitrary, then $g = g_1 + g_2$, where $g_1 \in Z$ and $g_2 \in Z^\perp$ ($Z^\perp$ is the orthogonal complement of Z) and $\mathsf{E}(gf_n) = \mathsf{E}(g_1 f_n) \to 0$ as $n \to \infty$. Thus,

$$f_n \to 0 \quad \text{weakly in } L_2 \quad (\text{as } n \to \infty). \tag{5.5}$$

Relations (5.4) and (5.5) show that the system $\{f_n\}_{n=1}^{\infty}$ has properties (1)–(3) in Theorem 5. Hence there exists a subsequence $\{\varphi_i\} \subset \{f_n\}$ equivalent in distribution to the Rademacher system on $[0,1]$. Hence, by Khintchine's inequality (see [3] or [6], § 4.5) we obtain

$$C_2^{-1}\|a\|_2 \leqslant \biggl\|\sum_{i=1}^{\infty} a_i\varphi_i\biggr\|_p \leqslant C_2\|a\|_2 \qquad (a=(a_i)_{i=1}^{\infty}\in l_2), \tag{5.6}$$

where the constant $C_2>0$ depends only on D, d, and $p\in[1,\infty)$.

At the same time, it follows from (5.2) that

$$C_1^{-1}\mathcal{K}_{1,2}(t,a) \leqslant \mathcal{K}\biggl(t,\sum_{i=1}^{\infty} a_i\varphi_i; L_\infty, L_2\biggr) \leqslant C_1\mathcal{K}_{1,2}(t,a) \qquad (a\in l_2,\quad t>0).$$

Hence, applying to the couples (L_∞, L_2) and (l_1,l_2) the interpolation functor $(\cdot,\cdot)_{\theta,p}$ with parameters $0<\theta<1$, $p=2/\theta$ (see § 1), in view of [11], Theorem 5.2.1 we obtain

$$\biggl\|\sum_{i=1}^{\infty} a_i\varphi_i\biggr\|_p \asymp \|a\|_{r,p} = \biggl\{\sum_{i=1}^{\infty}(a_i^*)^p i^{p/r-1}\biggr\}^{1/p},$$

where $r=2/(2-\theta)<2$ (the constant involved in the last equivalence depends only on C_1 and θ). Since $\|a\|_{r,p}\not\asymp\|a\|_2$, the last relation is in contradiction with inequalities (5.6) for $p=2/\theta$.

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Bibliography

[1] S. V. Astashkin, "Selection of subsystems 'majorized' by the Rademacher system", *Mat. Zametki* **65**:4 (1999), 483–495; English transl. in *Math. Notes* **65** (1999).

[2] V. F. Gaposhkin, "Lacunary series and independent functions", *Uspekhi Mat. Nauk* **21**:6 (1966), 3–82; English transl. in *Russian Math. Surveys* **21** (1966).

[3] A. Khintchine, "Über dyadische Brüche", *Math. Z.* **18** (1923), 109–116.

[4] A. Zygmund, "Sur les séries trigonométriques lacunaires", *J. London Math. Soc.* (2) **5**:2 (1930), 138–145.

[5] S. Banach, "Sur les séries lacunaires", *Bull. Int. Acad. Polon. Sci. A.* **1933**:4/8, 149–154.

[6] S. Kaczmarz and G. Steinhaus, *Theorie der Orthogonalreihen*, Subwencji Funduszu Kultury Narodowej, Warsaw 1935.

[7] A. Zygmund, "On lacunary trigonometric series", *Trans. Amer. Math. Soc.* **34** (1932), 435–446.

[8] S. Kaczmarz and G. Steinhaus, "Le système orthogonal de M. Rademacher", *Studia Math.* **2** (1930), 231–247.

[9] B. S. Kashin, "On certain properties of the space of trigonometric polynomials with uniform norm", *Trudy Mat. Inst. Steklov.* **145** (1980), 111–116; English transl. in *Proc. Steklov Inst. Math.* **145** (1981).

[10] B. S. Kashin and A. A. Saakyan, *Orthogonal series*, Nauka, Moscow 1984; English transl., Amer. Math. Soc., Providence, RI 1989.

[11] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin 1976.

[12] S. J. Montgomery-Smith, "The distribution of Rademacher sums", *Proc. Amer. Math. Soc.* **109** (1990), 517–522.

[13] T. Holmstedt, "Interpolation of quasi-normed spaces", *Math. Scand.* **26** (1970), 177–199.
[14] E. D. Gluskin and S. Kwapien, "Tail and moment estimates for sums of independent random variables with logarithmically concave tails", *Studia Math.* **114** (1995), 303–309.
[15] J.-P. Kahane, *Some random series of functions*, Heath, Lexington, MA 1968.
[16] J. Jakubowski and S. Kwapien, "On multiplicative systems of functions", *Bull. Acad. Pol. Sci. Ser. Sci. Math.* **27** (1979), 689–694.
[17] G. Pisier, "Les inégalités de Khintchin–Kahane d'après C. Borell", *Sémin. Géom. des Espaces de Banach*, vol. 7, 1978, pp. 1–14.
[18] N. H. Asmar and S. Montgomery-Smith, "On the distribution of Sidon series", *Ark. Mat.* **31**:1 (1993), 13–26.
[19] A. S. Belov and V. A. Rodin, "Norms of lacunary polynomials in function spaces", *Mat. Zametki* **51** (1992), 483–495; English transl. in *Math. Notes* **51** (1992).
[20] V. A. Rodin and E. M. Semyonov [Semenov], "Rademacher series in symmetric spaces", *Anal. Math.* **1**:3 (1975), 207–222.
[21] S. G. Kreĭn, Yu. I. Petunin, and E. M. Semenov, *Interpolation of linear operators*, Nauka, Moscow 1978; English transl., Amer. Math. Soc., Providence, RI 1982.
[22] G. Alexits, *Convergence problems of orthogonal series*, Pergamon Press, New York 1961.
[23] S. B. Stechkin, "Absolute convergence of Fourier series", *Izv. Akad. Nauk SSSR Ser. Mat.* **20** (1956), 385–412. (Russian)
[24] S. V. Astashkin, "Interpolation of subspaces of symmetric spaces spanned by the Rademacher system", *Izv. Ross. Akad. Estestv. Nauk Ser. Mat. Mat. Model. Inform. Upr.* **1**:1 (1997), 18–35.
[25] S. V. Astashkin, "On series with respect to the Rademacher system in rearrangement invariant spaces 'close' to L_∞", *Funktsional. Anal. i Prilozhen.* **32**:3, 62–65; English transl. in *Functional Anal. Appl.* **32** (1998).
[26] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, Berlin 1977.

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