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Multiple Rademacher Series in Rearrangement Invariant Spaces

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Let

$$r_k(t) = \operatorname{sign} \sin 2^{k-1}\pi t \qquad (k = 1, 2, \dots)$$

be the system of Rademacher functions on the interval $I = [0, 1]$. We shall consider the set of functions $x(t)$ representable in the form

$$x(t) = \sum_{i \neq j} a_{i,j} r_i(t) r_j(t) \qquad (t \in I). \tag{1}$$

Just as in the case of the usual Rademacher series (e.g., see [1, pp. 148–150 of the Russian translation]), the series (1) converges almost everywhere on I (i.e., there exists a limit of its rectangular partial sums) if and only if the sequence of coefficients $a = (a_{i,j})_{i \neq j}$ belongs to the space l_2.

The system $\{r_i r_j\}_{i \neq j}$ is orthonormal on I but the functions forming this system are not independent, in contrast with the usual Rademacher system. Nevertheless, the properties of this system in many respects are similar to those of systems of independent uniformly bounded functions. For example, the condition $\sum_{i \neq j} a_{i,j}^2 < \infty$ implies the integrability of the function $\exp(\alpha |x(t)|)$ for any $\alpha > 0$ [2, p. 105]. At the same time, there are also essential differences there. For example, the multiple system $\{r_i r_j\}_{i \neq j}$ is not a Sidon system [3]. To carry out a more comprehensive study of its properties (in particular, of the integrability of the series (1)), we shall consider general rearrangement invariant spaces.

Let us recall that a Banach space X of Lebesgue measurable functions $x = x(t)$ on I is called a rearrangement invariant space if the relations $x^*(t) \leqslant y^*(t)$ $(t \in [0, 1])$ and $y \in X$ imply the relations $x \in X$ and $\|x\|_X \leqslant \|y\|_X$ (here $z^*(t)$ is the nonincreasing rearrangement of a function $|z(u)|$ [4, p. 83]). Important examples of rearrangement invariant spaces are given by L_p-spaces $(1 \leqslant p \leqslant \infty)$ as well as Orlicz and Marcinkiewicz spaces.

If $S(t)$ is a nonnegative convex continuous function on $[0, \infty)$ such that $S(0) = 0$, then the Orlicz space L_S consists of all measurable functions $x = x(t)$ such that

$$\|x\|_S = \inf \left\{ u > 0 : \int_0^1 S\left(\frac{|x(t)|}{u} \right) dt \leqslant 1 \right\} < \infty.$$

If $\varphi(t)$ is a nonnegative increasing concave function on $(0, 1]$, then the Marcinkiewicz space $M(\varphi)$ consists of all measurable functions $x = x(t)$ such that

$$\|x\|_{M(\varphi)} = \sup \left\{ \frac{1}{\varphi(t)} \int_0^t x^*(s)\, ds : 0 < t \leqslant 1 \right\} < \infty.$$

1. The integrability of multiple Rademacher series. Let H be the closure of the space L_∞ in the Orlicz space L_M, where $M(t) = e^t - 1$.

Theorem 1. *The system $\{r_i r_j\}_{i \neq j}$ in a rearrangement invariant space X is equivalent to the canonical basis of l_2, i.e.,*

$$\left\| \sum_{i \neq j} a_{i,j} r_i r_j \right\|_X \asymp \left(\sum_{i \neq j} a_{i,j}^2 \right)^{1/2} \tag{1}$$

(there exists a two-sided estimate with constants that depend only on X) if and only if $X \supset H$.

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Remark 1. Theorem 1 shows that the assertion on the exponential integrability of multiple Rademacher series in the beginning of the paper is sharp.

Remark 2. For the usual Rademacher system, a similar result holds with H replaced by the closure G of the space L_∞ in the Orlicz space L_N, where $N(t) = e^{t^2} - 1$ [5].

2. The complementability of the subspace generated by the multiple Rademacher system. Recall that the norm of a rearrangement invariant space X is said to be *order semicontinuous* if the monotone convergence $x_n(t) \uparrow x(t)$ as $n \to \infty$ almost everywhere on I ($x_n \geqslant 0$, $x \in X$) implies the relation $\|x_n\|_X \to \|x\|_X$.

Let $\mathscr{R}(X)$ be the subspace of a rearrangement invariant space X consisting of all functions of the form (1) such that $x \in X$.

Theorem 2. *Let X be a rearrangement invariant space with order semicontinuous norm. The subspace $\mathscr{R}(X)$ is complemented in X if and only if $H \subset X \subset H^*$.*

Remark 3. It follows from the theory of Orlicz spaces [6] that $H^* = L_{\widetilde{M}}$, where $\widetilde{M}(u)u^{-1}\ln^{-1} u \to 1$ as $u \to +\infty$.

Remark 4. It is well known that the subspace generated by the usual Rademacher system is complemented in a rearrangement invariant space X with order semicontinuous norm if and only if $G \subset X \subset G^*$, where $G^* = L_{\widetilde{N}}$, $\widetilde{N}(t)u^{-1}\ln^{-1/2} u \to 1$ as $u \to +\infty$ (see [7] or [8, 2.b.4]).

3. The multiple Rademacher system in rearrangement invariant spaces "close" to L_∞. The behavior of the multiple Rademacher system is far more complicated in the space L_∞ and spaces "close" to it.

For arbitrary $n \in \mathbb{N}$ and $\theta = \{\theta_{i,j}\}_{i,j=1}^n$, $\theta_{i,j} = \pm 1$, we set

$$\varphi_n(\theta) = \left\| \sum_{i,j=1}^n \theta_{i,j} r_i r_j \right\|_\infty,$$

where $\|\cdot\|_\infty$ is the norm on L_∞.

It follows from the definition of Rademacher functions that $\sup_\theta \varphi_n(\theta)$ is attained at $\theta_{i,j} = 1$ ($i, j = 1, \dots, n$) and is equal to n^2.

Theorem 3. *The following estimates hold with constants independent of $n \in \mathbb{N}$:*

$$\inf_\theta \varphi_n(\theta) \asymp 2^{-n^2} \sum_\theta \varphi_n(\theta) \asymp n^{3/2}.$$

Remark 5. In particular, it follows from Theorem 3 that the system $\{r_i r_j\}_{i \neq j}$ is not a Sidon system on the interval I. Earlier this statement was proved in [3].

A basis sequence $\{x_n\}_{n=1}^\infty$ in a Banach space X is said to be *unconditional* if the convergence of a series $\sum_{n=1}^\infty a_n x_n$ ($a_n \in \mathbb{R}$) in X implies the convergence in X of the series $\sum_{n=1}^\infty \theta_n a_n x_n$ for arbitrary signs $\theta_n = \pm 1$ ($n = 1, 2, \dots$).

In view of Theorem 1, the multiple Rademacher system is not only unconditional but also symmetric in a rearrangement invariant space X if $X \supset H$. The situation is completely different for spaces "close" to L_∞. We suggest an approach that permits one to understand to what extent the multiple system is not unconditional in these spaces.

For any signs $\theta = \{\theta_{i,j}\}_{i,j=1}^\infty$, $\theta_{i,j} = \pm 1$, on $\mathscr{R}(L_\infty)$, we introduce the operator

$$T_\theta x(t) = \sum_{i \neq j} \theta_{i,j} a_{i,j} r_i(t) r_j(t),$$

where

$$x(t) = \sum_{i \neq j} a_{i,j} r_i(t) r_j(t) \in L_\infty, \qquad a = (a_{i,j})_{i \neq j} \in l_2.$$

Theorem 4. *For arbitrary $0 < \varepsilon < 1/2$, there exists a choice of signs $\theta = \{\theta_{i,j}\}$ such that*

$$T_\theta : \mathscr{R}(L_\infty) \not\to M(\varphi_\varepsilon),$$

where $M(\varphi_\varepsilon)$ is the Marcinkiewicz space with $\varphi_\varepsilon(t) = t\log_2^{1/2-\varepsilon}(2/t)$.

Using this theorem and the results of [5], we obtain the following statement.

Corollary. *If $p \in [1,2)$, then there is no rearrangement invariant space X for which*

$$\Big\| \sum_{i\neq j} a_{i,j} r_i r_j \Big\|_X \asymp \Big(\sum_{i\neq j} |a_{i,j}|^p \Big)^{1/p}.$$

Remark 6. The situation for the usual Rademacher system is completely different [5, 9]. For example, it was proved in [9] that for any sequence space E that is an interpolation space between l_1 and l_2, there exists a rearrangement invariant space X on I such that

$$\Big\| \sum_{j=1}^{\infty} c_j r_j \Big\|_X \asymp \|(c_j)\|_E.$$

In the proofs of Theorems 1–4, the main idea is to separate the variables. This means that, instead of the functions (1), we consider functions of the form

$$y(s,t) = \sum_{i\neq j} b_{i,j} r_i(s) r_j(t) \qquad (s \in I,\ t \in I). \tag{2}$$

Statements similar to Theorems 1–4 hold for functions of the form (2) and for the rearrangement invariant space $X(I \times I)$ on the square $I \times I$.

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