

RADEMACHER CHAOS IN SYMMETRIC SPACES

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Abstract

In this paper we study properties of series with respect to orthogonal systems $\{r_i(t)r_j(t)\}_{i \neq j}$ and $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$ in symmetric spaces on interval and square, respectively. Necessary and sufficient conditions for the equivalence of these systems with the canonical base in l_2 and also for the complementability of the corresponding generated subspaces, usually called *Rademacher chaos*, are derived. The results obtained allow, in particular, to establish the unimprovability of the exponential integrability of functions from Rademacher chaos. Besides, it is shown that for spaces that are "close" to L_∞ , the systems considered, in contrast to the ordinary Rademacher system, do not possess the property of unconditionality. The degree of this non-unconditionality is explained in the space L_∞ .

1 Introduction

Let

$$r_k(t) = \text{sign} \sin 2^{k-1} \pi t \quad (k = 1, 2, \dots)$$

be a Rademacher system on $I = [0, 1]$. The set of all real functions $y(t)$ which can be presented in the form

$$(1) \quad y(t) = \sum_{i \neq j} b_{i,j} r_i(t) r_j(t) \quad (t \in I)$$

is called the *Rademacher chaos* of degree 2 with respect to this system. The orthonormalized system $\{r_i(t)r_j(t)\}_{i < j}$ on I , unlike the ordinary Rademacher system, consists of functions which are not independent. Nevertheless, its properties remind in many aspects the properties of a family of independent and uniformly bounded functions. This concerns, in particular, the integrability of the functions of form (1): The condition $\sum_{i,j} b_{i,j}^2 < \infty$ implies the summability of the function $\exp(\alpha|y(t)|)$ for each $\alpha > 0$ [11, p.105]. In the same time, there are substantial differences. For example, the system $\{r_i(t)r_j(t)\}_{i < j}$ is not a Sidon system (see [14]).

The main purpose of this paper is a study of the behaviour of the Rademacher chaos in arbitrary symmetric spaces of functions defined on an interval. This will allow us to specify essentially the above mentioned results and to obtain new assertions about the geometric structure of subspaces of the symmetric spaces consisting of functions of the form (1). The essence of the method we use in the sequel is the passage to the

so called "decoupling" chaos, that is, the set of functions, defined on the square $I \times I$, of the form

$$(2) \quad x(s, t) = \sum_{i,j=1}^{\infty} a_{i,j} r_i(s) r_j(t).$$

Note that the study of the decoupling Rademacher chaos (i.e., multiple series with respect to this system) is of interest by itself.

Let us recall that a Banach space X of Lebesgue measurable on I functions $x = x(t)$ is called symmetric (s.s.), if:

- 1) The condition $|x(t)| \leq |y(t)|$, $y \in X$ implies $x \in X$ and $\|x\| \leq \|y\|$;
- 2) The assumption $\mu\{t \in I : |x(t)| > \tau\} = \mu\{t \in I : |y(t)| > \tau\}$ ($\tau > 0$) (μ is the Lebesgue measure on I), and $y \in X$, implies: $x \in X$ and $\|x\| = \|y\|$.

Given any s.s. X on the interval I , one can construct a space $X(I \times I)$ on the square $I \times I$, having properties which are similar to 1) and 2). Indeed, if $\Pi : I \rightarrow I \times I$ is one-to-one mapping that preserves the measure, then $X(I \times I)$ (we shall call it s.s. too) consists of all Lebesgue measurable on $I \times I$ functions $x = x(s, t)$ for which $x(\Pi(u)) \in X$ and $\|x\|_{X(I \times I)} = \|x(\Pi)\|_X$.

In the first part of the paper we obtain necessary and sufficient conditions for the equivalence of the systems $\{r_i(t)r_j(t)\}_{i < j}$ and $\{r_i(s)r_j(t)\}_{i,j=1}^{\infty}$ (in what follows we call them the multiple Rademacher systems) with the canonical base in l_2 , and also, for the complementability of the subspaces generated by them. The theorems proved resemble, in the essence, the analogous results about usual Rademacher series in s.s. on an interval (see [15], [16], [12, 2.b.4]). In the same time, there are some differences: The role of the "extreme" space is played by the space H , the closure of L_{∞} in the Orlicz space L_M , $M(t) = e^t - 1$, instead of G , the closure in the Orlicz space L_N , $N(t) = e^{t^2} - 1$ (see Theorems 1 – 4).

The behaviour of the multiple Rademacher systems "close" to the space L_{∞} , and in particular, in the same space, becomes more complicated. It is well-known [19] that the usual Rademacher system is a symmetric basic sequence in every s.s. X , that is, for arbitrary numerical sequence $(c_j)_{j=1}^{\infty}$, we have

$$\left\| \sum_{j=1}^{\infty} c_j r_j \right\|_X = \left\| \sum_{j=1}^{\infty} c_j^* r_j \right\|_X$$

((c_j^*) $_{j=1}^{\infty}$ is the rearrangement of $(|c_j|)_{j=1}^{\infty}$ in decreasing order). On the contrary, the multiple systems in L_{∞} and in the "close" spaces do not possess even the weaker unconditionality property. Another example: many s.s. sequences, which are intermediate between l_1 and l_2 , are spaces of coefficient sequences of Rademacher series from functional s.s. on the interval ([15], [1]). This is not true for multiple systems, for example in the case of the space l_p ($1 \leq p < 2$) (see Theorems 5 – 8 and the corollaries from them).

Let us recall certain definitions. If X is a s.s. on I , then we shall denote by X^0 the closure of L_{∞} in X . For any two measurable on I functions $x(t)$ and $y(t)$ we set:

$$\langle x, y \rangle = \int_0^1 x(t)y(t) dt$$

(if the integral exists). The last notation is similarly interpreted in the case of functions, defined in the square $I \times I$. The *associated* space X' with X is defined as the space of all measurable functions $y(t)$ for which the norm

$$\|y\|_{X'} = \sup\{< x, y > : \|x\|_X \leq 1\}$$

is finite. It is not difficult to verify that for $x \in L_\infty$, $\|x\|_X = \|x\|_{X''}$, and thus $X^0 = (X'')^0$ [6, p.255].

The norm in s.s. X is said to be *order semi-continuous*, if the conditions $x_n = x_n(t) \geq 0$ ($n = 1, 2, \dots$), $x_n \uparrow x$ almost everywhere on I , $x \in X$, imply: $\|x_n\|_X \rightarrow \|x\|_X$. If the norm in X is order semi-continuous, then it is isometrically embedded in X'' [6, p.255], that is,

$$\|x\|_X = \sup\{< x, y > : \|y\|_{X'} \leq 1\}.$$

Important examples of s.s. are the Orlicz spaces L_S ($S(t) \geq 0$ is a convex and continuous function on $[0, \infty)$) with the norm

$$\|x\|_{L_S} = \inf \left\{ u > 0 : \int_0^1 S(|x(t)|/u) dt \leq 1 \right\}$$

and the Marcinkiewicz space $M(\varphi)$ ($\varphi(t) \geq 0$ is a concave increasing function on $[0, 1]$) with the norm

$$\|x\|_{M(\varphi)} = \sup \left\{ \frac{1}{\varphi(t)} \int_0^t x^*(s) ds : 0 < t \leq 1 \right\}$$

($x^*(s)$ is the decreasing left continuous rearrangement of the function $|x(t)|$ [9, p.83]).

2 Equivalence of the multiple Rademacher systems to the canonical base in l_2

Theorem 1. *The system $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$ in the s. s. $X(I \times I)$ is equivalent to the canonical base in l_2 , if and only if $X \supset H$ where $H = L_M^0$, $M(t) = e^t - 1$.*

Proof. The equivalence of the system $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$ to the canonical base in l_2 means that for arbitrary numerical sequence $a = (a_{i,j})_{i,j=1}^\infty$, we have

$$(3) \quad \left\| \sum_{i,j=1}^\infty a_{i,j} r_i(s) r_j(t) \right\|_{X(I \times I)} \asymp \|a\|_2,$$

where $\|a\|_2 = (\sum_{i,j=1}^\infty a_{i,j}^2)^{1/2}$ (that is, a two-sided estimate takes place with constants that depend only on X).

Assume first that $X \supset H$. For $a = (a_{i,j})_{i,j=1}^\infty \in l_2$ denote

$$(4) \quad x(s, t) = \sum_{i,j=1}^\infty a_{i,j} r_i(s) r_j(t).$$

If $q \geq 2$, then after a second application of Khinchin's inequality [4, p.153] and the Minkowski integral inequality [18, p. 318] we obtain:

$$\begin{aligned}
\|x\|_q &= \left(\int_0^1 \int_0^1 |x(s, t)|^q ds dt \right)^{1/q} \\
&\leq \sqrt[q]{\int_0^1 \left(\sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{i,j} r_j(t) \right|^2 \right)^{q/2} dt}^{1/q} \\
&\leq \sqrt[q]{\sum_{i=1}^{\infty} \left(\int_0^1 \left| \sum_{j=1}^{\infty} a_{i,j} r_j(t) \right|^q dt \right)^{2/q}}^{1/2} \leq q \|a\|_2.
\end{aligned}$$

The inequality $\|x\|_q \leq q \|a\|_2$ obviously holds also for $1 \leq q < 2$.

Hence, making use of the expansion

$$\exp(uz) = \sum_{k=0}^{\infty} \frac{u^k}{k!} z^k \quad (u > 0),$$

we estimate:

$$\begin{aligned}
&\int_0^1 \int_0^1 \{ \exp(u|x(s, t)|) - 1 \} ds dt \\
&= \sum_{k=1}^{\infty} \frac{u^k}{k!} \int_0^1 \int_0^1 |x(s, t)|^k ds dt \leq \sum_{k=1}^{\infty} \frac{u^k}{k!} k^k \|a\|_2^k.
\end{aligned}$$

Now if $u < (2e\|a\|_2)^{-1}$, then the sum of the series on the right hand side of the last inequality does not exceed 1. Therefore, by the definition of the space H , the linear operator

$$Ta(s, t) = \sum_{i,j=1}^{\infty} a_{i,j} r_i(s) r_j(t)$$

is bounded from l_2 in $H(I \times I)$, and, in addition, $\|Ta\|_H = \|x\|_H \leq 2e\|a\|_2$. Since $X \supset H$, we derive $\|x\|_X \leq 2Ce\|a\|_2$.

On the other hand, for the function x , defined by the relation (4), in view of the embedding $X \subset L_1$ which holds for arbitrary s.s. on I [9, p.124], and also by virtue of Khinchin's inequality for the L_1 -norm with the constant from [20], and Minkowski inequality, we obtain :

$$\begin{aligned}
\|x\|_{X(I \times I)} &\geq D^{-1} \|x\|_1 \\
&\geq (\sqrt{2}D)^{-1} \int_0^1 \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{i,j} r_j(t) \right)^2 \right]^{1/2} dt \\
(5) \quad &\geq (\sqrt{2}D)^{-1} \left\{ \sum_{i=1}^{\infty} \left[\int_0^1 \left| \sum_{j=1}^{\infty} a_{i,j} r_j(t) \right| dt \right]^2 \right\}^{1/2} \\
&\geq (2D)^{-1} \|a\|_2
\end{aligned}$$

Thus (3) is proved.

For the proof of the inverse statement we need two lemmas.

In what follows

$$n_x(z) = \mu\{\omega : |x(\omega)| > z\}, \quad x^*(t) = \inf\{z > 0 : n_x(z) < t\}$$

are, respectively, the distribution function and the decreasing, left side continuous rearrangement of the measurable on I or on $I \times I$ function $|x(\omega)|$ [9, p.83].

Lemma 1. *If $n_{x_k}(z) \rightarrow n_x(z)$ for $k \rightarrow \infty$, then $x_k^*(t) \rightarrow x^*(t)$ at all points of continuity of the function $x^*(t)$ (and hence, almost everywhere).*

Proof. Let $x^*(t)$ be continuous at the point t_0 and $\varepsilon > 0$. For $z_1 = x^*(t_0) + \varepsilon$ we have $n_x(z_1) < t_0$ and thus $n_{x_k}(z_1) < t_0$, if $k \geq k_1$. Then, from the definition of the rearrangement,

$$(6) \quad x_k^*(t_0) \leq x^*(t_0) + \varepsilon \quad (k \geq k_1).$$

On the other hand, by the continuity of $x^*(t)$ at t_0 , there exists a $\delta > 0$ such that

$$(7) \quad x^*(t_0 + \delta) \geq x^*(t_0) - \varepsilon$$

If $z_2 = x^*(t_0 + \delta) - \varepsilon$, then $n_x(z_2) \geq t_0 + \delta$ and there exists k_2 such that $n_{x_k}(z_2) \geq t_0$ for $k \geq k_2$. Therefore, in view of (7),

$$x_k^*(t_0) \geq x^*(t_0) - 2\varepsilon \quad (k \geq k_2).$$

For $k \geq \max(k_1, k_2)$, both the previous inequality and inequality (6) hold. Combining all them we complete the proof.

Denote

$$L(z) = \mu\{(s, t) \in I \times I : \ln(e/s) \ln(e/t) > z\} \quad (z > 0).$$

Lemma 2. *For any $z \geq 1$, we have*

$$(8) \quad 1/2 \exp(-2\sqrt{z} + 2) \leq L(z) \leq 2 \exp(-\sqrt{z} + 2)$$

Proof. If $g_z = u + z/u$, $h_z = \max(z/u, u)$, then

$$(9) \quad h_z(u) \leq g_z(u) \leq 2h_z(u) \quad (u > 0).$$

After the change of variable $u = \ln(e/s)$ we obtain

$$\begin{aligned} L(z) &= \int_0^1 \mu\{t \in I : \ln(e/t) > z \ln^{-1}(e/s)\} ds \\ (10) \quad &= e \int_0^1 \exp\left(-\frac{z}{\ln(e/s)}\right) ds = e^2 \int_1^\infty \exp(-u - z/u) du \\ &= e^2 \int_1^\infty \exp(-g_z(u)) du. \end{aligned}$$

Integrating by parts, we estimate

$$\int_{\sqrt{z}}^z e^{-u} \frac{du}{u} = \frac{\exp(-\sqrt{z})}{\sqrt{z}} - \frac{\exp(-z)}{z} - \int_{\sqrt{z}}^z e^{-u} \frac{du}{u^2} \leq \frac{\exp(-\sqrt{z})}{\sqrt{z}}.$$

Thus, in view of (9) and (10), changing the variable, we arrive at the bound

$$e^{-2} L(z) \leq \int_1^\infty \exp(-h_z(u)) du = \int_1^{\sqrt{z}} \exp(-z/u) du + \exp(-\sqrt{z})$$

$$\begin{aligned}
&= \exp(-\sqrt{z}) + z \int_{\sqrt{z}}^z e^{-u} du/u^2 \\
&\leq \exp(-\sqrt{z}) + \sqrt{z} \int_{\sqrt{z}}^z e^{-u} du/u \leq 2 \exp(-\sqrt{z}).
\end{aligned}$$

For the reverse estimate, again with the help of (9) and (10), we obtain

$$e^{-2}L(z) \geq \int_1^\infty \exp(-2h_z(u)) \geq \int_{\sqrt{z}}^\infty \exp(-2u) du = 2^{-1} \exp(-2\sqrt{z}),$$

and inequality (8) is proved.

Returning now to the proof of Theorem 1, assume that the system $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$ is equivalent in $X(I \times I)$ to the canonical base in l_2 . If $a_{i,j} = b_i c_j$, then

$$\left\| \sum_{i,j=1}^\infty a_{i,j} r_i(s) r_j(t) \right\|_{X(I \times I)} \leq C \| (a_{i,j}) \|_2 = C \| (b_i) \|_2 \| (c_j) \|_2.$$

In particular, for $b_i = c_i = 1/\sqrt{n} (1 \leq i \leq n)$ and $b_i = c_i = 0 (i > n)$ ($n = 1, 2, \dots$), in view of the previous inequality and the symmetry of X ,

$$(11) \quad \|v_n^*(s)v_n^*(t)\|_{X(I \times I)} \leq C (n = 1, 2, \dots),$$

where $v_n(s) = 1/\sqrt{n} \sum_{i=1}^n r_i(s)$.

By the central limit theorem (see, for example, [3, p.161]),

$$\lim_{n \rightarrow \infty} \mu\{s \in I : v_n^*(s) > z\} = \Phi(z)$$

where

$$\Phi(z) = 2/\sqrt{2\pi} \int_z^\infty \exp(-u^2/2) du (z > 0).$$

Applying Lemma 1, we then obtain

$$\lim_{n \rightarrow \infty} v_n^*(s) = \Phi^{-1}(s) (s > 0)$$

($\Phi^{-1}(s)$ is the inverse function of $\Phi(z)$). Therefore, in view of (11) and the properties of the second associated space X'' [6, p.256],

$$\|\Phi^{-1}(s)\Phi^{-1}(t)\|_{X''(I \times I)} \leq C.$$

Since $\Phi(z) \leq \exp(-z^2/2)$, then $\ln^{1/2}(e/s) \ln^{1/2}(e/t) \in X''(I \times I)$ and thus, by Lemma 2, in view of the symmetry of X'' , we have

$$g(t) = \ln(e/t) \in X''.$$

If now $\chi_{(0,t)}$ is the characteristic function of the interval $(0, t)$, then, as known (see [8, p.88]),

$$h(t) = 1/\|\chi_{(0,t)}\|_{L_M} = M^{-1}(1/t) = \ln(1 + (e - 1)/t) \asymp g(t).$$

Therefore $h \in X''$ as well. For arbitrary function $x \in L_\infty$ and all $t \in I$, in view of [9, p.144], we have

$$x^*(t) \leq 1/t \int_0^t x^*(s) ds \leq t^{-1} \|x\|_{L_M} \|\chi_{(0,t)}\|_{(L_M)'} = \|x\|_{L_M} h(t).$$

Therefore, $\|x\|_X = \|x\|_{X''} \leq K \|x\|_{L_M}$ where $K = \|h\|_{X''}$. This means that $H = L_M^0 \subset X^0 \subset X$ and Theorem 1 is proved.

We shall prove a similar result for the "undecoupling" Rademacher chaos.

Theorem 2. *The system $\{r_i(t)r_j(t)\}_{i < j}$ in the s.s. X on I is equivalent to the canonical base in l_2 if and only if $X \supset H$.*

Proof. The idea of the proof is a passage to the "decoupling" chaos and then application of Theorem 1.

Let the function $y(t)$ be of the form (1) and

$$(12) \quad y^N(t) = \sum_{1 \leq i \neq j \leq N} b_{i,j} r_i(t) r_j(t)$$

for $N = 1, 2, \dots$. The next representation follows by combinatorics arguments.

$$(13) \quad y^N(t) = 2^{1-N} \sum_{D \subset \{1, \dots, N\}} \left(\sum_{i \in D, j \notin D} a_{i,j} r_i(t) r_j(t) \right),$$

where $a_{i,j} = b_{i,j} + b_{j,i}$ and the summation is taken over all $D \subset \{1, \dots, N\}$ [11, p.108]. Clearly the functions

$$\sum_{i \in D, j \notin D} a_{i,j} r_i(t) r_j(t) \quad (t \in I) \quad \text{and} \quad \sum_{i \in D, j \notin D} a_{i,j} r_i(s) r_j(t) \quad (s, t \in I)$$

are equimeasurable, that is, they have equal distribution functions. Hence, if s.s. X belongs to H , then it follows from Theorem 1 and equality (13) that

$$\|y^N\|_X \leq 2^{1-N} C \sum_{D \subset \{1, \dots, N\}} \left(\sum_{i \in D, j \notin D} a_{i,j}^2 \right)^{1/2} \leq 2C \|a\|_2.$$

Going to the limit as $N \rightarrow \infty$, we obtain $\|y\|_X \leq 2C \|a\|_2$.

On the other hand, it is well-known (see, for example, [13, p.149]), that for a certain $C_1 > 0$, $\|a\|_2 \leq C_1 \|y\|_1$. Therefore, in view of the embedding $X \subset L_1$, the reverse inequality also holds: $\|a\|_2 \leq C_1 D \|y\|_X$.

For the proof of the inverse proposition we need the following.

Lemma 3. *Let*

$$(14) \quad x(s, t) = \sum_{i \in A, j \in B} a_{i,j} r_i(s) r_j(t) \quad (s, t \in I),$$

where A, B are finite subsets of the set of natural numbers $\mathcal{N}$. Then the distribution function $n_x(z)$ will not change after the replacement of the summation sets A and B by the sets D and E respectively, where $|D| = |A|$, $|E| = |B|$ ($|G|$ is the number of elements of the set G).

Proof. Indeed, by the Fubini theorem

$$n_x(z) = \int_0^1 \mu \left\{ t \in I : \left| \sum_{j \in B} \left(\sum_{i \in A} a_{i,j} r_i(s) \right) r_j(t) \right| > z \right\} ds.$$

But it follows from the definition of the Rademacher function (see also [19]) that for a fixed $s \in I$ the set measure under the integral sign will not change after the substitution of B by E , provided $|E| = |B|$.

Continuing now the proof of Theorem 2, assume that the system of functions $\{r_i(t)r_j(t)\}_{i < j}$ is equivalent to the canonical base in l_2 .

For every function $x(s, t)$ of the form (14) there exists an equimeasurable, in absolute value, function

$$y(t) = \sum_{i \in D, j \in E} a_{i,j} r_i(t) r_j(t)$$

such that $|D| = |A|$ and $|E| = |B|$. It suffices to take $D = A$ and E so that $E \cap A = \emptyset$, $|E| = |B|$. Then, by Lemma 3, the absolute value of the function

$$z(s, t) = \sum_{i \in D, j \in E} a_{i,j} r_i(s) r_j(t)$$

is equimeasurable to $|x(s, t)|$, and in view of the independence of the Rademacher functions and the choice of D and E , to $|y(t)|$ ($t \in I$).

It follows from the assumptions and the presented reasonings that for a certain $C > 0$ and all $N = 1, 2, \dots$,

$$\left\| \sum_{i,j=1}^N a_{i,j} r_i(s) r_j(t) \right\|_{X(I \times I)} \leq C \|a\|_2.$$

After a passage to the limit as $N \rightarrow \infty$ we obtain

$$\left\| \sum_{i,j=1}^{\infty} a_{i,j} r_i(s) r_j(t) \right\|_{X(I \times I)} \leq C \|a\|_2.$$

Since the inverse inequality takes place always, the system $\{r_i(s)r_j(t)\}_{i,j=1}^{\infty}$ is also equivalent to the canonical base in l_2 . Therefore, by Theorem 1, $X \supset H$ and Theorem 2 is proved.

Remark 1. If the function $y(t)$ can be presented in the form (1), then, as we mentioned in the introduction, the condition $\sum_{i,j} b_{i,j}^2 < \infty$ implies the summability of the function $\exp(\alpha|y(t)|)$ for each $\alpha > 0$ [11, p.105]. Theorem 2 gives not only a new proof of this assertion, it shows also its unimprovability in the class of all s.s.

3 Complementability of the Rademacher chaos in symmetric space

Let $\tilde{M}(u)$ be the function which is complementary to $e^u - 1$, that is, $\tilde{M}(u) == \sup_{v>0} (uv - M(v))$. It can be easily shown that $\tilde{M}(u) \asymp u \ln u$ as $u \rightarrow \infty$. As

known (see [8, p.97]), the space $H^* = H' = L'_M = L_{\tilde{M}}$ is the conjugate one to H and $L'_{\tilde{M}} = L_M$.

Theorem 3. *Let X be a s.s. with order semi-continuous norm. The subspace $\mathcal{R}(X)$ of all functions from $X(I \times I)$, which admit a presentation of the form (2), is complemented in this space if and only if $H \subset X \subset H'$.*

Proof. If $H \subset X \subset H'$, then by duality, $H \subset X'$. Hence, in view of Theorem 1, for arbitrary $a = (a_{i,j})_{i,j=1}^\infty \in l_2$, we have

$$(15) \quad \left\| \sum_{i,j=1}^\infty a_{i,j} r_i(s) r_j(t) \right\|_{X(I \times I)} \leq C \|a\|_2$$

and

$$(16) \quad \left\| \sum_{i,j=1}^\infty a_{i,j} r_i(s) r_j(t) \right\|_{X'(I \times I)} \leq C \|a\|_2.$$

With any given $x = x(s, t) \in X(I \times I)$ we associate the system

$$(17) \quad a = (a_{i,j}(x))_{i,j=1}^\infty, \quad a_{i,j}(x) = \int_0^1 \int_0^1 x(s, t) r_i(s) r_j(t) ds dt$$

We shall show that the orthogonal projector

$$(18) \quad Px(s, t) = \sum_{i,j=1}^\infty a_{i,j}(x) r_i(s) r_j(t)$$

is bounded in $X(I \times I)$.

Indeed, in view of (16),

$$\begin{aligned} \|a\|_2^2 &= \sum_{i,j=1}^\infty (a_{i,j}(x))^2 = \int_0^1 \int_0^1 x(s, t) Px(s, t) ds dt \\ &\leq \|x\|_X \|Px\|_{X'} \leq C \|a\|_2 \|x\|_X. \end{aligned}$$

Then it follows from relations (15) that

$$\|Px\|_X \leq C \|a\|_2 \leq C^2 \|x\|_X \quad (x \in X).$$

Since the image of P coincides with $\mathcal{R}(X)$, then this subspace is complemented.

For the proof of the inverse statement we shall need the following.

Lemma 4. *If the projector P defined by equality (18) is bounded in the s.s. $X(I \times I)$, then $H \subset X \subset H'$.*

Proof. Denote by $\|P\|$ the norm of the projector P in the space $X(I \times I)$. Since

$$(19) \quad \langle Py, x \rangle = \langle y, Px \rangle \quad (x \in X, y \in X'),$$

then P is bounded also in $X'(I \times I)$ with a norm not exceeding $\|P\|$. Let the sequences $(a_{i,j}(x)), (a_{i,j}(y))$ be defined as in (17). Then, in view of the order semicontinuity of

X , the equality (19), and the pairwise orthogonality of the functions $r_i(s)r_j(t)$ ($i, j = 1, 2, \dots$), for arbitrary $x \in X(I \times I)$ we have

$$\begin{aligned} \|Px\|_X &= \sup\{\langle Px, y \rangle : \|y\|_{X'} \leq 1\} = \\ &= \sup\{\langle Px, Py \rangle : \|y\|_{X'} \leq 1\} = \\ &= \sup\left\{\left|\sum_{i,j} a_{i,j}(x)a_{i,j}(y)\right| : \|y\|_{X'} \leq 1\right\} \leq \\ &\leq \sup\{\|(a_{i,j}(x))\|_2 \|(a_{i,j}(y))\|_2 : \|y\|_{X'} \leq 1\}. \end{aligned}$$

Applying now (5) to the subspace X' and the function Py , we obtain:

$$\|(a_{i,j}(y))\|_2 \leq C\|Py\|_{X'} \leq C\|P\|\|y\|_{X'}.$$

Therefore, $\|Px\|_X \leq C\|P\|\|(a_{i,j}(x))\|_2$, that is,

$$\left\|\sum_{i,j=1}^{\infty} a_{i,j}r_i(s)r_j(t)\right\|_{X(I \times I)} \leq C\|P\|\|(a_{i,j})\|_2.$$

Just in the same manner one can show that

$$\left\|\sum_{i,j=1}^{\infty} a_{i,j}r_i(s)r_j(t)\right\|_{X'(I \times I)} \leq C\|A\|\|(a_{i,j})\|_2.$$

Since the reverse inequalities always hold (see (5)), then by Theorem 1 we conclude that $X \supset H$ and $X' \supset H$. Using the fact that $X \subset X''$, by the duality we obtain $H \subset X \subset H'$.

Next we continue the proof of Theorem 3. Assume that the subspace $\mathcal{R}(X)$ is complemented in $X(I \times I)$. In view of Lemma 4, it suffices to show that the projector P is bounded in this space.

Let $s = \sum_{i=1}^{\infty} s_i 2^{-i}$, $u = \sum_{i=1}^{\infty} u_i 2^{-i}$ ($s_i, u_i = 0, 1$) be the binary expansion of the numbers s and u from $I = [0, 1]$. Following [7, p.159] (see also [16] and [2]), we set

$$s \dot{+} u = \sum_{i=1}^{\infty} |s_i - u_i| 2^{-i}.$$

This operation transforms the interval I , as well as the square $I \times I$, in a compact Abelian group.

For $u, v \in I$ we define on $I \times I$ the shift transforms

$$\psi_{u,v}(s, t) = (s \dot{+} u, t \dot{+} v).$$

Since they preserve the Lebesgue measure on $I \times I$, the operators

$$T_{u,v}x(s, t) = x(\psi_{u,v}(s, t))$$

act in the s.s. $X(I \times I)$ isometrically.

Introduce the sets

$$U_i = \left\{u \in I : u = \sum_{k=1}^{\infty} u_k 2^{-k}, u_i = 0\right\}, \quad \bar{U}_i = I \setminus U_i \quad (i = 1, 2, \dots).$$

Since $r_1(s \dot{+} u) = r_1(s)$ and

$$(20) \quad r_i(s \dot{+} u) = \begin{cases} r_i(s), & \text{if } u \in U_{i-1} \\ -r_i(s), & \text{if } u \in \bar{U}_{i-1} \end{cases} \quad (i = 2, 3, \dots)$$

then the subspace $\mathcal{R}(X)$ is invariant with respect to the transforms $T_{u,v}$. Therefore, by Rudin's theorem [17, p.152], there exists a bounded projector Q from $X(I \times I)$ to this subspace which commutes with all operators $T_{u,v}$ ($u, v \in I$). We shall show that $Q = P$.

Let us present Q in the form

$$(21) \quad Qx(s, t) = \sum_{i,j=1}^{\infty} Q_{i,j}(x) r_i(s) r_j(t),$$

where $Q_{i,j}$ are linear functionals in $X(I \times I)$. Since Q is a projector, then

$$(22) \quad Q_{i,j}(r_k(s) r_m(t)) = \begin{cases} 1, & k = i, m = j \\ 0, & \text{otherwise} \end{cases}.$$

From equality (20) and the property $T_{u,v}Q = QT_{u,v}$, we obtain for $Q'_{i,j} = Q_{i+1,j+1}$ and $i, j = 1, 2, \dots$

$$Q'_{i,j}(T_{u,v}x) = \begin{cases} Q'_{i,j}(x), & u \in U_i, v \in U_j \text{ or } u \in \bar{U}_i, v \in \bar{U}_j \\ -Q'_{i,j}(x), & u \in U_i, v \in \bar{U}_j \text{ or } u \in \bar{U}_i, v \in U_j. \end{cases}$$

Hence, taking into account that $\mu(U_i) = 1/2$ ($i = 1, 2, \dots$), we conclude that

$$(23) \quad \int_{U_i} \int_{U_j} Q'_{i,j}(T_{u,v}x) du dv = \int_{\bar{U}_i} \int_{\bar{U}_j} Q'_{i,j}(T_{u,v}x) du dv = \frac{1}{4} Q'_{i,j}(x),$$

$$(23') \quad \int_{U_i} \int_{\bar{U}_j} Q'_{i,j}(T_{u,v}x) du dv = \int_{\bar{U}_i} \int_{U_j} Q'_{i,j}(T_{u,v}x) du dv = -\frac{1}{4} Q'_{i,j}(x).$$

The functionals $Q_{i,j}$ are bounded in $X(I \times I)$. Indeed, since $\{r_i(s) r_j(t)\}_{i,j=1}^{\infty}$ is an orthonormal system in $L_2(I \times I)$, then by the virtue of Khinchin's inequality with the constant from [20], and the embedding $X \subset L_1$,

$$|Q_{i,j}(x)| \leq \|Qx\|_2 \leq 2\|Qx\|_1 \leq 2D\|Qx\|_{X(I \times I)} \leq 2D\|Q\| \|x\|_{X(I \times I)},$$

where $\|Q\|$ is the norm of the projector Q in the space $X(I \times I)$. Therefore, in relations (23) and (23'), the functional can be taken out from the integral sign.

It is easily verified that for any $s \in I$,

$$\{z \in I : z = s \dot{+} u, u \in U_i\} = \begin{cases} U_i, & \text{if } s \in U_i \\ \bar{U}_i, & \text{if } s \in \bar{U}_i, \end{cases}$$

$$\{z \in I : z = s \dot{+} u, u \in \bar{U}_i\} = \begin{cases} \bar{U}_i, & \text{if } s \in U_i \\ U_i, & \text{if } s \in \bar{U}_i. \end{cases}$$

Then, introducing the notations

$$k_{i,j} = \int_{U_i} \int_{U_j} x(u, v) du dv, l_{i,j} = \int_{\bar{U}_i} \int_{U_j} x(u, v) du dv,$$

$$\begin{aligned}
m_{i,j} &= \int_{U_i} \int_{\bar{U}_j} x(u, v) du dv, n_{i,j} = \int_{\bar{U}_i} \int_{\bar{U}_j} x(u, v) du dv, \\
K_{i,j}(s, t) &= \chi_{U_i \times U_j}(s, t), L_{i,j}(s, t) = \chi_{\bar{U}_i \times U_j}(s, t), \\
M_{i,j}(s, t) &= \chi_{U_i \times \bar{U}_j}(s, t), N_{i,j}(s, t) = \chi_{\bar{U}_i \times \bar{U}_j}(s, t),
\end{aligned}$$

we can write

$$\begin{aligned}
\int_{U_i} \int_{U_j} T_{u,v} x du dv &= k_{i,j} K_{i,j} + l_{i,j} L_{i,j} + m_{i,j} M_{i,j} + n_{i,j} N_{i,j}, \\
\int_{\bar{U}_i} \int_{U_j} T_{u,v} x du dv &= l_{i,j} K_{i,j} + k_{i,j} L_{i,j} + n_{i,j} M_{i,j} + m_{i,j} N_{i,j}, \\
\int_{U_i} \int_{\bar{U}_j} T_{u,v} x du dv &= m_{i,j} K_{i,j} + n_{i,j} L_{i,j} + k_{i,j} M_{i,j} + l_{i,j} N_{i,j}, \\
\int_{\bar{U}_i} \int_{\bar{U}_j} T_{u,v} x du dv &= n_{i,j} K_{i,j} + m_{i,j} L_{i,j} + l_{i,j} M_{i,j} + k_{i,j} N_{i,j}.
\end{aligned}$$

Thus, the relations (23) and (23'), the linearity of $Q'_{i,j}$, and the equalities

$$r_{i+1}(s) = \chi_{U_i}(s) - \chi_{\bar{U}_i}(s) \quad (i = 1, 2, \dots)$$

yield

$$\begin{aligned}
Q'_{i,j}(x) &= Q'_{i,j} \left((K_{i,j} + N_{i,j} - L_{i,j} - M_{i,j}) \int_0^1 \int_0^1 x(u, v) r_{i+1}(u) r_{j+1}(v) du dv \right) \\
&= Q'_{i,j} \left(\int_0^1 \int_0^1 x(u, v) r_{i+1}(u) r_{j+1}(v) du dv r_{i+1}(s) r_{j+1}(t) \right) \\
&= \int_0^1 \int_0^1 x(u, v) r_{i+1}(u) r_{j+1}(v) du dv.
\end{aligned}$$

Therefore

$$Q_{i,j}(x) = a_{i,j}(x) \quad (i, j = 2, 3, \dots).$$

A similar reasoning shows that the last equality remains true also for $i, j = 1, 2, \dots$. Thus, in view of the relations (18) and (21), we obtain $Q = P$ and hence the theorem is proved.

Theorem 4. *Let X be a s.s. with an order semi-continuous norm in I . The subspace $\bar{\mathcal{R}}(X)$ of all functions from X which can be presented in the form (1), is complemented in X if and only if $H \subset X \subset H'$.*

The proof is similar to that in Theorem 3. Omitting the details we only remark that instead of the projector P one should consider also the orthogonal projector

$$Sx(t) = \sum_{1 \leq i < j < \infty} \int_0^1 x(u) r_i(u) r_j(u) du r_i(t) r_j(t),$$

and instead of the transforms $\psi_{u,v}$ of the square $I \times I$, the transforms $\psi_u(s) = s \dot{+} u$ ($u \in I$) of the interval I .

4 Rademacher chaos in L_∞ and in "close" s.s.

For any given $n \in \mathcal{N}$ and $\theta = \{\theta_{i,j}\}_{i,j=1}^n$, $\theta_{i,j} = \pm 1$, we introduce the quantity

$$\varphi_n(\theta) = \left\| \sum_{i,j=1}^n \theta_{i,j} r_i(s) r_j(t) \right\|_\infty$$

($\|\cdot\|_\infty$ is the norm in the space $L_\infty(I \times I)$).

It follows from the definition of the Rademacher functions that $\sup_\theta \varphi_n(\theta)$ is attained for $\theta_{i,j} = 1$ ($i, j = 1, \dots, n$) and it equals n^2 .

Theorem 5. *We have*

$$(24) \quad \inf_\theta \varphi_n(\theta) \asymp 2^{-n^2} \sum_\theta \varphi_n(\theta) \asymp n^{3/2}$$

with constants which do not depend on $n \in \mathcal{N}$.

Proof. Note first that by virtue of Khinchin's inequality with the constant from [20], for any $\theta_{i,j} = \pm 1$, we have

$$\begin{aligned} \left\| \sum_{i,j=1}^n \theta_{i,j} r_i(s) r_j(t) \right\|_\infty &= \sup_{0 < s \leq 1} \sum_{j=1}^n \left| \sum_{i=1}^n \theta_{i,j} r_i(s) \right| \geq \\ &\geq \sum_{j=1}^n \int_0^1 \left| \sum_{i=1}^n \theta_{i,j} r_i(s) \right| ds \geq \\ &\geq \frac{1}{\sqrt{2}} \sum_{j=1}^n \left\{ \int_0^1 \left(\sum_{i=1}^n \theta_{i,j} r_i(s) \right)^2 ds \right\}^{1/2} = 1/\sqrt{2} n^{3/2}. \end{aligned}$$

Therefore, $\inf_\theta \varphi_n(\theta) \geq 1/\sqrt{2} n^{3/2}$.

To prove the opposite inequality, note first that

$$(25) \quad 2^{-n^2} \sum_\theta \varphi_n(\theta) = \int_0^1 \left\| \sum_{i,j=1}^n r_{i,j}(u) r_i(s) r_j(t) \right\|_\infty du,$$

where $\{r_{i,j}\}_{i,j=1}^n$ are the first n^2 Rademacher functions numbered in an arbitrary order.

Apply the known theorem about the distribution of the L_∞ -norm of polynomials with random coefficients (see [5, p. 97]) to the linear space B of functions of the form

$$f(s, t) = \sum_{i,j=1}^n a_{i,j} r_i(s) r_j(t)$$

defined on $I \times I$. Since for every function $f \in B$ we have $|f(s, t)| = \|f\|_\infty$ on the square $K \subset I \times I$ with a measure $\mu(K) \geq 2^{-2(n-1)}$, then for $z \geq 2$,

$$\mu \left\{ u \in I : \left\| \sum_{i,j=1}^n r_{i,j}(u) r_i(s) r_j(t) \right\|_\infty \geq 3 \left[\sum_{i,j=1}^n \ln(2^{2n-1} z) \right]^{1/2} \right\} \leq 2/z.$$

Now, after some not complicated transformations for the functions

$$X(u) = \left\| \sum_{i,j=1}^n r_{i,j}(u) r_i(s) r_j(t) \right\|_{\infty}$$

we arrive at the estimate

$$\mu\{u \in I : X(u) \geq 3\sqrt{2}n^{3/2}\tau\} \leq e^{-\tau^2} (\tau \geq 2).$$

Therefore, in view of (25),

$$\begin{aligned} 2^{-n^2} \sum_{\theta} \varphi_n(\theta) &= \|X\|_1 = 3\sqrt{2}n^{3/2} \int_0^{\infty} \mu\{u \in I : X(u) \geq 3\sqrt{2}n^{3/2}\tau\} d\tau \\ &\leq 3\sqrt{2}n^{3/2} \left(2 + \int_2^{\infty} e^{-\tau^2} d\tau \right) \leq 9\sqrt{2}n^{3/2}, \end{aligned}$$

and the theorem is proved.

The "probability" proof of Theorem 5 does not yield the specific arrangement of the signs for which the exact lower bound in (24) is attained. That is why we give additionally one more proposition where, for simplicity, only the case $n = 2^k$ is considered.

Proposition. *For any $k = 0, 1, 2, \dots$ there exists an arrangement of the signs $\theta = \{\theta_{i,j}\}_{i,j=1}^{2^k}$ for which*

$$\varphi_{2^k}(\theta) \leq 2^{3k/2}.$$

Proof. Consider the Walsh matrix, that is, the matrix constructed for the first 2^k Walsh functions

$$w_1(t), w_2(t), \dots, w_{2^k}(t)$$

($w_1(t) = 1, w_{2^i+j}(t) = r_{i+2}(t)w_j(t), i = 0, 1, \dots; j = 1, \dots, 2^i$) [7, p.158]. Since on the intervals $\Delta_i^k = ((i-1)2^{-k}, i2^{-k})$ ($1 \leq i \leq 2^k$) these functions are constant and equal $+1$ or -1 , then one can determine the signs

$$\theta_{i,j} = \text{sign } w_j(t), \quad t \in \Delta_i^k.$$

For each $s \in I$ consider the function

$$x_s(u) = \sum_{i=1}^{2^k} r_i(s) \chi_{\Delta_i^k}(u) \quad (u \in I).$$

Since $|x_s(u)| = 1$, then $\|x_s\|_2 = 1$. For $1 \leq j \leq 2^k$ the Fourier – Walsh coefficients of the function $x_s(u)$ are given by

$$c_j(x_s) = \sum_{i=1}^{2^k} \int_{\Delta_i^k} w_j(u) du r_i(s) = 2^{-k} \sum_{i=1}^{2^k} \theta_{i,j} r_i(s).$$

Then, by virtue of Hölder's and Bessel's inequalities, for the chosen $\theta_{i,j}$ we have

$$\left\| \sum_{i,j=1}^{2^k} \theta_{i,j} r_i(s) r_j(t) \right\|_{\infty} = \sup_{0 < s \leq 1} \sum_{j=1}^{2^k} \left| \sum_{i=1}^{2^k} \theta_{i,j} r_i(s) \right| = 2^k \sup_{0 < s \leq 1} \sum_{j=1}^{2^k} |c_j(x_s)| \leq$$

$$\leq 2^{3k/2} \sup_{0 < s \leq 1} \left\{ \sum_{j=1}^{2^k} (c_j(x_s))^2 \right\}^{1/2} \leq 2^{3k/2} \|x_s\|_2 = 2^{3k/2},$$

and the proposition is proved.

For the "non-decoupling" chaos, introduce the following quantity which is analogous to $\varphi_n(\theta)$.

$$(26) \quad \bar{\varphi}_n(\theta) = \left\| \sum_{i,j=1}^n \theta_{i,j} r_i(t) r_j(t) \right\|_{\infty}$$

($\theta = \{\theta_{i,j}\}_{i,j=1}^n$ is the symmetric arrangement of the signs, i.e., $\theta_{i,j} = \theta_{j,i}$, $\|\cdot\|_{\infty}$ is the norm in L_{∞} on I).

Theorem 6. *With a certain $C > 0$, independent of $n \in \mathcal{N}$,*

$$(27) \quad \inf_{\theta} \bar{\varphi}_n(\theta) \asymp 2^{-n^2} \sum_{\theta} \bar{\varphi}_n(\theta) \asymp n^{3/2}$$

Proof. Observe that for arbitrary $n \in \mathcal{N}$ and $a = (a_{i,j})_{i,j=1}^n$ the following inequality holds:

$$\left\| \sum_{i,j=1}^n a_{i,j} r_i(t) r_j(t) \right\|_{L_{\infty}} \leq \left\| \sum_{i,j=1}^n a_{i,j} r_i(s) r_j(t) \right\|_{L_{\infty}(I \times I)},$$

which, by Theorem 5, yields the inequalities $\leq$ in relation (27).

Let θ be an arbitrary symmetric arrangement of the signs. By Theorem 5, there exists a constant $C > 0$ such that $\|x\| \geq Cn^{3/2}$ for all natural n and the functions

$$x(s, t) = \left\| \sum_{i,j=1}^n \theta_{i,j} r_i(s) r_j(t) \right\|_{\infty}.$$

Denote

$$x_1(s, t) = \sum_{1 \leq i < j \leq n} \theta_{i,j} r_i(s) r_j(t), \quad x_2(s, t) = \sum_{1 \leq j < i \leq n} \theta_{i,j} r_i(s) r_j(t),$$

$$x_3(s, t) = x(s, t) - x_1(s, t) - x_2(s, t).$$

Since $\|x_3\|_{\infty} \leq n$ and $x_1(s, t) = x_2(t, s)$, then

$$(28) \quad \|x_1\|_{\infty} \geq \frac{1}{4} Cn^{3/2}$$

for all sufficiently large n . Further we need a theorem from [10] about the comparison of the distribution functions of quadratic and bilinear forms. Let us formulate it: *Let $X = (X_1, X_2, \dots, X_n)$ be a vector with coordinates that are independent symmetrically distributed random variables on a certain probability space with a measure P , and let $Y_1, Y_2, \dots, Y_n$ be independent copies of $X_1, X_2, \dots, X_n$, respectively. Then there exist constants K_1, K_2, k_1, k_2 such that for arbitrary forms*

$$Q(X) = \sum_{1 \leq i < j \leq n} a_{i,j} X_i X_j, \quad \tilde{Q}(X, Y) = \sum_{1 \leq i < j \leq n} a_{i,j} X_i Y_j$$

and any $z > 0$,

$$K_1 P\{k_1 |Q(X)| > z\} \leq P\{|\tilde{Q}(X, Y)| > z\} \leq K_2 P\{k_2 |Q(X)| > z\}.$$

Let us apply this theorem in the case $X_i = r_i(t), Y_i = r_i(s) (t, s \in I)$. Then, in view of inequality (28) and the fact that the norm of a function in L_∞ is defined by its distribution function, we obtain

$$\left\| \sum_{1 \leq i < j \leq n} \theta_{i,j} r_i(t) r_j(t) \right\|_\infty \geq C_1 n^{3/2}$$

with certain $C_1 > 0$ and for all sufficiently large n . Since the arrangement of the signs is symmetric and the norm of the diagonal terms in the sum do not exceed n , we conclude that

$$\left\| \sum_{i,j=1}^n \theta_{i,j} r_i(t) r_j(t) \right\|_\infty \geq C_2 n^{3/2}.$$

Diminishing the constant C_2 one can make the last inequality hold for all natural n . The theorem is proved.

Remark 2. The relation (27), obviously remains true also in the case when the summation in (26) is expanded only over $i < j$.

Recall that an orthonormal system of functions $\{u_k(t)\}_{k=1}^\infty$, defined on a certain probability space, is called a *Sidon system* (see, for example, [7, p.327]), if for every generalized polynomial $\mathcal{P}(t) = \sum_{k=1}^n a_k u_k(t)$ with respect to this system holds the estimate

$$C^{-1} \sum_{k=1}^n |a_k| \leq \|\mathcal{P}\|_\infty \leq C \sum_{k=1}^n |a_k|,$$

where the constant $C > 0$ does not depend on the polynomial $\mathcal{P}(t)$.

Corollary. *The multiple systems $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$ and $\{r_i(t)r_j(t)\}_{i < j}$ are not Sidon systems on $I \times I$ and I , respectively.*

Remark 3. The assertion in the last corollary concerning the system $\{r_i(t)r_j(t)\}_{i < j}$ was proved in [14].

The basic sequence $\{x_n\}_{n=1}^\infty$ in a Banach space X is said to be unconditional if the convergence of the series $\sum_{n=1}^\infty a_n x_n$ ($a_n \in \mathcal{R}$) in X implies the convergence in the same space of the series $\sum_{n=1}^\infty \theta_n a_n x_n$ for arbitrary signs $\theta_n = \pm 1$ ($n = 1, 2, \dots$) [7, p.22]. It is not difficult to verify that in this case there exists a constant $C > 0$ such that

$$(29) \quad \left\| \sum_{n \in F} a_n x_n \right\|_X \leq C \left\| \sum_{n=1}^\infty a_n x_n \right\|_X$$

for arbitrary $F \subset \mathcal{N}$. Besides, the smallest C for which (29) is true, is called the *constant of unconditionality* for the sequence $\{x_n\}$.

As seen from Theorems 1 and 2, the basic sequences $\{r_i(s)r_j(t)\}_{i,j=1}^\infty$ and $\{r_i(t)r_j(t)\}_{i < j}$ are not only unconditional, they are symmetric as well (see Introduction) in the s.s.

$X(I \times I)$ and X , respectively, provided $X \supset H$. The situation is completely different in the space L_∞ and in s.s. which are "close" to it.

We prove first one more auxiliary proposition.

Lemma 5. *Let $n_0 = 0 < n_1 < n_2 < \dots$; $c_{i,j} \in \mathcal{R}$ and*

$$y_k = \sum_{i,j=n_k+1}^{n_{k+1}} c_{i,j} r_i(s) r_j(t) \neq 0.$$

Then $\{y_k\}_{k=1}^\infty$ is an unconditional basic sequence in any s.s. $X(I \times I)$ with a constant of unconditionality equal to 1.

Proof. It suffices to verify (see [7, p.23]) that for arbitrary $m = 1, 2, \dots$, $\theta_k = \pm 1$ ($k = 1, 2, \dots, m$) and real a_k , the norms of the functions $y = \sum_{k=1}^m a_k y_k$ and $y_\theta = \sum_{k=1}^m \theta_k a_k y_k$ in $X(I \times I)$ coincide.

The values of the Rademacher functions $r_i(s)$ ($i = 1, 2, \dots, n_{m+1}$) give all possible arrangements of the signs (up to multiplication by -1), corresponding to the intervals $((k-1)2^{-n_{m+1}+1}, k2^{-n_{m+1}+1})$ ($k = 1, 2, \dots, 2^{n_{m+1}-1}$). That is why the distribution function

$$\begin{aligned} n_{y_\theta}(z) &= \mu\{(s, t) \in I \times I : |y_\theta(s, t)| > z\} = \int_0^1 \mu\{s \in I : |y_\theta(s, t)| > z\} dt \\ &= \int_0^1 \mu\left\{s \in I : \left| \sum_{k=1}^m \theta_k a_k \sum_{i=n_k+1}^{n_{k+1}} \left(\sum_{j=n_k+1}^{n_{k+1}} c_{i,j} r_j(t) \right) r_i(s) \right| > z\right\} dt \end{aligned}$$

does not depend on the signs θ_k . Therefore $\|y_\theta\|_{X(I \times I)} = \|y\|_{X(I \times I)}$ and the lemma is proved.

For any arrangement of the signs $\theta = \{\theta_{i,j}\}_{i,j=1}^\infty$, $\theta_{i,j} = \pm 1$, define the operator

$$T_\theta x(s, t) = \sum_{i,j=1}^\infty \theta_{i,j} a_{i,j} r_i(s) r_j(t)$$

on $\mathcal{R}(L_\infty)$, where

$$x(s, t) = \sum_{i,j=1}^\infty a_{i,j} r_i(s) r_j(t) \in L_\infty(I \times I), a = (a_{i,j})_{i,j=1}^\infty \in l_2.$$

Theorem 7. *For arbitrary $0 < \varepsilon < 1/2$, there exists an arrangement of the signs $\theta = \{\theta_{i,j}\}$, for which*

$$T_\theta : \mathcal{R}(L_\infty) \not\rightarrow M(\varphi_\varepsilon)(I \times I),$$

where $M(\varphi_\varepsilon)$ is the Marcinkiewicz space defined by the function

$$\varphi_\varepsilon(t) = t \log_2^{-\varepsilon+1/2}(2/t)$$

(see Introduction).

Proof. By Theorem 5 and Lemma 3, for every $k = 0, 1, 2, \dots$ one can find $\theta_{i,j} = \pm 1$ ($2^k + 1 \leq i, j \leq 2^{k+1}$) so that the associated function

$$z_k(s, t) = \sum_{i,j=2^k+1}^{2^{k+1}} \theta_{i,j} r_i(s) r_j(t)$$

satisfies

$$(30) \quad \|z_k\|_\infty \asymp 2^{3k/2}.$$

Set $x_k(s, t) = 2^{-(3+\varepsilon)k/2} z_k(s, t)$ and

$$x(s, t) = \sum_{k=0}^{\infty} x_k(s, t) = \sum_{i,j=1}^{\infty} a_{i,j} r_i(s) r_j(t),$$

where $a_{i,j} = 2^{-(3+\varepsilon)k/2} \theta_{i,j}$ ($2^k + 1 \leq i, j \leq 2^{k+1}$) and $a_{i,j} = 0$ otherwise.

It follows from (30) that

$$\|x\|_\infty \leq C \sum_{k=0}^{\infty} 2^{-\varepsilon k/2} = C 2^{\varepsilon/2} / (2^{\varepsilon/2} - 1),$$

i.e., $x \in \mathcal{R}(L_\infty)$.

Let the arrangement of the signs θ consists of the values $\theta_{i,j}$ ($2^k + 1 \leq i, j \leq 2^{k+1}$) found above and arbitrary $\theta_{i,j} = \pm 1$ in the other cases. Then

$$y(s, t) = T_\theta x(s, t) = \sum_{k=0}^{\infty} 2^{-(3+\varepsilon)k/2} y_k(s, t)$$

where

$$y_k(s, t) = \sum_{i,j=2^k+1}^{2^{k+1}} r_i(s) r_j(t).$$

Applying Lemma 5 to this sequence and $X = M(\varphi_\varepsilon)$, in view of the relation (29), we obtain

$$(31) \quad \|y\|_{M(\varphi_\varepsilon)} \geq 2^{-(3+\varepsilon)k/2} \|y_k\|_{M(\varphi_\varepsilon)} \quad (k = 0, 1, 2, \dots).$$

It is clear from the definition of the Rademacher functions that $y_k(s, t) = 2^{2k}$ for $0 < s, t < 2^{-2^{k+1}+1}$. Thus, if $u_k = 2^{-2^{k+2}+1}$ ($k = 0, 1, \dots$), then the rearrangement satisfies $y_k^*(u_k) \geq 2^{2k}$.

Since, according to [9, p.156],

$$\|x\|_{M(\varphi_\varepsilon)} \asymp \sup_{0 < u \leq 1} x^*(u) \log_2^{\varepsilon-1/2}(2/u),$$

we obtain from the last inequality that

$$\|y_k\|_{M(\varphi_\varepsilon)} \geq C 2^{2k} \log_2^{\varepsilon-1/2}(2/u_k) \geq C 2^{(\varepsilon+3/2)k-1}.$$

Then it follows from (31) that for every $k = 0, 1, \dots$,

$$\|y\|_{M(\varphi_\varepsilon)} \geq 2^{\varepsilon k/2-1}$$

and thus $y = T_\theta x \notin M(\varphi_\varepsilon)(I \times I)$.

Corollary 2. *If $p \in [1, 2)$, then there is no a s.s. X for which*

$$(32) \quad \left\| \sum_{i,j=1}^{\infty} a_{i,j} r_i(s) r_j(t) \right\|_{X(I \times I)} \asymp \left(\sum_{i,j=1}^{\infty} |a_{i,j}|^p \right)^{1/p}.$$

Proof. If $p = 1$, then the assertion follows immediately from Theorem 5, since for every s.s. X on I , $X \supset L_\infty$ (see [9, p.124]).

Assume that (32) is true for $p \in (1, 2)$. Then, taking $a_{1,j} = c_j$ and $a_{i,j} = 0$ ($i \neq 1$), we obtain

$$\left\| \sum_{j=1}^{\infty} c_j r_j \right\|_X \asymp \left(\sum_{j=1}^{\infty} |c_j|^p \right)^{1/p}.$$

Therefore, by Theorem 3 from [15], $X = \Lambda_p(\varphi)$ where $\Lambda_p(\varphi)$ is the Lorentz space with the norm

$$\|x\| = \left\{ \int_0^1 (x^*(t))^p d\varphi(t) \right\}^{1/p}, \quad \varphi(t) = \log_2^{1-p}(2/t).$$

Since the fundamental function of this space satisfies

$$\|\chi_{(0,t)}\|_{\Lambda_p(\varphi)} = \varphi^{1/p}(t) = \log_2^{-1+1/p}(2/t),$$

and the Marcinkiewicz space is maximal among the s.s. with the same fundamental function [9, p.162], then $X \subset M(\varphi_\varepsilon)$ where $\varepsilon = 1/p - 1/2$. Applying Theorem 7 we obtain $\theta = \{\theta_{i,j}\}$, $\theta_{i,j} = \pm 1$ and therefore

$$T_\theta : \mathcal{R}(L_\infty) \not\hookrightarrow M(\varphi_\varepsilon)(I \times I).$$

Moreover,

$$T_\theta : \mathcal{R}(L_\infty) \not\hookrightarrow X(I \times I),$$

and hence there exists a function $x \in \mathcal{R}(L_\infty) \subset X(I \times I)$ such that $T_\theta x \notin X(I \times I)$. Since this is in a contradiction with the relation (32), the corollary is proved.

Similar propositions hold in the "undecoupling" chaos.

Recall that $\bar{\mathcal{R}}(L_\infty)$ is a subspace of L_∞ , consisting of all functions of the form

$$y(t) = \sum_{1 \leq i \neq j < \infty} b_{i,j} r_i(t) r_j(t), \quad b = (b_{i,j})_{i,j=1}^\infty \in l_2.$$

For any arrangement of the signs $\theta = \{\theta_{i,j}\}_{i,j=1}^\infty$, $\theta_{i,j} = \pm 1$, we define the operator

$$\bar{T}_\theta y(t) = \sum_{i,j=1}^{\infty} \theta_{i,j} b_{i,j} r_i(t) r_j(t)$$

in the space $\bar{\mathcal{R}}(L_\infty)$.

Theorem 8. *For arbitrary $0 < \varepsilon < 1/2$ there exists arrangement of the signs $\theta = \{\theta_{i,j}\}$ for which*

$$\bar{T}_\theta : \bar{\mathcal{R}}(L_\infty) \not\hookrightarrow M(\varphi_\varepsilon),$$

where $M(\varphi_\varepsilon)$ is the Marcinkiewicz space defined by the function $\varphi_\varepsilon(t) = t \log_2^{-\varepsilon+1/2}(2/t)$.

Proof. Assume that the theorem is not true, that is, for a certain $\varepsilon > 0$,

$$(33) \quad \bar{T}_\theta : \bar{\mathcal{R}}(L_\infty) \rightarrow M(\varphi_\varepsilon)$$

for any arrangement of the signs θ .

Let

$$x(s, t) = \sum_{i,j=1}^{\infty} a_{i,j} r_i(s) r_j(t) \in L_\infty(I \times I), \quad a = (a_{i,j})_{i,j=1}^{\infty} \in l_2.$$

By Lemma 3, for arbitrary natural n , the absolute values of the functions

$$x_n(s, t) = \sum_{i,j=1}^n a_{i,j} r_i(s) r_j(t) \quad \text{and} \quad y_n(t) = \sum_{i,j=1}^n a_{i,j} r_i(t) r_{j+n}(t)$$

are equimeasurable.

Let us introduce the following arrangement of the signs $\bar{\theta} = \{\bar{\theta}_{i,j}\}$, $\bar{\theta}_{i,j} = \theta_{i,j-n}$, if $j > n$, and $\bar{\theta}_{i,j}$ arbitrary, if $j \leq n$. Then the absolute values of the functions

$$T_\theta x_n(s, t) = \sum_{i,j=1}^n \theta_{i,j} a_{i,j} r_i(s) r_j(t)$$

and

$$\bar{T}_{\bar{\theta}} y_n(t) = \sum_{i,j=1}^n \bar{\theta}_{i,j+n} a_{i,j} r_i(t) r_{j+n}(t) = \sum_{i,j=1}^n \theta_{i,j} a_{i,j} r_i(t) r_{j+n}(t)$$

are also equimeasurable.

Summarizing, from relations (33) we obtain for the arrangement $\bar{\theta}$:

$$\|T_\theta x_n\|_{M(\varphi_\varepsilon)(I \times I)} = \|\bar{T}_{\bar{\theta}} y_n\|_{M(\varphi_\varepsilon)} \leq C \|y_n\|_\infty = C \|x_n\|_\infty.$$

Since the norm in the spaces L_∞ and $M(\varphi_\varepsilon)$ is order semi-continuous (see [9]), a passage to the limit as $n \rightarrow \infty$ implies

$$\|T_\theta x\|_{M(\varphi_\varepsilon)(I \times I)} \leq C \|x\|_\infty,$$

that is, for every arrangement of signs θ ,

$$T_\theta : \mathcal{R}(L_\infty) \rightarrow M(\varphi_\varepsilon)(I \times I),$$

which contradicts Theorem 7.

Corollary 3. *If $p \in [1, 2)$, then there is no s.s. X on I for which*

$$\left\| \sum_{i,j=1}^{\infty} b_{i,j} r_i r_j \right\|_X \asymp \left(\sum_{i,j=1}^{\infty} |b_{i,j}|^p \right)^{1/p} \quad (b_{i,i} = 0).$$

The proof is similar to that of Corollary 2, with the only difference that instead of Theorem 7 one has to apply Theorem 8.

Remark 4. Propositions similar to Corollaries 2 and 3 can be established for other s.s. sequences, for example, for the spaces of Lorentz, Marcinkiewicz and Orlicz. In the case of usual Rademacher system the situation is completely different (see [15], [1]). For instance, it was shown in [1] that for any space of sequences E , interpolation between l_1 and l_2 , one can find a functional s.s. X on $[0, 1]$ such that

$$\left\| \sum_{j=1}^{\infty} c_j r_j \right\|_X \asymp \|(c_j)\|_E.$$

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