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# Extraction of Subsystems "Majorized" by the Rademacher System

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ABSTRACT. In this paper it is proved that from any uniformly bounded orthonormal system $\{f_n\}_{n=1}^{\infty}$ of random variables defined on the probability space $(\Omega, \Sigma, \mathsf{P})$, one can extract a subsystem $\{f_{n_i}\}_{i=1}^{\infty}$ majorized in distribution by the Rademacher system on $[0,1]$. This means that

$$\mathsf{P}\left\{\omega\in\Omega:\left|\sum_{i=1}^{m}a_if_{n_i}(\omega)\right|>z\right\}\le C\left|\left\{t\in[01]:\left|\sum_{i=1}^{m}a_ir_i(t)\right|>\frac{z}{C}\right\}\right|,$$

where $C>0$ is independent of $m\in\mathbb{N}$, $a_i\in\mathbb{R}$ $(i=1,\dots,m)$ and $z>0$.


KEY WORDS: systems of random variables, probability space, Banach space, majorization by the Rademacher system, orthonormal system of random variables, Holmstedt's formula, equivalence principle for distribution functions.

## Introduction

Suppose that $r_i(t)=\operatorname{sign}\sin 2^{i-1}\pi t$ $(i=1,2,\dots)$ is the system of Rademacher functions on $[0,1]$. In what follows, we shall say that a system $\{f_i\}_{i=1}^{\infty}$ of random variables defined on the probability space $(\Omega,\Sigma,\mathsf{P})$ ***is majorized in distribution by the Rademacher system*** if there exists a constant $C>0$ such that for arbitrary $m\in\mathbb{N}$, $a_i\in\mathbb{R}$ $(i=1,\dots,m)$, and $z>0$ we have

$$\mathsf{P}\left\{\omega\in\Omega:\left|\sum_{i=1}^{m}a_if_i(\omega)\right|>z\right\}\le C\left|\left\{t\in[0,1]:\left|\sum_{i=1}^{m}a_ir_i(t)\right|>\frac{z}{C}\right\}\right|$$

(from now on, $|e|$ is the Lebesgue measure of the set $e\subset[0,1]$).

Suppose that $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded orthonormal system of random variables on the probability space $(\Omega,\Sigma,\mathsf{P})$. It is well known (see, for example, [1, p. 287 of the Russian translation]) that from such a system we can extract a subsystem $\{f_{n_i}\}_{i=1}^{\infty}$ that is an $S_p$-system simultaneously for all $p<\infty$. This means that

$$\left\|\sum_{i=1}^{m}a_if_{n_i}\right\|_p\le K_p\left(\sum_{i=1}^{m}a_i^2\right)^{1/2}$$

where $K_p>0$ is independent of $m\in\mathbb{N}$ and $a_i\in\mathbb{R}$ $(i=1,\dots,m)$.

The main result of this paper is a strengthening, in a certain sense, of the stated assertion. It will be proved that from any uniformly bounded orthonormal system of random variables one can extract a subsystem that is majorized in distribution by the Rademacher system.

To prove this result, we need some assertions that are of interest in themselves. First, let us recall a few definitions.

If $X$ is a random variable defined on the probability space $(\Omega,\Sigma,\mathsf{P})$, then $n_{|X|}(z)=\mathsf{P}\{\omega\in\Omega: |X(\omega)|>z\}$ and $X^*(t)$ $(t\in[0,1])$ is a nonincreasing left-continuous rearrangement of the function $|X|$. As is well known [2, p. 83], $|X|$ and $X^*$ are equimeasurable, i.e., for all $z>0$ we have $n_{|X|}(z)=n_{X^*}(z)$. The Banach space $E$ of random variables $X=X(\omega)$ on $(\Omega,\Sigma,\mathsf{P})$ is called *symmetric* if

1) from of the fact that $|X|\le|Y|$ almost everywhere and $Y\in E$, it follows that $X\in E$ and $\|X\|\le\|Y\|$;
2) from of the fact that $n_{|X|}(z)=n_{|Y|}(z)$ $(z>0)$ and $Y\in E$, it follows that $X\in E$ and $\|X\|=\|Y\|$.

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In addition to $L_p$-spaces ($1 \leq p \leq \infty$), some important examples of symmetric spaces are: the Orlicz space $L_S$

$$\|x\|_S = \inf\left\{u > 0 : \int_\Omega S\left(\frac{|X(\omega)|}{u}\right) dP(\omega) \leq 1\right\},$$

the Marcinkiewicz space $M(\varphi)$:

$$\|x\|_{M(\varphi)} = \sup\left\{\frac{1}{\varphi(t)} \int_0^t X^*(s)\, ds : 0 < t \leq 1\right\},$$

the Lorentz space $\Lambda_p(\varphi)$:

$$\|x\|_{\varphi,p} = \left\{\int_0^1 (X^*(s))^p\, d\varphi(s)\right\}^{1/p}.$$

Here $S(t) \geq 0$ is a convex continuous function on $[0,\infty)$ and $\varphi(t) \geq 0$ is a concave increasing function on $(0,1]$.

An important role in the theory of interpolation of operators is played by the Peetre $\mathcal{K}$-functional (see, for example, [3]):

$$\mathcal{K}(t, x; E_0, E_1) = \inf\{\|x_0\|_{E_0} + t\|x_1\|_{E_1} : x = x_0 + x_1,\ x_i \in E_i\},$$

where $E_0$ and $E_1$ are symmetric spaces, $x \in E_0 + E_1$, $t > 0$. It is readily shown that for a fixed $x \in E_0 + E_1$ the $\mathcal{K}$-functional is a concave increasing function of $t$ [3, p. 55 of the Russian translation].

In what follows, the expression $F_1 \asymp F_2$ indicates that for some $C > 0$ we have the inequality $C^{-1}F_2 \leq F_1 \leq CF_2$; moreover, as a rule, the constant $C$ is independent of all or part of the arguments of $F_1$ and $F_2$. If $1 \leq p < \infty$ and $X$ is a random variable on $(\Omega\Sigma, \mathsf{P})$, then

$$\|X\|_p = \left\{\int_\Omega |X(\omega)|^p\, dP(\omega)\right\}^{1/p};$$

but if $a = (a_i)_{i=1}^\infty$, then

$$\|a\|_p = \left\{\sum_{i=1}^\infty |a_i|^p\right\}^{1/p}.$$

## §1. The $\mathcal{K}$-functional on the Rademacher sums

In [4] (see also [5]) it was shown that for functions of the form

$$x(t) = \sum_{i=1}^\infty a_i r_i(t) \qquad a = (a_i)_{i=1}^\infty \in \ell_2, \tag{1}$$

the following relation is valid:

$$\mathcal{K}(t, x; L_\infty, G) \asymp \mathcal{K}(t, a; \ell_1, \ell_2),$$

where $G$ is the closure of $L_\infty$ in the Orlicz space $L_N$, $N(t) = e^{t^2} - 1$. We obtain a similar relation for the $\mathcal{K}$-functional $\mathcal{K}(t, x; L_1, L_\infty)$.

**Theorem 1.** *The following relation is valid:*

$$\mathcal{K}\Big(u, \sum a_i r_i; L_1, L_\infty\Big) \asymp u\mathcal{K}\left(\ln^{1/2}\frac{3}{u}, a; \ell_1, \ell_2\right), \tag{2}$$

*where the constants are independent of* $a = (a_i)_{i=1}^\infty$ *and* $0 < u \leq 1$.

**Proof.** For $a = (a_i)_{i=1}^{\infty} \in \ell_2$ we set $\varphi_a(t) = \mathcal{K}(t, a; \ell_1, \ell_2)$. In [6] it was shown that for functions of the form (1) the following inequalities hold:

$$n_{|x|}(\varphi_a(t)) \le 2e^{-t^2/2} \tag{3}$$

$$n_{|x|}(A^{-1}\varphi_a(t)) \ge A^{-1}e^{-At^2}, \tag{4}$$

where $A > 1$ is independent of $a$ and $t > 0$.

It follows from inequality (3) by the definition of the rearrangement $x^*(3e^{-t^2/2}) \le \varphi_a(t)$ or after the substitution $s = 3e^{-t^2/2}$ that

$$x^*(s) \le \sqrt{2}\varphi_a\Big(\ln^{1/2}\frac{3}{s}\Big), \qquad 0 < s \le 1.$$

Hence

$$\int_0^u x^*(s)\,ds \le \sqrt{2}\int_0^u \varphi_a\Big(\ln^{1/2}\frac{3}{s}\Big)\,ds, \qquad 0 < u \le 1.$$

On the other hand, inequality (4) yields

$$x^*(s) \ge A^{-1}\varphi_a(A^{-1/2}\ln^{1/2}(A^{-1}s^{-1})), \qquad 0 < s < A^{-1}.$$

Since the function $\varphi_a(u)$ is concave, we have

$$x^*\Big(\frac{t}{3A}\Big) \ge A^{-3/2}\varphi_a\Big(\ln^{1/2}\frac{3}{t}\Big), \qquad 0 < t < 1.$$

Therefore, for all $0 < u \le 1$

$$\int_0^u x^*(s)\,ds \ge \int_0^{u/(3A)} x^*(s)\,ds \ge 3^{-1}A^{-5/2}\int_0^u \varphi_a\Big(\ln^{1/2}\frac{3}{t}\Big)\,dt.$$

Thus we have

$$\int_0^u x^*(s)\,ds \asymp \int_0^u \varphi_a\Big(\ln^{1/2}\frac{3}{s}\Big)\,ds, \tag{5}$$

where the constant is independent of $a = (a_i)_{i=1}^{\infty}$ and $0 < u \le 1$,

Let us now prove that

$$\int_0^u \varphi_a\Big(\ln^{1/2}\frac{3}{s}\Big)\,ds \asymp u\varphi_a\Big(\ln^{1/2}\frac{3}{u}\Big). \tag{6}$$

First, suppose that $u = 3^{-k}$, $k \in \mathbb{N}$. Since $\varphi_a(t)$ is concave, we can write

$$\int_0^{3^{-k}} \varphi_a\Big(\ln^{1/2}\frac{3}{s}\Big)\,ds = \sum_{i=k}^{\infty}\int_{3^{-i-1}}^{3^{-i}} \varphi_a\Big(\ln^{1/2}\frac{3}{s}\Big)\,ds \asymp \sum_{i=k}^{\infty}\varphi_a(\sqrt{i\ln 3})3^{-i} \asymp \sum_{i=k}^{\infty}\varphi_a(\sqrt{i})3^{-i}. \tag{7}$$

At the same time, we have

$$\sum_{i=k}^{\infty}\varphi_a(\sqrt{i})3^{-i} = \sum_{j=0}^{\infty}\sum_{i=k3^j}^{k3^{j+1}-1}\varphi_a(\sqrt{i})3^{-i} \asymp \sum_{j=0}^{\infty}\varphi_a(\sqrt{k}3^{j/2})3^{-k3^j}.$$

Since the last sum is greater than $\varphi_a(\sqrt{k})3^{-k}$ and less than

$$\varphi_a(\sqrt{k})3^{-k}\sum_{j=0}^{\infty}3^{j/2+k-k3^j} \le \varphi_a(\sqrt{k})3^{-k}\sum_{j=0}^{\infty}3^{1-j/2} \le 6\sqrt{3}\varphi_a(\sqrt{k})3^{-k},$$

it follows that for $u = 3^{-k}$, $k \in \mathbb{N}$, relation (7) implies (6).

If $u \in (0, 1]$ is arbitrary, choose $k \in \mathbb{N}$ so that $3^{-k} \le u \le 3^{-k+1}$. Then, since the functions in question are concave, we have

$$\int_0^u \varphi_a\Big(\ln^{1/2}\frac{3}{s}\Big)\,ds \asymp \int_0^{3^{-k}} \varphi_a\Big(\ln^{1/2}\frac{3}{s}\Big)\,ds \asymp 3^{-k}\varphi_a(\ln^{1/2}3^{k+1}) \asymp u\varphi_a\Big(\ln^{1/2}\frac{3}{u}\Big),$$

and relation (6) is proved.

It follows from (5) and (6) that

$$\int_0^u x^*(s)\,ds \asymp u\varphi_a\Big(\ln^{1/2}\frac{3}{u}\Big).$$

In view of of the well-known relation (see, for example, [3, p. 142 of the Russian translation])

$$\mathcal{K}(u, x;\, L_1, L_\infty) = \int_0^u x^*(s)\,ds,$$

the last relation is equivalent to (2), and the theorem is proved. □

A sufficiently good approximation to the function $\varphi_a(t) = \mathcal{K}(t, a;\, \ell_1, \ell_2)$ is given by the well-known Holmstedt formula [7]

$$\mathcal{K}(t, a;\, \ell_1, \ell_2) \asymp \sum_{i=1}^{[t^2]} a_i^* + t\Big(\sum_{i=[t^2]+1}^{\infty}(a_i^*)^2\Big)^{1/2}, \tag{8}$$

where $(a_i^*)$ is a nonincreasing rearrangement of the sequence $(|a_i|)$, and $[z]$ is the integral part of the number $z$.

Thus we obtain the following result.

**Corollary 1.** *The following relation is valid:*

$$\int_0^u \Big(\sum_{i=1}^{\infty} a_i r_i(t)\Big)^* dt \asymp u\Big\{\sum_{i=1}^{[\ln 3/u]} a_i^* + \ln^{1/2}\frac{3}{u}\Big(\sum_{i=[\ln 3/u]+1}^{\infty}(a_i^*)^2\Big)^{1/2}\Big\}.$$

*where the constants are independent of* $a = (a_i)_{i=1}^{\infty} \in \ell_2$ *and* $0 < u \le 1$*,*

**Remark 1.** Since for arbitrary $E_0, E_1$ and $u > 0$ we have

$$\mathcal{K}(u, x;\, E_0, E_1) = u\mathcal{K}\Big(\frac{1}{u}, x;\, E_1, E_0\Big),$$

it follows from relation (2) that

$$\mathcal{K}(u, x;\, L_\infty, L_1) \asymp \varphi_a(\ln^{1/2}3u), \qquad u \ge 1,$$

or equivalently

$$\mathcal{K}(u, x;\, L_\infty, L_1) \asymp \varphi_a(\ln^{1/2}(1+u^2)), \qquad u \ge 1.$$

But if $0 < u < 1$, then it follows from the definition of the $\mathcal{K}$-functional, as well as from the inequalities $\|x\|_1 \le \|x\|_\infty$, $\|a\|_2 \le \|a\|_1$, and $\|x\|_1 \le \|a\|_2 \le \sqrt{2}\|x\|_1$ (Khinchine's inequality for the $L_1$-norm with a constant; see [8]) that

$$\mathcal{K}(u, x;\, L_\infty, L_1) \asymp u\|x\|_1 \asymp u\|a\|_2 \asymp \varphi_a(u) \asymp \varphi_a(\ln^{1/2}(1+u^2)).$$

As a result, we obtain the relation

$$\mathcal{K}(u, x;\, L_\infty, L_1) \asymp \mathcal{K}(\ln^{1/2}(1+u^2), a;\, \ell_1, \ell_2), \tag{9}$$

which is now valid for all $u > 0$.

**Remark 2.** In exactly the same way, for functions of the form (1) we can prove that

$$\mathcal{K}(ux\,;L_p,L_\infty)\asymp\mathcal{K}\Big(\ln^{1/2}\frac{3}{u},a\,;\ell_1,\ell_2\Big),\qquad 0<u\le 1, \tag{2$'$}$$

$$\mathcal{K}(u,x\,;L_\infty,L_p)\asymp\mathcal{K}(\ln^{1/2}(1+u^2),a\,;\ell_1,\ell_2),\qquad u>0, \tag{9$'$}$$

where $1\le p<\infty$ (naturally, the constants of these equivalences depend on $p$).

This implies the following result.

**Corollary 2.** *The following equivalence is valid:*

$$\mathcal{K}\Big(u,\sum a_i r_i\,;L_p,L_\infty\Big)\asymp\mathcal{K}\Big(u,\sum a_i r_i\,;L_1,L_\infty\Big),$$

*where the constants are independent of* $a=(a_i)_{i=1}^\infty$ *and* $0<u\le 1$.

## §2. On comparing distribution functions

In what follows, we shall need a version (proved in [9]) of the equivalence principle for distribution functions.

Its proof is exactly the same as that of the principle itself (see [9]).

**Proposition 1.** *Suppose that* $X\ge 0$ *and* $Y\ge 0$ *are two random variables defined, in general, on different probability spaces, and* $\{X_i\}_{i=1}^\infty$ *and* $\{Y_i\}_{i=1}^\infty$ *are independent "copies" of* $X$ *and* $Y$, *respectively, (i.e.,* $n_{X_i}(z)=n_X(z)$ *and* $n_{Y_i}(z)=n_Y(z)$*). We assume that*

$$\big\|\max_{i=1\dots,n}Y_i\big\|_1\le C_1\big\|\max_{i=1,\dots,n}X_i\big\|_1, \tag{10}$$

$$\big\|\max_{i=1,\dots,n}X_i\big\|_2\le C_2\big\|\max_{i=1,\dots,n}X_i\big\|_1, \tag{11}$$

*where* $C_1>0$ *and* $C_2>0$ *are independent of* $n\in\mathbb{N}$.
*Then there exists a* $C>0$ *depending only on* $C_1$ *and* $C_2$ *such that for all* $z>0$

$$n_Y(z)\le Cn_X\Big(\frac{z}{C}\Big).$$

Let us show that conditions (10) and (11) can be stated using the notion of a $\mathcal{K}$-functional.

**Proposition 2.** *We assume that* $\{X_i\}_{i=1}^\infty$ *is a sequence of independent "copies" of a random variable* $X\ge 0$ *on the probability space* $(\Omega,\Sigma,\mathsf{P})$ $1\le p<\infty$.
*Then for any* $n\in\mathbb{N}$

$$\big\|\max_{i=1,\dots,n}X_i\big\|_p\asymp n\mathcal{K}\Big(\frac{1}{n},X^*\,;L_p[0,1],L_\infty[0,1]\Big),$$

*where the constant is independent of* $X$ *and* $n$.

**Proof.** Since the $X_i$ $(i=1,2,\dots,n)$ are independent, for $\overline{X}=\max_{i=1,\dots,n}X_i$ we have

$$\begin{aligned}\|\overline{X}\|_p^p&=\int_\Omega\cdots\int_\Omega\max\{X^p(\omega_1),\dots,X^p(\omega_n)\}\,dP(\omega_1)\cdots dP(\omega_n)\\&=n\int\cdots\int_{\{X(\omega_i)\le X(\omega_1)\}}X^p(\omega_1)\,dP(\omega_n)\cdots dP(\omega_2)\,dP(\omega_1)\\&=n\int_\Omega X^p(\omega_1)(\mathsf{P}\{\omega:X(\omega)\le X(\omega_1)\})^{n-1}\,dP(\omega_1)=n\int_\Omega X^p(\omega_1)\big(1-n_X(X(\omega_1))\big)^{n-1}\,dP(\omega_1).\end{aligned}$$

Since the functions $X(\omega_1)$ $(\omega_1 \in \Omega)$ and $X^*(t)$ $(t \in [0,1])$ are equimeasurable, and since $n_X(X^*(t)) = t$ for almost all $t \in [0,1]$ [2, p. 83], we obtain

$$\|\overline{X}\|_p^p = n \int_0^1 (X^*)^p(t)\left(1 - n_X(X^*(t))\right)^{n-1} dt = n \int_0^1 (X^*)^p(t)(1-t)^{n-1}\, dt.$$

The last equality implies

$$\|\overline{X}\|_p = \|X^*\|_{\Lambda_p(\varphi_n)}, \tag{12}$$

where $\Lambda_p(\varphi_n)$ is the symmetric Lorentz space constructed for the function $\varphi_n(t) = 1 - (1-t)^n$.

Let us show that for any $n \in \mathbb{N}$ and $t \in [0,1]$ we have

$$e^{-1}\psi_n(t) \le \varphi_n(t) \le \psi_n(t), \tag{13}$$

where $\psi_n(t) = \min\{1, nt\}$.

Since $\varphi_1(t) = \psi_1(t) = 1$, we can assume that $n \ge 2$. If $t \ge 1/n$, then $\varphi_n(t) \le 1 = \psi_n(t)$, and $\varphi_n(t) \ge \varphi_n(1/n) = 1 - (1-1/n)^n \ge 1 - e^{-1} \ge \psi_n(t)/e$ (clearly, $(1-1/n)^n \uparrow 1/e$). In the case $0 < t \le 1/n$, consider the function

$$f_n(t) = \psi_n(t) - \varphi_n(t) = nt - 1 + (1-t)^n.$$

Since $f_n(0) = 0$ and $f_n'(t) = n - n(1-t)^{n-1} \ge 0$, we have $f_n(t) \ge 0$, which is equivalent to the right-hand side of inequality (13) in the case $0 \le t \le 1/n$. At the same time, for the function

$$g_n(t) = \frac{\psi_n(t)}{e} - \varphi_n(t) = \frac{nt}{e} - 1 + (1-t)^n$$

the following relations are valid:

$$g_n(0) = 0, \qquad g_n'(t) = \frac{n}{e} - n(1-t)^{n-1} = n\left(\frac{1}{e} - (1-t)^{n-1}\right) \le n\left(\frac{1}{e} - \left(1 - \frac{1}{n}\right)^{n-1}\right) \le 0$$

(clearly, $(1-1/n)^{n-1} \downarrow 1/e$). Therefore, $g_n(t) \le 0$, which is equivalent to the left-hand side (13) for $0 \le t \le 1/n$; the proof of this relation is thus complete.

It follows from (13) that for an arbitrary measurable function $x(t)$ on $[0,1]$

$$\frac{1}{e}\|x\|_{\Lambda(\psi_n)} \le \|x\|_{\Lambda(\varphi_n)} \le \|x\|_{\Lambda(\psi_n)}.$$

Hence, in view of (12) and the formula for the $\mathcal{K}$-functional [3, p. 142 of the Russian translation], we have

$$\|\overline{X}\|_p \asymp \|X^*\|_{\Lambda_p(\psi_n)} = n\left\{\int_0^{1/n} (X^*(t))^p\, dt\right\}^{1/p} \asymp n\mathcal{K}\left(\frac{1}{n}, X^*; L_p, L_\infty\right),$$

where the constant is independent of $X$ and $n$. □

The following theorem is a consequence of Propositions 1 and 2.

**Theorem 2.** *Suppose that $X \ge 0$ and $Y \ge 0$ are two random variables defined, in general, on different probability spaces. We assume that*

$$\int_0^t Y^*(s)\, ds \le C_1 \int_0^t X^*(s)\, ds, \qquad 0 < t \le 1, \tag{14}$$

$$\int_0^t (X^*(s))^2\, ds \le C_2\left(\int_0^t X^*(s)\, ds\right)^2, \qquad 0 < t \le 1. \tag{15}$$

*Then there exists a $C > 0$ depending only on $C_1$ and $C_2$ such that for all $z > 0$*

$$n_Y(z) \le C n_X\left(\frac{z}{C}\right).$$

**Corollary 3.** *Suppose that $E$ is a symmetric space on $[0,1]$, $x = x(t)$ and $y = y(t)$ are measurable (on $[0,1]$) functions for which conditions* (14) *and* (15) *are valid. Then if $x \in E$, then $y \in E$ and $\|y\|_E \le C\|x\|_E$, where $C > 0$ depends only on $C_1$ and $C_2$.*

**Proof.** Using Corollary 2, we find that for all $z > 0$

$$n_{|y|}(z) \le C' n_{|x|}\Big(\frac{z}{C'}\Big),$$

or equivalently $y^*(t) \le C'\sigma_{C'}x^*(t)$ $(t \in (0,1])$, where $\sigma_\tau x(t) = x(t/\tau)$ is the dilatation operator. Since $\sigma_\tau$ is bounded in any symmetric space and $\|\sigma_\tau\|_{E\to E} \le \max\{1,\tau\}$ [2, p. 133], it follows that $\|y\|_E \le C\|x\|_E$, where $C = C'\max\{1, C'\}$. □

**Remark 3.** It is easily verified that condition (15) in Corollary 3 is essential. Moreover, if $E$ is a symmetric space and not the interpolation space between the spaces $L_1$ and $L_\infty$, then we can always find a function $x \in E$ and a linear operator $T$ boundedly acting into $L_1$ and $L_\infty$ such that $y = Tx \notin E$ [2, pp. 130, 166].

We now apply Theorem 2 for the case in which

$$X(t) = \Big|\sum_{i=1}^m a_i r_i(t)\Big|, \quad Y(\omega) = \Big|\sum_{i=1}^m a_i f_i(\omega)\Big|, \qquad m \in \mathbb{N}, \quad a_i \in \mathbb{R},$$

where the $r_i(t)$ are Rademacher functions on $[0,1]$, and the $f_i(\omega)$ are random variables on some probability space $(\Omega, \Sigma, \mathsf{P})$.

**Theorem 3.** *Suppose that $\{f_i\}_{i=1}^\infty$ is an orthonormal system of random variables on the probability space $(\Omega,\Sigma,\mathsf{P})$, $|f_i(\omega)| \le M$, $\omega \in \Omega$, $i = 1,2,\dots$ .*

*The following conditions are equivalent:*

1) *for an arbitrary sequence $a = (a_i)_{i=1}^\infty \in \ell_2$, the function $f = \sum_{i=1}^\infty a_i f_i$ belongs to $L_N$, where $N(t) = e^{t^2} - 1$;*
2) *there exists a constant $C > 0$ independent of $m \in \mathbb{N}$, $a = (a_i)_{i=1}^m$, and $t \in [0,1]$ such that*

$$\int_0^t \Big(\sum_{i=1}^m a_i f_i\Big)^*(s)\,ds \le C_1 t \mathcal{K}\Big(\ln^{1/2}\frac{3}{t}, a; \ell_1, \ell_2\Big); \tag{16}$$

3) *the system $\{f_i\}_{i=1}^\infty$ is majorized in distribution by the Rademacher system.*

**Proof.** First, let us prove the implication 1) $\Longrightarrow$ 2). As is well known [10], the Orlicz space $L_N$ coincides with the Marcinkiewicz space $M(\varphi)$, where $\varphi(t) = t\log_2^{1/2}(2/t)$. Therefore, by the definition of the norm in $M(\varphi)$, we have

$$\int_0^t \Big(\sum_{i=1}^\infty a_i f_i\Big)^*(s)\,ds \le C_1 t \log_2^{1/2}\frac{2}{t}\,\|a\|_2, \tag{17}$$

where $C_1$ is independent of $a = (a_i)_{i=1}^\infty \in \ell_2$ and $t \in [0,1]$.

Suppose that $a_k^* = |a_{i_k}|$ $(k = 1,2,\dots)$ is a nonincreasing rearrangement of the sequence $(|a_i|)_{i=1}^\infty$. In view of inequality (17) and of the assumptions of the theorem, for an arbitrary $j \in \mathbb{N}$ we obtain

$$\begin{aligned}
\int_0^{2^{-j}} \Big(\sum_{i=1}^\infty a_i f_i\Big)^*(s)\,ds &\le \int_0^{2^{-j}} \Big(\sum_{k=1}^j a_{i_k} f_{i_k}\Big)^*(s)\,ds + \int_0^{2^{-j}} \Big(\sum_{k=j+1}^\infty a_{i_k} f_{i_k}\Big)^*(s)\,ds \\
&\le 2^{-j} M \sum_{k=1}^j |a_{i_k}| + C_1 2^{-j}\log_2^{1/2}(2^{j+1})\Big(\sum_{k=j+1}^\infty |a_{i_k}|^2\Big)^{1/2} \\
&\le 2^{-j}\max\{M, 2C_1\}\Big(\sum_{k=1}^j a_k^* + \sqrt{j}\Big(\sum_{k=j+1}^\infty (a_k^*)^2\Big)^{1/2}\Big) \\
&\le 2^{-j} C_2 \max\{M, 2C_1\}\mathcal{K}(\sqrt{j}, a; \ell_1, \ell_2),
\end{aligned}$$

where the last inequality follows from Holmstedt's formula (8).

Thus relation (16) is satisfied for $t = 2^{-j}$ ($j \in \mathbb{N}$). Since the functions in question are concave, this relation is also valid for all $t \in [0, 1]$.

The implication 2) $\Longrightarrow$ 3) follows from Theorem 1, Corollary 2, and Theorem 2.

Finally, we assume that condition 3) is satisfied. It is well known (see, for example, [11, p. 342 of the Russian translation]) that the function $g = \sum_{i=1}^{\infty} a_i r_i$ belongs to $L_N$ if $a = (a_i)_{i=1}^{\infty} \in \ell_2$. By assumption, the following inequality is valid for the function $f = \sum_{i=1}^{\infty} a_i f_i$:

$$n_{|f|}(z) \le C n_{|g|}\left(\frac{z}{C}\right).$$

Therefore, since the space $L_N$ is symmetric, we find that $f \in L_N$ as in the proof of Corollary 3, and the theorem is thereby proved. □

We need the following notation (for details, see [12, Chap. 1]). Suppose that $(\Omega, \Sigma, \mathsf{P})$ is a probability space, $\mathfrak{N}$ is a $\sigma$-subalgebra of the $\sigma$-algebra $\Sigma$. If $f\colon \Omega \to \mathbb{R}$ is a random variable, then $\mathsf{E}[f|\mathfrak{N}]$ is the conditional expectation of $f$ with respect to $\mathfrak{N}$. If $\mathfrak{N}$ is the $\sigma$-subalgebra generated by the random variables $g_1, g_2, \dots, g_n$, then the conditional expectation with respect to them will be denoted by $\mathsf{E}[f|g_1, \dots, g_n]$. In particular,

$$\mathsf{E}[f] = \int_{\Omega} f(\omega)\, dP(\omega).$$

Let us recall that the sequence $\{f_n\}_{n=1}^{\infty}$ of random variables on the probability space $(\Omega, \Sigma, \mathsf{P})$ is called *multiplicative* if for any $k \in \mathbb{N}$ and $n_1 < n_2 < \dots < n_k$

$$\mathsf{E}[f_{n_1} f_{n_2} \cdots f_{n_k}] = 0.$$

By Theorem 3 and Corollary 3 from [13], we obtain the following result.

**Corollary 4.** *If $\{f_i\}_{i=1}^{\infty}$ is a uniformly bounded multiplicative orthonormal system on $(\Omega, \Sigma, \mathsf{P})$, then it can be majorized in distribution by the Rademacher system.*

**Remark 4.** In [13] it was shown that under the conditions of Corollary 4 the vector $(f_1, f_2, \dots, f_n)$ has the same distribution as the conditional expectation $\mathsf{E}[(g_1, g_2, \dots, g_n)|\mathfrak{N}]$ of the vector $(g_1, g_2, \dots, g_n)$, which has the same distribution as the vector $(r_1, r_2, \dots, r_n)$. This does not imply the assertion of Corollary 4, since the operator of the conditional expectation can even be unbounded in a symmetric space if it is not an interpolation space between $L_1$ and $L_\infty$ (see, for example, [14, pp. 128–129]).

## §3. Isolation of the exponentially integrable subsystems

In the proof of the following assertion an idea from [15, Lemma A] is used.

**Lemma.** *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of random variables defined on the probability space $(\Omega, \Sigma, \mathsf{P})$, $|f_n(\omega)| \le M$, $\omega \in \Omega$, $n \in \mathbb{N}$, and $f_n \to 0$ weakly in $L_2(\Omega)$. Then there exist $\{f_{n_i}\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ and a sequence of random variables $\{g_k\}_{k=1}^{\infty}$ such that*

1) $|g_k(\omega)| \le 2M$, $\omega \in \Omega$, $k \in \mathbb{N}$;
2) $\mathsf{E}[g_k|g_1, \dots, g_{k-1}] = 0$ *almost everywhere on* $\Omega$;
3) $\|f_{n_k} - g_k\|_k \le 2^{-k}$, $k \in \mathbb{N}$.

**Proof.** Let us prove this assertion by induction. Since the $f_n$ tend to zero weakly in $L_2(\Omega)$, we can find an $n_1$ such that

$$|\mathsf{E}[f_{n_1}]| \le \frac{1}{6}. \tag{18}$$

There exists a finite-valued function $h_1$ for which $|h_1(\omega)| \le M$ and

$$\mathsf{E}[|f_{n_1} - h_1|] \le \frac{1}{6}. \tag{19}$$

We set $g_1 = h_1 - \mathsf{E}[h_1]$. Then $\mathsf{E}[g_1] = 0$ and $|g_1(\omega)| \le 2M$. Moreover, it follows from inequalities (18) and (19) that

$$\|f_{n_1} - g_1\|_1 \le \mathsf{E}[|f_{n_1} - h_1|] + \mathsf{E}[|h_1 - g_1|] = \mathsf{E}[|f_{n_1} - h_1|] + |\mathsf{E}[h_1]|$$
$$\le 2\mathsf{E}[|f_{n_1} - h_1|] + |\mathsf{E}[f_{n_1}]| \le \frac{1}{2}.$$

We assume that $k > 1$ and that the obtained values $f_{n_1}, \dots, f_{n_{k-1}}$ $1 \le n_1 < \cdots < n_{k-1}$ and $g_1, \dots, g_{k-1}$ are such that for all $s = 1, \dots, k-1$

a) $|g_s(\omega)| \le 2M$;

b) $g_s$ is a constant on the sets $e_i^{(s)} \subset \Omega$, $i = 1, \dots, i_s$, $e_i^{(s)} \bigcap e_j^{(s)} = \varnothing$ $(i \ne j)$ and $\bigcup_i e_i^{(s)} = \Omega$; moreover, $e_i^{(s)}$ lie entirely in the sets of constancy $e_i^{(s-1)}$, $i = 1, \dots, i_{s-1}$, of the function $g_{s-1}$;

c) $\mathsf{E}[g_s | g_1, \dots, g_{s-1}] = 0$ almost everywhere on $\Omega$;

d) $\|f_{n_s} - g_s\|_s \le 2^{-s}$.

Then we obtain the next pair of values $f_{n_k}$ and $g_k$ satisfying the same conditions. Since $f_n \to 0$ weakly in $L_2(\Omega)$, there exists an $n_k > n_{k-1}$ such that for all $i = 1, \dots, i_{k-1}$ we have

$$\left| \int_{e_i^{(k-1)}} f_{n_k}(\omega)\, dP(\omega) \right| \le \frac{2^{-k}}{3} \mathsf{P}\big(e_i^{(k-1)}\big). \tag{20}$$

Suppose that $h_k$ is a finite-valued function, $|h_k(\omega)| \le M$, whose sets of constancy are all contained entirely in the sets $e_i^{(k-1)}$ $(i = 1, \dots, i_{k-1})$, and moreover for all $i = 1, \dots, i_{k-1}$ we can write

$$\int_{e_i^{(k-1)}} |f_{n_k}(\omega) - h_k(\omega)|^k \, dP(\omega) \le \left(\frac{2^{-k}}{3}\right)^k \mathsf{P}\big(e_i^{(k-1)}\big). \tag{21}$$

Set $g_k = h_k - \mathsf{E}[h_k | g_1, \dots, g_{k-1}]$. Then $\mathsf{E}[g_k | g_1, \dots, g_{k-1}] = 0$ almost everywhere on $\Omega$ and we have $|g_k(\omega)| \le 2M$. Since for $\omega \in e_i^{(k-1)}$ and an arbitrary random variable $f$ on $\Omega$ we can write

$$\mathsf{E}[f | g_1, \dots, g_{k-1}](\omega) = \mathsf{P}\big(e_i^{(k-1)}\big)^{-1} \int_{e_i^{(k-1)}} f(u)\, dP(u),$$

in view of (21), it follows that

$$|\mathsf{E}[f_{n_k} - h_k | g_1, \dots, g_{k-1}]| \le \max_{i=1,\dots,i_{k-1}} \mathsf{P}\big(e_i^{(k-1)}\big)^{-1} \int_{e_i^{(k-1)}} |f_{n_k}(u) - h_k(u)|\, dP(u)$$
$$\le \max_{i=1,\dots,i_{k-1}} \mathsf{P}\big(e_i^{(k-1)}\big)^{-1/k} \left( \int_{e_i^{(k-1)}} |f_{n_k}(u) - h_k(u)|^k \, dP(u) \right)^{1/k} \le \frac{2^{-k}}{3}.$$

Therefore, using (20), we obtain

$$|\mathsf{E}[h_k | g_1, \dots, g_{k-1}]| \le |\mathsf{E}[f_{n_k} - h_k | g_1, \dots, g_{k-1}]| + |\mathsf{E}[f_{n_k} | g_1, \dots, g_{k-1}]| \le \frac{2^{-k+1}}{3}.$$

Combining this with inequalities (21), we find that

$$\|f_{n_k} - g_k\|_k \le \|f_{n_k} - h_k\|_k + \|h_k - g_k\|_k \le \frac{2^{-k}}{3} + \|\mathsf{E}[h_k | g_1, \dots, g_{k-1}]\|_k \le 2^{-k},$$

and the proof of the lemma is complete. □

The following assertion was stated without proof by V. F. Gaposhkin [16, Theorem 1.3.2]. The proof was communicated to me by the author and is given here with his permission.

**Theorem 4.** *Suppose that $\{f_n\}_{n=1}^{\infty}$ is an orthonormal sequence of random variables on the probability space $(\Omega, \Sigma, \mathsf{P})$, $|f_n(\omega)| \le M$, $\omega \in \Omega$, $n \in \mathbb{N}$. There exists a subsequence $\{f_{n_i}\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that for any sequence $a = (a_i)_{i=1}^{\infty} \in \ell_2$ the function $f = \sum_{i=1}^{\infty} a_i f_{n_i}$ belongs to $L_N$, where $L_N$ is the Orlicz space constructed for the function $N(t) = e^{t^2} - 1$.*

**Proof.** Since $f_n \to 0$ weakly in $L_2(\Omega)$, by the lemma there exist $\{f_{n_i}\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ such that $|g_k(\omega)| \le 2M$, $\mathsf{E}[g_k | g_1, \dots, g_{k-1}] = 0$ almost everywhere and

$$\|g_k - f_{n_k}\|_k \le 2^{-k}, \qquad k \in \mathbb{N}. \tag{22}$$

Note that the sequence $\{g_k\}$ is multiplicative. In fact, since for arbitrary $k_1 < \cdots < k_m$ (see [12, p. 13 of the Russian translation])

$$\mathsf{E}[g_{k_m} g_{k_1} \cdots g_{k_{m-1}} | g_{k_1}, \dots, g_{k_{m-1}}] = g_{k_1} \cdots g_{k_{m-1}} \mathsf{E}[g_{k_m} | g_{k_1}, \dots, g_{k_{m-1}}] = 0,$$

it follows that

$$\mathsf{E}[g_{k_1} \cdots g_{k_{m-1}} g_{k_m}] = \int_{\Omega} \mathsf{E}[g_{k_m} g_{k_1} \cdots g_{k_{m-1}} | g_{k_1} \cdots g_{k_{m-1}}] \, dP(\omega) = 0.$$

In view of [13, Corollary 3], for multiplicative systems we have

$$\left\| \sum_{k=1}^{\infty} b_k g_k \right\|_p \le C\sqrt{p} \|(b_k)\|_2,$$

where $C = C(M) > 0$ is independent of $(b_k) \in \ell_2$ and $p \in (1, \infty)$. Therefore, using (22), for $f = \sum_{i=1}^{\infty} a_i f_{n_i}$ and arbitrary $k \in \mathbb{N}$ we obtain

$$\begin{aligned} \|f\|_k &\le \left\| \sum_{i=1}^{k} a_i f_{n_i} \right\|_k + \left\| \sum_{i=k+1}^{\infty} a_i (f_{n_i} - g_i) \right\|_k + \left\| \sum_{i=k+1}^{\infty} a_i g_i \right\|_k \le \left( \sum_{i=1}^{k} a_i^2 \right)^{1/2} \left\| \left( \sum_{i=1}^{k} f_{n_i}^2 \right)^{1/2} \right\|_k \\ &\quad + \left( \sum_{i=k+1}^{\infty} a_i^2 \right)^{1/2} \left\| \left( \sum_{i=k+1}^{\infty} (f_{n_i} - g_i)^2 \right)^{1/2} \right\|_k + \left\| \sum_{i=k+1}^{\infty} a_i g_i \right\|_k \\ &\le M\sqrt{k} \left( \sum_{i=1}^{k} a_i^2 \right)^{1/2} + \left( \sum_{i=k+1}^{\infty} a_i^2 \right)^{1/2} \sum_{i=k+1}^{\infty} \|f_{n_i} - g_i\|_i + C\sqrt{k} \left( \sum_{i=k+1}^{\infty} a_i^2 \right)^{1/2} \\ &\le (C + M + 1)\sqrt{k} \|a\|_2. \end{aligned}$$

The expansion

$$e^{uz^2} = \sum_{k=0}^{\infty} \frac{u^k}{k!} z^{2k}, \qquad u > 0,$$

yields

$$\int_{\Omega} (e^{u|f(\omega)|^2} - 1) \, dP(\omega) = \sum_{k=1}^{\infty} \frac{u^k}{k!} \int_{\Omega} |f(\omega)|^{2k} \, dP(\omega) \le \sum_{k=1}^{\infty} (C + M + 1)^{2k} \frac{u^k}{k!} (2k)^k \|a\|_2^{2k}.$$

Since this series is convergent for $0 < u < (2e)^{-1}(C + M + 1)^{-2}(\|a\|_2)^{-2}$, it follows that $f$ belongs to the Orlicz space $L_N$, $N(t) = e^{t^2} - 1$, and the theorem is proved. $\square$

Theorems 3 and 4 imply the main result of the paper.

**Theorem 5.** *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is an orthonormal sequence of random variables on the probability space* $(\Omega, \Sigma, \mathsf{P})$, $|f_n(\omega)| \le M$, $\omega \in \Omega$, $n \in \mathbb{N}$.

*Then from* $\{f_n\}_{n=1}^{\infty}$ *we can extract a subsequence* $\{f_{n_i}\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ *that is majorized in distribution by the Rademacher system. This means that for some* $C = C(M) > 0$ *independent of* $m \in \mathbb{N}$, $a = (a_i)_{i=1}^{m}$, *and* $z > 0$ *the following relation is valid:*

$$\mathsf{P}\left\{\omega \in \Omega : \left|\sum_{i=1}^{m} a_i f_{n_i}(\omega)\right| > z\right\} \le C\left|\left\{t \in [0,1] : \left|\sum_{i=1}^{m} a_i r_i(t)\right| > \frac{z}{C}\right\}\right|.$$

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