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BRIEF COMMUNICATIONS

On the Selection of Subsystems Equivalent in Distribution to the Rademacher System

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Introduction

We say that systems $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ of random variables defined on probability spaces $(\Omega, \Sigma, \mathbb{P})$ and $(\Omega', \Sigma', \mathbb{P}')$, respectively, are *equivalent in distribution* (denoted $\{f_n\} \overset{\mathbb{P}}{\sim} \{g_n\}$) if there exists a $C > 0$ such that for arbitrary $m \in \mathbb{N}$, $a_n \in \mathbb{R}$ $(n = 1, 2, \dots, m)$, and $z > 0$ we have

$$C^{-1}\mathbb{P}\left\{\left|\sum_{n=1}^m a_n f_n(\omega)\right| > Cz\right\} \le \mathbb{P}'\left\{\left|\sum_{n=1}^m a_n g_n(\omega')\right| > z\right\} \le C\mathbb{P}\left\{\left|\sum_{n=1}^m a_n f_n(\omega)\right| > C^{-1}z\right\}.$$

If only the second inequality holds, then we say that the system $\{g_n\}_{n=1}^\infty$ is *majorized in distribution* by the system $\{f_n\}_{n=1}^\infty$. For the case of finite systems $\{f_n\}_{n=1}^N$ and $\{g_n\}_{n=1}^N$ these notions are defined in a similar way.

In [1] it is shown that in any uniformly bounded sequence $\{f_n\}_{n=1}^\infty$ of random variables (r.v.'s) one can select a subsequence $\{f_{n_k}\}_{k=1}^\infty$ that is majorized in distribution by the Rademacher system $\{r_n\}_{n=1}^\infty$, $r_n(x) = \operatorname{sign}\sin(2^{n-1}\pi x)$ $(x \in [0, 1])$.

In this note we present some assertions concerning the possibility of choosing a subsequence equivalent to the system $\{r_n\}_{n=1}^\infty$ under the same conditions. Moreover, we find necessary and sufficient conditions for a system of r.v.'s to contain a subsystem equivalent in distribution to the Rademacher system. In the case of finite systems, we are interested in the density of their subsystems with a similar property.

The problems concerning the selection of "nearly independent" subsystems (i.e., possessing some property that is characteristic for systems of independent functions) was studied in detail in the well-known paper of V. F. Gaposhkin [2]. Let us recall the most important results concerning integrability and absolute convergence of the sum of a series constructed from a given system of functions.

By the classical inequality proved by A. Ya. Khinchin [3], the sum of the series $\sum_{n=1}^\infty a_n r_n(x)$, where $(x \in [0, 1])$, belongs to all spaces L_p $(p > 2)$ if the sequence of coefficients $a = (a_n)_{n=1}^\infty$ belongs to l_2. In the case of lacunary trigonometric series, a similar result was obtained by A. Zygmund [4]. In this connection, somewhat later the notion of S_p-system appeared in the works of S. Banach and S. Sidon. A sequence of r.v.'s $\{f_n\}_{n=1}^\infty$, $f_n \in L_p$ $(p > 2)$, is called a *S_p-system* if we have

$$\left\|\sum_{n=1}^m a_n f_n\right\|_p \le K_p \left\|\sum_{n=1}^m a_n f_n\right\|_2,$$

where the constant $K_p > 0$ is independent of $m \in \mathbb{N}$ and $a_n \in \mathbb{R}$ $(n = 1, \dots, m)$. If $\{f_n\}$ is a S_p-system for any $p > 2$, then it is called a *S_∞-system.* A classical result on selecting S_p-subsystems belongs to S. Banach ([5] or [6, 7.2]). It implies, in particular, that any uniformly bounded orthonormal

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system of functions contains an S_∞-subsystem. A more precise result in this direction was proved by S. B. Stechkin [2, Theorem 1.3.1], i.e., a sequence of functions $\{f_n\}_{n=1}^\infty$ on $[0,1]$ contains an S_p-subsystem $(p>2)$ if and only if there exists a subsequence $\{f_{n_k}\}$ satisfying the following conditions:

1) $\|f_{n_k}\|_p \le D \quad (k=1,2,\dots)$;
2) $f_{n_k} \to 0$ weakly in L_2.

Another important property of the lacunary series is absolute convergence. For trigonometric series, the classical Sidon theorem on absolute convergence of a lacunary (in the sense of Hadamard) Fourier series of a bounded function is well known. A. Zygmund proved a local version of this theorem [7]. More recently, similar results were established for the Rademacher system and some other specific lacunary systems of functions [8]. As a natural generalization, the following notion appeared: a sequence of r.v.'s $\{f_n\}_{n=1}^\infty$ is called a *Sidon system* if

$$\sum_{n=1}^m |a_n| \le C\Big\|\sum_{n=1}^m a_n f_n\Big\|_\infty,$$

where the constant $C>0$ is independent of $m\in\mathbb{N}$ and $a_n\in\mathbb{R}\quad(n=1,2,\dots,m)$.

In [2, Theorem 1.4.1] V. F. Gaposhkin obtained broad sufficient conditions for the possibility of selecting Sidon subsystems. Suppose that a system $\{f_n\}_{n=1}^\infty$ of measurable functions on $[0,1]$ possesses the following properties:

1) $\|f_n\|_2 = 1 \quad (n=1,2,\dots)$;
2) $|f_n(x)| \le D \quad (n=1,2,\dots;\ x\in[0,1])$;
3) there exists a subsequence $\{f_{n_k}\}\subset\{f_n\}$ such that $f_{n_k}\to 0$ weakly in L_2; then $\{f_n\}_{n=1}^\infty$ contains a Sidon subsystem.

The above definitions and the properties of the Rademacher system imply that if $\{f_n\} \overset{\mathbb{P}}{\sim} \{r_n\}$, then the following assertions hold:

1) $\{f_n\}$ is an S_∞-system, where the constant of p-lacunarity has the same growth rate with respect to p as in the case of the Rademacher system, i.e., $K_p \asymp \sqrt{p}$;
2) $\{f_n\}$ is a Sidon system.

In this connection, the assertions stated in §2 ensure "good" properties of the selected subsystem in both senses, refining and strengthening the results mentioned above.

Along with the question of selecting infinite subsequences, other problems, i.e., concerning finite systems of r.v.'s, are of considerable interest. Suppose that $\{f_n\}_{n=1}^N$ is a family of r.v.'s. We seek the maximal number $s=s(N)$ with the following property: there exists a family $\{f_{n_i}\}_{i=1}^s \subset \{f_n\}_{n=1}^N$ satisfying some gap condition with the constant independent of N. Here we only mention a result on the density of finite Sidon subsystems proved by B. S. Kashin (see [9] or [10, Theorem 8.9]), i.e., if $\{f_n\}_{n=1}^N$ is an orthonormal family of functions defined on $[0,1]$, $|f_n(x)|\le D\quad(n=1,\dots,N;\ x\in[0,1])$, then there is a subsystem $\{f_{n_i}\}_{i=1}^s\quad(1\le n_1<\dots<n_s\le N)$ such that $s\ge\max\{[1/6\log_2 N],1\}$ and for any $a_i\in\mathbb{R}$ $(i=1,\dots,s)$ we have

$$D^{-1}\Big\|\sum_{i=1}^s a_i f_{n_i}\Big\|_\infty \le \sum_{i=1}^s |a_i| \le 4D\Big\|\sum_{i=1}^s a_i f_{n_i}\Big\|_\infty.$$

In §3 we state a similar assertion on selecting a subsystem $\{f_{n_i}\}_{i=1}^s$ also of "logarithmic" density, but satisfying a stronger condition, i.e., equivalent to the system $\{r_i\}_{i=1}^s$ with the constant depending only on D.

In what follows, by writing $F_1\asymp F_2$ we mean that $C^{-1}F_1\le F_2\le CF_1$ for some $C>0$, where the constant C is, as a rule, independent of all or part of the arguments of F_1 and F_2. If $1\le p\le\infty$, $f=f(\omega)$ is a r.v. on the probability space $(\Omega,\Sigma,\mathbb{P})$, and $a=(a_n)_{n=1}^\infty$ is a number sequence, then the notations $\mathbb{E}(f)$, $\|f\|_p$, $\|a\|_p$ (respectively L_p, l_p) are understood in the standard way, $|E|$ is the Lebesgue measure of the set $E\subset[0,1]$.

§1. Systems of r.v.'s that are equivalent in distribution to the Rademacher system

Here we state a criterion for an arbitrary system of r.v.'s to be equivalent in distribution to the Rademacher system, and also present its consequences.

In what follows, an important part is played by the Peetre $\mathcal{K}$-functional from the interpolation theory of operators [11]. If (X_0, X_1) is a Banach pair, $x \in X_0 + X_1$, $t > 0$, then by definition we have

$$\mathcal{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}.$$

We shall only need the functional $\mathcal{K}_{1,2}(t, a) = \mathcal{K}(t, a; l_1, l_2)$ constructed by the Banach pair (l_1, l_2).

In [12] it is proved that $\| \sum_{n=1}^{\infty} a_n r_n \|_t \asymp \mathcal{K}_{1,2}(\sqrt{t}, a)$ with the constant independent of $t \in [1, \infty)$, and $a = (a_n)_{n=1}^{\infty} \in l_2$ (implicitly this relation is already contained in [13]). The following assertion shows that a relation similar to the one above determines systems of r.v.'s equivalent in distribution to the Rademacher system.

Theorem 1. *Let $\{f_n\}_{n=1}^{\infty}$ be a system of r.v.'s defined on a probability space $(\Omega, \Sigma, \mathbb{P})$. The following conditions are equivalent:*

1) $\{f_n\} \overset{\mathbb{P}}{\sim} \{r_n\}$;

2) *the equivalence*

$$\left\|\sum_{n=1}^{m} a_n f_n\right\|_t \asymp \left\|\sum_{n=1}^{m} a_n r_n\right\|_t$$

holds with the constant independent of $t \in [1, \infty)$, $m \in \mathbb{N}$, and $a_n \in \mathbb{R}$ $(n = 1, \dots, m)$;

3) *the equivalence*

$$\left\|\sum_{n=1}^{\infty} a_n f_n\right\|_t \asymp \mathcal{K}_{1,2}(\sqrt{t}, a)$$

holds with the constant independent of $t \in [1, \infty)$ and $a = (a_n)_{n=1}^{\infty} \in l_2$.

Let us apply this theorem to two more specific cases.

Recall that a system of r.v.'s $\{f_n\}_{n=1}^{\infty}$ is called *multiplicative* if for any pairwise distinct $n_1, n_2, \dots, n_k$ $(k \in \mathbb{N})$ we have $\mathbb{E}(f_{n_1} f_{n_2} \cdots f_{n_k}) = 0$. If we additionally have $\mathbb{E}(f_{n_1} f_{n_2} \cdots f_{n_k} f_n^2) = 0$ for the same $n_1, n_2, \dots, n_k$ and $n \neq n_s$ $(s = 1, \dots, k)$, then the system $\{f_n\}$ is called *strongly multiplicative.*

An important example of a strongly multiplicative system is the sequence $\{f_n\}_{n=1}^{\infty}$ of independent r.v.'s such that $f_n \in L_2$ and $\mathbb{E}(f_n) = 0$ $(n = 1, 2, \dots)$.

From the results of [14] and Theorem 1 we obtain the following assertion:

Theorem 2. *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a strongly multiplicative system of r.v.'s such that $|f_n(\omega)| \leq D$ $(n = 1, 2, \dots ; \omega \in \Omega)$ and $d = \inf_{n=1,2,\dots} \mathbb{E}(f_n^2) > 0$. Then $\{f_n\} \overset{\mathbb{P}}{\sim} \{r_n\}$, where the constant of this equivalence depends only on D and d.*

Corollary 1. *Each sequence of independent r.v.'s $\{f_n\}_{n=1}^{\infty}$ such that $f_n \in L_2$, $\mathbb{E}(f_n) = 0$, $|f_n(\omega)| \leq D$ $(n = 1, 2, \dots ; \omega \in \Omega)$, and $d = \inf_{n=1,2,\dots} \mathbb{E}(f_n^2) > 0$ is equivalent in distribution to the Rademacher system.*

Now let us consider another situation. Suppose that G is a compact Abelian group, Γ is its group of characters, and μ is the Haar measure on G. In accordance with the definition above (see the introduction), the set $F \subset \Gamma$ is called a *Sidon set* if for some $C = C(F)$ we have $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| \leq C\|f\|_\infty$ for any $f \in C(G)$ such that $\hat{f}(\gamma) = \int_G f \bar{\gamma} d\mu = 0$ $(\gamma \notin F)$.

G. Pisier proved that for an arbitrary Sidon set $F = \{\gamma_n\} \subset \Gamma$ we have

$$\| \sum_{n=1}^{m} a_n \gamma_n \|_t \asymp \| \sum_{n=1}^{m} a_n r_n \|_t \qquad t \geq 1$$

(the equivalence constant depends only on the Sidon constant $C(F)$) [15]. Hence by Theorem 1 we immediately obtain the following assertion, which was proved in [16] in a different way (but also with the help of the theorem of G. Pisier):

Theorem 3. *Any infinite Sidon system $F = \{\gamma_n\}_{n=1}^{\infty}$ of characters defined on a compact Abelian group is equivalent in distribution to the Rademacher system. Here the equivalence constant depends only on the Sidon constant $C(F)$.*

Corollary 2. *The sequences $f_n(x) = \sin(2\pi k_n x)$ and $g_n(x) = \cos(2\pi k_n x)$, where $x \in [0,1]$ and $k_{n+1}/k_n \geq \lambda > 1$, are equivalent in distribution to the Rademacher system.*

Remark 1. Note that it was the analogy in behavior between the Rademacher series and lacunary trigonometric series (see the introduction) that served as the starting point in discovering their deeper relationship and led to studying systems equivalent in distribution to the "model" Rademacher system.

§2. Selecting subsystems equivalent in distribution to the Rademacher system

The main role in the proof of the possibility of selecting subsystems equivalent in distribution to the Rademacher system is played by the following assertion:

Theorem 4. *Let a system of r.v.'s $\{f_n\}_{n=1}^{\infty}$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ contain a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ satisfying the following conditions:*
1) $|f_{n_k}(\omega)| \leq D \quad (k = 1, 2, \dots; \omega \in \Omega)$;
2) $d = \inf_{k=1,2,\dots} \|f_{n_k}\|_2 > 0$;
3) $f_{n_k} \to 0$ *weakly in* L_2.

Then there is a subsystem $\{\varphi_i\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that

$$\left\|\sum_{i=1}^{\infty} a_i \varphi_i\right\|_t \asymp \mathcal{K}_{1,2}(\sqrt{t}, a) \qquad (a = (a_i)_{i=1}^{\infty} \in l_2, \quad t \geq 1)$$

with the constant depending only on D and d.

Remark 2. An essential part of the proof consists in finding upper bounds for L_q-norms ($q > 1$) of the modified Riesz products (in selecting a Sidon system, it suffices to estimate the L_1-norm [2, Theorem 1.4.1]). The latter have a block structure, which follows by application of an approximation formula for the $\mathcal{K}$-functional $\mathcal{K}_{1,2}(t, a)$ due to S. Montgomery-Smith [13].

The following two assertions are direct consequences of Theorems 1 and 4.

Theorem 5. *A system $\{f_n\}_{n=1}^{\infty}$ of r.v.'s defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ contains a subsystem $\{\varphi_i\}_{i=1}^{\infty}$ equivalent in distribution to the Rademacher system on $[0,1]$ if and only if there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ satisfying conditions 1)–3) of Theorem 4. Besides, the equivalence constant depends only on D and d.*

Theorem 6. *If $\{f_n\}_{n=1}^{\infty}$ is an orthonormal sequence of r.v.'s defined on a probability space $(\Omega, \Sigma, \mathbb{P})$, $|f_n(\omega)| \leq D$ $(n = 1, 2, \dots; \omega \in \Omega)$, then it is possible to select a subsequence $\{\varphi_i\}_{i=1}^{\infty}$ in it which is equivalent in distribution to the Rademacher system. The constant of this equivalence depends only on D.*

§3. Density of subsystems equivalent to the Rademacher system

Consider the problem of selecting subsystems in finite uniformly bounded orthonormal families $\{f_n\}_{n=1}^{N}$ of functions defined on the segment $[0,1]$. The result stated below means that any such family contains a subsystem $\{f_{n_i}\}_{i=1}^{s}$ of "logarithmic" density (i.e., $s \geq C \log_2 N$) equivalent in distribution to the system of the first s Rademacher functions, with the constant independent of s.

Theorem 7. *Let $\{f_n\}_{n=1}^{N}$ be an orthonormal family of functions defined on $[0,1]$, $|f_n(x)| \leq D$ $(n = 1, 2, \dots, N)$. Then there exists a family*

$$\{f_{n_i}\}_{i=1}^{s}, \qquad 1 \leq n_1 < n_2 < \dots < n_s \leq N, \quad s \geq \max\{[1/6 \log_2 N], 1\},$$

such that for some $C > 0$ depending only on D, for all $a_i \in \mathbb{R}$ $(i = 1, 2, \dots, s)$ and $z > 0$ we have

$$C^{-1}\left|\left\{\left|\sum_{i=1}^{s} a_i r_i(x)\right| > Cz\right\}\right| \leq \left|\left\{\left|\sum_{i=1}^{s} a_i f_{n_i}(x)\right| > z\right\}\right| \leq C\left|\left\{\left|\sum_{i=1}^{s} a_i r_i(x)\right| > C^{-1}z\right\}\right|.$$

Remark 3. The example of the trigonometric system shows that Theorem 7 (just as the theorem of B. S. Kashin mentioned in the introduction) is unimprovable with respect to order. Indeed, as was proved by S. B. Stechkin [18], if the sequence $\{\sqrt{2}\cos 2\pi n_k x\}_{k=1}^{\infty}$ $(x \in [0,1])$ is a Sidon system, then we have $\sum_{k\,:\,n_k<N} 1 \le C \ln N$ $(N = 2, 3, \dots)$.

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On Interpolation in the Classes E^p

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§1. We say that a simply connected Jordan domain G on the complex plane $\overline{\mathbb{C}}$ belongs to the class $\mathbf{A}$ (Ahlfors class) if its boundary $\gamma = \partial G$ is locally rectifiable and $\sup\{\operatorname{mes}_1(\gamma \cap d)/\operatorname{diam}(d)\} < \infty$, where the supremum is taken over all open disks $d \subset \mathbb{C}$ (see, for example, [1, 2]). Let $\lambda = \{z_j\}_{j=1}^{\infty}$ be a sequence of points in G. This sequence is called *interpolating* (*in the sense of Carleson*) if for any family of complex numbers $\{a_j\}_{j=1}^{\infty} \in \mathrm{l}^{\infty}$ with $\|a_j\|_{\mathrm{l}^\infty} = \sup\{|a_j| : j \in \mathbb{N}\} \le 1$ the interpolation problem $f(z_j) = a_j$ $(\forall j \in \mathbb{N})$ has solutions f in the Smirnov class $\mathbf{E}^{\infty}(G)$ (see [3–5]). By $w = \varphi_j(z)$ denote one-sheeted conformal mappings of G onto the disk $\{w : |w| < 1\}$ with the conditions $\varphi_j(z_j) = 0$ and $\varphi_j(v) \ge 0$ at a fixed point $v \in G$. Further, suppose that the product $\mathbf{B}_\lambda(z) := \prod_{j=1}^{\infty} \varphi_j(z)$ converges in G. For $j \in \mathbb{N}$ set $\mathbf{B}_{\lambda,j}(z) := \mathbf{B}_\lambda(z)/\varphi_j(z)$. It is known that the property $|\mathbf{B}_{\lambda,j}(z_j)| > \delta$ $(\forall j \in \mathbb{N})$ for a fixed $\delta = \delta(\lambda) > 0$ is characteristic for a sequence λ to be interpolating (Carleson's theorem, see [3, 4]).

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